When is Separate Sales Optimal?

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This paper studies a multi-product monopolist facing a privately informed consumer with additive and type-increasing valuations. Separate monopoly pricing is robustly optimal with respect to the type distribution if and only if the marginal rate of substitution between each pair of goods is monotone. This result is obtained by extending the conventional Myersonian approach.

Keywords: Multi-product monopoly, bundling, separate monopoly pricing, ratio monotonicity

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1 Introduction

Bundling is a common practice among firms selling multiple products. It can be profitable under a variety of circumstances, even when there is no complementarity between the goods. When the consumer valuations are negatively correlated (Stigler, 1963; Adams and Yellen, 1976) or independently distributed across many goods (Bakos and Brynjolfsson, 1999; Armstrong, 1999), the profit-maximizing strategy involves selling the goods together in bundles. It is also straightforward to find examples where bundling outperforms selling each good separately when the valuations are positively correlated or, in the extreme, increase together (Carroll, 2017). Separate sales can be suboptimal even in settings that appear most favorable to it.

But then, a natural question arises. Can separate sales be optimal, and if so, is there a general condition for this? The theory of multi-product monopoly pricing has yet to provide an exact answer. The main difficulty is the analytical complexity of the problem (Rochet and Choné, 1998; Rochet and Stole, 2003). Neither the standard Myersonian approach developed for the single-good case (Myerson, 1981) nor the more recent duality framework introduced by Carroll (2017) and Cai, Devanur, and Weinberg (2016) has yielded a general solution thus far. The aim of this paper is to characterize the boundary between profitable and unprofitable bundling by developing a canonical extension of the Myersonian approach to the multi-good setting.

I consider a multi-product setting in which the buyer's valuation is additive and type-increasing, and the production cost is normalized to zero. Although the optimal form of separate sales, or "separate monopoly pricing (SMP)", may appear to be a natural extension of the posted price mechanism in the single-good case (Riley and Zeckhauser, 1983), the Myersonian approach cannot be extended directly.

When a good is sold alone, a buyer with a higher valuation is more willing to purchase a larger quantity because of the higher (constant) marginal utility of consumption. This so-called "single-crossing property (SCP)" ensures type-increasing allocations and serves as a key requirement for applying the Myersonian approach. In the multi-good case, however, a buyer with a higher overall valuation may wish to purchase less of one good, say A, when it is bundled with more of another, say B, if the benefit from B offsets the loss from A.

The seller may exploit such preferences around a threshold type above which demand for A becomes *in*elastic, to extract additional revenue from higher-valuation types.[1] Raising the price of A above the threshold excludes some types from consuming A, but by instead

[1] If there is no such threshold for any good, every profit function is quasi-concave, or single-peaked, and the Myersonian approach applies directly, yielding the optimality of SMP (Carroll, 2017).

offering more of B, which is more elastic, deviations to the allocations of types below the threshold can be prevented. This explains why bundling may dominate separate sales, and why the optimal allocations may be non-monotone, even under type-increasing valuations.

A monotonicity condition on the *marginal rate of substitution* (MRS) rules out such non-monotone allocations and implies that separate monopoly pricing is robustly optimal for any type distribution (the "If" part of Theorem 1). Monotone MRS means that, for every pair of goods, the relative attractiveness of one good to another changes monotonically with type. Under type-increasing valuations, this condition further implies that the demand for one good is uniformly more elastic than that for another. Since a more elastic good is less attractive to higher-valuation types, assigning it first saves information rent for a given increase in surplus. Therefore, the more elastic good is optimally allocated before the less elastic one, creating a "prefix structure" in allocations. This one-dimensional structure restores the optimality of type-increasing allocations in the multi-good case and makes bundling unnecessary.

Conversely, if monotone MRS does not hold, separate monopoly pricing cannot be robustly optimal (the "Only if" part of Theorem 1). For SMP to be optimal, the reasoning above implies that, as type increases, the profit function for each good must not switch from elastic to inelastic demand across the monopoly quantities of the other goods. This requirement, referred to as "quasi-concavity at profit maxima", is guaranteed by monotone MRS.[2] However, in the absence of the monotonicity condition, there must exist a type distribution under which the requirement fails.

Most of the papers in the bundling literature focuses on deriving various forms of bundling that is more profitable than separate sales under additive valuations, including pure bundling (Schmalensee, 1984; Fang and Norman, 2006; Pavlov, 2011; Menicucci et al., 2015), mixed bundling (Adams and Yellen, 1976; McAfee et al., 1989), and probabilistic bundling (Thanassoulis, 2004; Manelli and Vincent, 2007; Daskalakis et al., 2017; Hart and Nisan, 2019). Among the few papers showing the optimality of SMP, the closest is Carroll (2017) on robust screening.[3] Despite the max-min nature of the seller's problem with only marginal distributions known, it essentially reduces to finding a type distribution consistent with those marginals for which SMP is optimal. The full-support distribution, derived by solving a system of linear equations, is specific but converges to one admitting monotone valuations assumed in this paper as the marginal distributions become regular.

Methodologically, this paper extends the Myersonian approach. Rather than relying

[2] This notion of quasi-concavity of the profit function at a type is weaker than the standard regularity condition, which requires concavity or quasi-concavity over the entire type set, as introduced by Myerson (1981) and Bergemann, Bonatti, Haupt, and Smolin (2021).

[3] See also Che and Zhong (2025), who extend the result of Carroll (2017) to categorical bundling, and Haberman, Jagadeesan, and Yang (2024), who consider the possibility of returns in a Bayesian setting.

on full incentive compatibility, the analysis focuses on *downward* incentive compatibility (downward IC) combined with optimality to derive the necessary monotonicity of allocations without assuming SCP. Dating back to Moore (1984) and Matthews and Moore (1987), focusing on downward IC constraints has proved useful for characterizing optimal screening mechanisms under certain ranking conditions on agents' preferences. In particular, under *ratio monotonicity*, it plays a central role in establishing the optimality of mechanisms with a *nested* structure.[4]

Haghpanah and Hartline (2021) apply a downward-IC-based duality approach to a bundling problem with non-additive valuations. They show that selling only the grand bundle is optimal independently of the type distribution if and only if the relative valuation of each smaller bundle increases with respect to the grand bundle.[5] This ratio monotonicity condition, however, does not generically hold under additive valuations. It is shown that monotone MRS and the robust optimality of separate sales (Theorem 1) are, in fact, additive-valuation counterparts to the result of Haghpanah and Hartline (2021).

Under non-additive and monotone valuations, Ghili (2023) applies the Myersonian approach to derive optimal pure bundling, while Yang (2025b) extends this result to nested bundling. Their regularity conditions take the form of quasi-concavity of the profit function for each bundle sold individually, or of the difference between such profit functions. These conditions generalize the regularity condition in the single-good problem of Myerson (1981) in that only local downward IC constraints are essential, leaving no room for ironing that concavifies the profit function. In contrast, the monotone MRS condition in this paper does not require any assumption on the type distribution, and both local and non-local downward IC are essential.

The closest complementary work to this paper is Bergemann, Bonatti, Haupt, and Smolin (2021) (henceforth, BBHS), despite their reliance on the duality approach. They consider additive valuations and introduce a regularity condition under which a weak form of monotone valuations ensures the optimality of upgrade pricing (or equivalently, nested bundling) assuming monotone MRS.[6] BBHS also show that under additive and monotone valuations, every upgrade pricing mechanism has an outcome-equivalent separate-sales mechanism, and vice versa. The present paper complements their analysis by establishing the sufficiency of monotone MRS and the necessity of regularity without it.

[4]See also Guo and Shmaya (2019) in persuasion, Kattwinkel (2019) in bilateral trade, and Yang (2025a) in the selection of screening instruments.

[5]Yang (2025b) extends this condition to establish robust optimality of nested bundling. See also Salant (1989), Deneckere and McAfee (1996), and Anderson and Dana (2009) for the role of ratio monotonicity under non-additive valuations.

[6]This condition guarantees that good-by-good ironing to obtain the quasi-concave closure of the profit function can be successfully carried out even when valuations are only partially monotone.

The rest of the paper is organized as follows. Section 2 introduces the general bundling problem under additive and type-increasing valuations. Section 3 illustrates the main idea through an example. Section 4 establishes the robust optimality of SMP under monotone MRS. Section 5 discusses the assumptions and possible extensions. Section 6 concludes. All omitted proofs are provided in the Appendix.

2 Model

There are two players: a buyer and a seller. The seller sells d goods with zero production cost. The buyer's type is drawn from a finite set $[N] := \{1, 2, \ldots, N\}$, and for each good $i \in [d] := \{1, 2, \ldots, d\}$, the buyer's valuation v_n^i is positive and strictly increasing in n. This assumption is known as *monotone valuations*.

The buyer is privately informed of his type, whereas the seller only knows the prior distribution over types, denoted by $f := (f_1, f_2, \ldots, f_N)$, with cumulative distribution $F_n := \sum_{k=1}^{n} f_k$. The buyer's utility is quasi-linear in price and additive across the goods. Specifically, for a bundle $q = (q^1, q^2, \cdots, q^d) \in [0,1]^d$ sold at a price p, the utility of type n is

$$U_n(q, p) := \sum_{i=1}^{d} v_n^i q^i - p.$$

The value of the buyer's outside option is normalized to zero.

By the revelation principle, I restrict attention to direct mechanisms $M = \{(q_n, p_n)\}_{n \in [N]}$ that satisfy incentive-compatibility (IC) and individual rationality (IR) constraints:

$$U_n(q_n, p_n) \geq U_n(q_{n'}, p_{n'}) \quad \forall n, n' \in [N]; \tag{IC}$$

$$U_n(q_n, p_n) \geq 0 \quad \forall n \in [N]. \tag{IR}$$

Let $\mathcal{M}$ denote the set of such mechanisms. Then, an optimal mechanism solves:

$$\max_{M \in \mathcal{M}} \Sigma_{n=1}^{N} f_n p_n. \tag{1}$$

This paper aims to characterize when separate sales is optimal. In this case, revenue is maximized by offering each good at a fixed price, with deterministic bundles priced at the sum of their components. Among separate-sales mechanisms, the optimal form is to sell each good at its monopoly price. This mechanism is referred to as *separate monopoly pricing* (SMP) and will be the main focus of the analysis. In particular, SMP is *robustly optimal* with respect to the type distribution if for every type distribution consistent with the given type space, the *corresponding* SMP mechanism is optimal.

To define SMP formally, let the monopoly profit function of each good $i \in [d]$ at type n as the profit from selling the good at price v_n^i:

$$\Pi_n^i := v_n^i \cdot (1 - F_{n-1}).$$

The monopoly type n^{i*} is any element of $\arg\max_{n\in[N]} \Pi_n^i$, and the corresponding monopoly price is $p^{i*} := v^i_{n^{i*}}$.[7] Then, a mechanism $M = \{(q_n, p_n)\}_{n\in[N]}$ implements SMP if, for each $i \in [d]$,

$$q_n^i = \begin{cases} 1 & \text{if } n \geq n^{i*}, \\ 0 & \text{otherwise,} \end{cases}$$

and $p_n = \sum_{i=1}^d q_n^i p^{i*}$. The prices are chosen so that the IC constraints between every pair of adjacent types n and $n' = n-1$ (the local downward IC constraints) and the IR constraint for the lowest type ($n = 1$) are binding. This ensures that all IC and IR constraints are satisfied under monotone valuations.

3 An Example

The key factor for the optimality of separate sales is whether the optimal quantities are type-increasing. That is, q_n^i is increasing in n for every good $i \in [d]$. Once attention can be restricted to type-increasing quantity schedules under monotone valuations, the multi-good problem reduces to separate single-good optimizations, yielding the optimality of SMP. In this warm-up section, I provide a simple example to highlight the intuition behind the role of *monotone* marginal rate of substitution (MRS) (i.e., $\frac{v_n^j}{v_n^i}$ changes monotonically in n for each pair of goods i, j) in allowing such a restriction without loss of optimal revenue.

Suppose that there are three types and two goods. Type 2's valuation is 1 for both goods. By monotone valuations, it follows that

$$v_1^1 < v_2^1 = 1 < v_3^1, \quad v_1^2 < v_2^2 = 1 < v_3^2.$$

Without loss of generality, let $v_1^1 > v_1^2$. For each type n, the MRS between the two goods is $\frac{v_n^2}{v_n^1}$, the value of good 2 relative to good 1. The MRS is monotone if and only if $v_3^2 \geq v_3^1$. In this case, the relative attractiveness of good 1 falls as type increases. Figure 1 illustrates the valuation structure.

[7]Note that the monopoly price must correspond to some v_n^i. Thus, the profit function Π_n^i is defined only over $n \in [N]$.

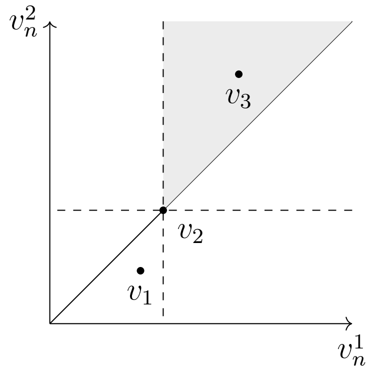

Figure 1: Monotone MRS

Under monotone MRS, one can always find an optimal selling mechanism that is type-increasing. To see this, consider non-monotone allocations that assign good 1 to type 1, good 2 to type 2, and the grand bundle to type 3, corresponding to bundles $(1,0)$, $(0,1)$, and $(1,1)$, respectively. This kind of mechanism is also known as *mixed bundling*. Let the corresponding prices be (p^1, p^2, p^G).

For this scheme to be incentive compatible, it must first be that

$$v_1^1 - v_1^2 \geq p^1 - p^2 \geq 0.$$

Also, type 1's individual rationality constraint must be binding, so $p^1 = v_1^1$. Thus, $p^2 \leq p^1 = v_1^1$. To induce type 3 to select $(1,1)$, p^G must be smaller than $p^1 + v_3^2$ and $v_3^1 + p^2$. Hence,

$$p^G \leq \min\{p^1 + v_3^2, v_3^1 + p^2\} \leq \min\{v_1^1 + v_3^2, v_3^1 + v_1^1\}.$$

If $v_3^2 \geq v_3^1$, the revenue-maximizing prices are $(p^1, p^2, p^G) = (v_1^1, v_1^1, v_3^1 + v_1^1)$. Type 2 is indifferent between $(1,0)$ and $(0,1)$, while type 3 is indifferent between $(1,1)$ and $(0,1)$, and therefore, obtains an information rent equal to $v_3^2 - v_1^1$.

Now, consider an alternative mechanism, which allocates $(1,0)$ to both types 1 and 2 and $(1,1)$ to type 3. The price of $(1,0)$ is set at v_1^1 as before, but the price of the grand bundle is increased to $v_1^1 + v_3^2$. This mechanism can be interpreted as a separate-sales mechanism with $p^1 = v_1^1$ and $p^2 = v_3^2$. It is clearly incentive-compatible, and moreover, increases expected revenue, regardless of the type distribution. Type 3's information rent is reduced to $v_3^1 - v_1^1 \leq v_3^2 - v_1^1$, while types 1 and 2 pay the same as before.

This simple example demonstrates that with monotone MRS, it is always cheaper to provide information rent to a high valuation type with the good whose relative attractiveness falls. In the former mechanism, allocating good 2 to type 2 meant that the seller extracts the full surplus of good 1 from type 3; in the latter, by using good 1 instead of good 2, the seller extracts the full surplus of good 2 from type 3, which is more valuable to this buyer. This implies a one-dimensional "prefix" structure in the optimal quantity schedule. For any pair of goods with monotone MRS, the seller allocates one good first and then the other to every buyer.

When the MRS is not monotone, the one-dimensional structure need not hold, and restricting attention to type-increasing quantities may reduce optimal revenue. As a result, SMP may fail to be optimal. A continuation of this example illustrating this case is provided in Appendix A.3.

4 Main Result

4.1 Statement of the Result

I first formally define the notion of monotone MRS.

Definition 1 (Monotone MRS) *For all $i \in [d-1]$, $\frac{v_n^{i+1}}{v_n^i}$ is non-decreasing in n.*

This means that a buyer with higher valuations for all products has a stronger relative valuation for higher-indexed products than for lower-indexed ones. More generally, if the MRS between any pair of goods is monotone in type, that is, for all $i, j \in [d]$, either $\frac{v_n^j}{v_n^i}$ or $\frac{v_n^i}{v_n^j}$ is non-decreasing in n, then the goods can be reindexed so that the ratio monotonicity property holds.[8]

Under monotone valuations and a fixed type distribution f, this condition implies a uniform ranking of demand elasticities across goods. Specifically, if the price elasticity of demand for good $i \in [d]$ at type $n \in [N-1]$ is measured by

$$e_n^i := \frac{f_n/(1-F_{n-1})}{(v_{n+1}^i - v_n^i)/v_n^i},$$

then demand for a higher-indexed good is uniformly less elastic, that is, $e_n^i \geq e_n^j$ for every $n \in [N-1]$ whenever $i < j$.

I now state the main result.

Theorem 1 *Under monotone valuations, separate monopoly pricing is robustly optimal with respect to the type distribution if and only if monotone MRS holds.*

[8]Formally, pairwise monotonicity of the MRS induces a transitive relation over goods: for any i, j, k, if v_n^j/v_n^i and v_n^k/v_n^j are non-decreasing in n, so is v_n^k/v_n^i. This yields a total ordering under which goods can be reindexed so that v_n^j/v_n^i is non-decreasing in n whenever $j > i$.

Thus, under monotone MRS, SMP is optimal for all type distributions, without requiring any regularity condition on the type distribution, such as quasi-concavity (or concavity) of the profit function Π^i for each good i. Conversely, if monotone MRS fails, there exists a type distribution for which any separate-sales mechanism is suboptimal. In other words, some form of regularity is required for the optimality of SMP when MRS is non-monotone.

4.2 Proof of Sufficiency (the "If" Part)

Outline Using an extended Myersonian approach, I show that separate monopoly pricing is optimal under monotone MRS. The main departure from the standard Myersonian approach is that monotonicity of allocations is not a property implied by incentive compatibility, but rather a property of optimal mechanisms. To establish this, I first solve a relaxed problem imposing only *downward* IC constraints and identify an optimal allocation rule for this problem that displays monotonicity. Given such monotonicity, the optimal selling mechanism can then be obtained by solving the single-good monopoly problem for each good separately. The resulting optimality of SMP in the relaxed problem implies its optimality in the original problem.

The proof proceeds through the following steps.

(a) Relax upward IC, keeping only downward IC and the lowest type's IR.

(b) Show that an optimal allocation rule can be chosen to be monotone (Lemma 1).

(c) Show that under monotone allocations, local downward IC and the lowest type's IR bind, so the problem reduces to virtual surplus maximization under monotonicity.

(d) Use separability of virtual values to reduce the problem to d independent single-good problems.

(e) Apply Myerson (1981) to show that SMP solves the relaxed problem, and conclude the optimality of SMP since it satisfies all original IC and IR constraints.

Without loss of generality, I assume that the type distribution f has full support.[9]

Monotone allocations (Steps (a)–(b)) Consider the following relaxed optimization problem, which imposes only the downward IC constraints and the IR constraint for the

[9] For any given mechanism (i.e., menu of bundle–price pairs), the revenue is unaffected by adding or removing zero-probability types from the type set. Thus, the optimality of SMP is also preserved.

lowest type:

$$\max_{\{(q_n,p_n)\}_{n\in[N]}} \sum_{n=1}^{N} f_n p_n \tag{2}$$

subject to

$$U_{n'}(q_{n'},p_{n'}) \geq U_{n'}(q_n,p_n) \quad \forall\, n,n' \in [N] \text{ with } n' \geq n \tag{Downward IC}$$

$$U_1(q_1,p_1) \geq 0 \tag{IR for $n=1$}$$

As for problem (1), existence of a solution follows from a compactness argument.

The following lemma presents the key implication of monotone MRS, generalizing the arguments in Section 3. It identifies an optimal mechanism in which each type's allocation is a "prefix" (i.e., all goods below/above some cutoff are included/excluded, with possible randomization at the cutoff) and higher types receive weakly larger sets of goods.[10] This implies type-increasing allocations.

Lemma 1 *Under monotone MRS, there exists a solution $M = \{(q_n,p_n)\}_{n\in[N]}$ to problem (2) such that the following hold:*

(i) For all $n \in [N]$, q_n is a "prefix", that is, there exists a cutoff good $i_n \in [d]$ such that

$$q_n^i \begin{cases} = 1 & \text{if } i < i_n, \\ = 0 & \text{if } i > i_n, \\ \in [0,1] & \text{if } i = i_n. \end{cases}$$

(ii) For all $i \in [d]$, q_n^i is non-decreasing in n.

Proof. See Appendix A.1 for a formal proof. ■

Lemma 1(i) means that a revenue-maximizing seller, when considering only downward IC constraints, would allocate good j to a type only after allocating goods $1,\ldots,j-1$, which are uniformly more elastic. The resulting prefix structure of q_n implies that optimal allocations lie on a one-dimensional line and are well-ordered under the standard component-wise order.

For an intuitive proof, consider type n's allocation (q_n,p_n). Figure 2 illustrates the argument in the two-good case ($d = 2$). Fix the price p_n so that the expected revenue

[10] The term *prefix* is used in the algorithmic mechanism design literature to describe allocation structures where items or agents are processed sequentially in a fixed order. See, for example, Cai and Daskalakis (2015) and Alaei, Fu, Haghpanah, Hartline, and Malekian (2019).

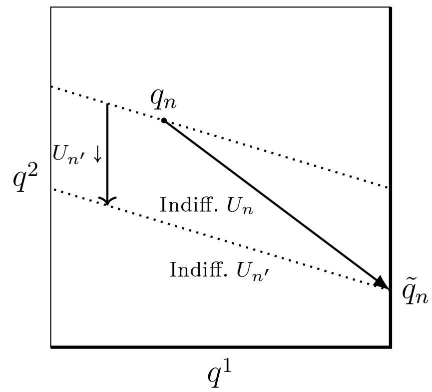

Figure 2: One-dimensional structure of the optimal allocations

remains unchanged. The corresponding indifference curve (diagonal arrowed line) then has slope $-\frac{v_n^1}{v_n^2}$, which equals the negative of the MRS of good 1 for good 2. Because the MRS is type-decreasing, the indifference curve of a higher type $n' > n$ (two dotted lines) is flatter than that of type n. Equivalently, this flattening reflects that good 1 is uniformly more elastic, and thus relatively less attractive for higher types. If q_n is shifted down to $\tilde{q}_n$ along type n's indifference curve, that is, by assigning good 1 first, type n's payoff stays the same, while the payoffs of all higher types from deviating to type n decrease. Thus, no downward IC constraints are violated, and some of the previously binding constraints may even become slack. Repeating this procedure for all types aligns their allocations on a one-dimensional line (bottom and right edges) with $q_n^1 = 1$ or $q_n^2 = 0$, which generates the prefix structure.

This argument generalizes the reasoning in the two-good, three-type example of Section 3. When the MRS equals 1, separate sales yields weakly higher revenue than mixed bundling (p^1, p^2, p^G) by assigning good 1 instead of good 2 to type 2. A strict revenue increase arises when the MRS is strictly monotone, that is, when $v_3^2 > v_3^1$. In this case, the downward IC constraint between types 2 and 3 becomes slack, which allows the seller to raise p^G.

The one-dimensionality of the allocation, combined with monotone valuations, ensures that the (strict) single-crossing property (SCP) holds among prefix allocations ordered by the strict component-wise order $>$. Specifically, for any $q > \tilde{q}$ and for every n and any $p, \tilde{p}$,

$$U_n(q,p) \geq U_n(\tilde{q},\tilde{p}) \implies U_{n'}(q,p) > U_{n'}(\tilde{q},\tilde{p}) \quad \forall n' > n,$$
$$U_n(q,p) \leq U_n(\tilde{q},\tilde{p}) \implies U_{n'}(q,p) < U_{n'}(\tilde{q},\tilde{p}) \quad \forall n' < n.[11]$$

When both upward and downward IC constraints are imposed, the SCP implies monotonicity of allocations. By contrast, under downward IC alone, monotonicity must be derived from

[11]When $q > \tilde{q}$ (that is, $q \geq \tilde{q}$ but $q \neq \tilde{q}$), $U_n(q,p) - U_n(\tilde{q},\tilde{p}) = v_n \cdot (q - \tilde{q}) - (p - \tilde{p})$ is strictly increasing in n.

the seller's revenue-maximizing behavior: decreasing quantities is unprofitable because it reduces total surplus without yielding any rent savings.

Suppose that the prefix quantity is increasing up to type n, so $q_{\tilde{n}} \leq q_n$ for every $\tilde{n} \leq n$. By the SCP and the downward IC constraints of type n against lower types, type $n+1$ then prefers (q_n, p_n) to any lower type's allocation $(q_{\tilde{n}}, p_{\tilde{n}})$. Consequently, a revenue-maximizing seller offers a bundle q_{n+1} with price p_{n+1} that makes type $n+1$ indifferent between its own allocation and that of type n:

$$U_{n+1}(q_{n+1}, p_{n+1}) = U_{n+1}(q_n, p_n).$$

The only possible reason for the seller to choose $q_{n+1} < q_n$ (with $p_{n+1} < p_n$), despite the resulting loss in total surplus, is to reduce the information rent of higher types. Yet this rent does not change, since all higher types will strictly prefer type n's allocation over type $n+1$'s, again due to the SCP. Therefore, the seller always chooses $q_{n+1} \geq q_n$, which yields a type-increasing quantity schedule across all types.

One important implication of Lemma 1 is that restricting attention to type-increasing quantity schedules in (2) does not reduce the seller's maximum revenue. Moreover, it even preserves the set of optimal mechanisms when MRS is strictly monotone, that is, when for all $i \in [d-1]$, the ratio $\frac{v_n^{i+1}}{v_n^i}$ is strictly increasing in n.[12]

Revenue maximization under monotonicity (Steps (c)–(e)) In the spirit of Myerson (1981), Lemma 1 reduces problem (2) to finding a type-increasing quantity schedule that maximizes total virtual surplus. Once monotonicity of allocations is imposed, all local downward IC constraints and the lowest type's IR constraint must bind, as in the standard single-good case. As a consequence, all remaining non-local downward IC constraints are redundant: under type-increasing allocations, they are implied by the binding constraints due to the SCP.

The argument parallels the standard single-good proof. Suppose that for a solution $\{(q_n, p_n)\}_{n \in [N]}$ to problem (2) with type-increasing q_n, some local downward IC constraint or the lowest type's IR constraint is slack. That is, assume that there exists some $n \geq 1$ such that

$$U_{n+1}(q_{n+1}, p_{n+1}) > U_{n+1}(q_n, p_n),$$

or the same holds for $n = 0$ with $(q_0, p_0) = (0, 0)$, so that monotonicity of allocations could be extended to all $n \geq 0$. Then, by the SCP implied by monotone valuations, each higher

[12]Section 5.2 discusses the unique optimality of separate sales under strictly monotone MRS.

type $n' \geq n+1$ strictly prefers its own allocation to that of any lower type $n'' \leq n$:

$$U_{n'}(q_{n'}, p_{n'}) \geq U_{n'}(q_{n+1}, p_{n+1}) > U_{n'}(q_n, p_n) \geq U_{n'}(q_{n''}, p_{n''}).$$

The first inequality follows from the downward IC constraint for type n' against type $n+1$, and the next two follow from the SCP across (q_{n+1}, p_{n+1}), (q_n, p_n), and $(q_{n''}, p_{n''})$. Hence, by uniformly increasing $p_{n'}$ for all $n' \geq n+1$ by a small amount $\epsilon > 0$, total revenue increases strictly while all downward IC constraints and the IR constraint for the lowest type remain satisfied.[13]

For good $i \in [d]$ and type $n \in [N-1]$, the (probability-weighted) virtual valuation can be defined as the change in monopoly profit from selling the good alone when the price decreases from v^i_{n+1} to v^i_n, i.e.,

$$V^i_n := \Pi^i_n - \Pi^i_{n+1} = f_n v^i_n - (1 - F_n)(v^i_{n+1} - v^i_n).$$

Let $V^i_N = f_N v^i_N$.

Using the binding local downward IC constraints and the binding IR constraint for the lowest type, the seller's expected revenue can be written as

$$\sum_{n=1}^{N} f_n p_n \;=\; \sum_{i=1}^{d} \sum_{n=1}^{N} V^i_n q^i_n.$$

To see this, note first that the binding local downward IC constraints imply

$$p_{n+1} - p_n \;=\; \sum_{i=1}^{d} v^i_{n+1} \left(q^i_{n+1} - q^i_n\right),$$

while the binding IR constraint for the lowest type yields

$$p_1 \;=\; \sum_{i=1}^{d} v^i_1 q^i_1.$$

[13] Such a perturbation is feasible for sufficiently small $\epsilon > 0$ because there are only finitely many downward IC constraints between higher types $n' \geq n+1$ and lower types $n'' \leq n$, and each of these constraints is slack.

Applying summation by parts then gives

$$\begin{aligned}\sum_{n=1}^{N} f_n p_n &= p_1 + \sum_{n=1}^{N-1} (1 - F_n)(p_{n+1} - p_n) \\ &= \sum_{i=1}^{d} \left[v_1^i q_1^i \ + \ \sum_{n=1}^{N-1} (1 - F_n)\, v_{n+1}^i (q_{n+1}^i - q_n^i) \right] \\ &= \sum_{i=1}^{d} \left[\sum_{n=1}^{N-1} \Big(f_n v_n^i - (1 - F_n)(v_{n+1}^i - v_n^i) \Big) q_n^i + f_N v_N^i q_N^i \right] \\ &= \sum_{i=1}^{d} \sum_{n=1}^{N} V_n^i q_n^i.\end{aligned}$$

Consequently, problem (2) with the monotonicity constraint reduces to the following:

$$\max_{\{q_n\}_{n\in[N]}} \sum_{i=1}^{d} \sum_{n=1}^{N} V_n^i q_n^i \tag{3}$$

subject to q_n^i being non-decreasing in n for every $i \in [d]$. By Lemma 1, any solution to problem (3), with prices $\{p_n\}_{n\in[N]}$ determined by binding local downward IC constraints and the binding IR constraint for the lowest type, also solves problem (2).

Because virtual surplus is additively separable across goods, (3) decomposes into d independent single-good monopoly problems. That is, for each $i \in [d]$, consider

$$\max_{\{q_n^i\}_{n\in[N]}} \sum_{n=1}^{N} V_n^i q_n^i \tag{4}$$

subject to q_n^i being non-decreasing in n. As shown by Myerson (1981), each single-good problem is optimally solved by allocating $q_n^i = 1$ for $n \geq n^{i*}$ and $q_n^i = 0$ otherwise, without requiring any regularity assumption on the type distribution. This allocation rule is equivalent to a posted-price mechanism at the monopoly price p^{i*}, with all local downward IC constraints and the IR constraint for the lowest type binding. Therefore, separate monopoly pricing solves problem (3), and hence the relaxed problem (2). Since it satisfies all IC and IR constraints, SMP is feasible for the original problem (1) and therefore optimal.

4.3 Proof of Necessity (the "Only If" Part)

Outline The proof of necessity proceeds in two steps. First, I establish a necessary condition on the type distribution for the optimality of separate monopoly pricing, referred to as "quasi-concavity at profit maxima". Second, I show that when monotone MRS fails, one can always construct a type distribution that violates this condition.[14]

Quasi-concavity at a type To state the necessary condition, I first introduce the preliminary notion of quasi-concavity *at a type*. This is weaker than the standard regularity assumption, which requires the profit function to be quasi-concave over the entire type space. A profit function is quasi-concave if and only if it is quasi-concave at every type.

Definition 2 (Quasi-concavity at a type) *The profit function Π_n^i for good i is quasi-concave at type n if for every $n' \leq n \leq n''$, $\Pi_n^i \geq \min\{\Pi_{n'}^i, \Pi_{n''}^i\}$.*

The above condition implies that, as type increases, a shift across a given type n from elastic demand to inelastic demand for good i cannot occur. Such a shift would correspond to the following pattern for some $n' < n < n''$: demand decreases elastically when the price increases from $v_{n'}^i$ to v_n^i (so that $\Pi_{n'}^i > \Pi_n^i$) and decreases inelastically when the price increases from v_n^i to $v_{n''}^i$ (so that $\Pi_n^i < \Pi_{n''}^i$).[15]

The types at which the profit function is quasi-concave can be fully characterized. Type n is a *left-dominant peak* for good i if

$$\Pi_n^i \geq \Pi_{n'}^i \quad \text{for all } n' < n.$$

Similarly, type n is a *right-dominant peak* if

$$\Pi_n^i \geq \Pi_{n'}^i \quad \text{for all } n' > n.$$

In words, a left-dominant (resp. right-dominant) peak is a type at which the profit Π_n^i is at least as high as that of all types to its left (resp. right). According to the definition of quasi-concavity, the left- and right-dominant peaks consist of types where the profit function is quasi-concave. This is summarized below.

Lemma 2 *The profit function Π_n^i is quasi-concave at type n if and only if it is either a left-dominant or right-dominant peak for good i.*

[14] Appendix A.3 revisits the two-good, three-type example from Section 3 under non-monotone MRS and derives a condition under which a more complex bundling mechanism dominates both SMP and mixed bundling, corresponding to a violation of quasi-concavity at profit maxima.

[15] With an extended notion of the price elasticity of demand, the shift from elastic to inelastic demand means that

$$e_{n',n}^i := \frac{(F_{n-1} - F_{n'-1})/(1 - F_{n'-1})}{(v_n^i - v_{n'}^i)/v_{n'}^i} > 1 > e_{n,n''}^i := \frac{(F_{n''-1} - F_{n-1})/(1 - F_{n-1})}{(v_{n''}^i - v_n^i)/v_n^i}.$$

Furthermore, a left-dominant (resp. right-dominant) peak is a monopoly type $n^{i*} \in \arg\max_n \Pi_n^i$ itself, or a type on the left (resp. right) side of a monopoly type around which *ironing* is unnecessary, as illustrated in Figure 3.

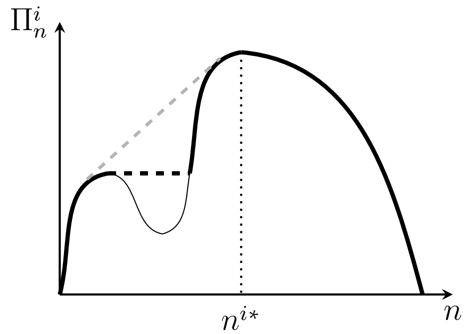

Figure 3: Left-dominant peaks (thick, left of n^{i*}), right-dominant peaks (thick, right of n^{i*}), the quasi-concave closure (black dashed), and the concave closure (gray dashed) of a profit function.

In Figure 3, the black dashed line illustrates the outcome of ironing for obtaining the quasi-concave closure of the profit function Π_n^i (i.e., the smallest quasi-concave function that lies above it) applied only to types that are not left- or right-dominant peaks. Note that for minimal ironing, it is sufficient to employ the quasi-concave closure of BBHS, which is weaker than the concave closure (gray dashed line) used in Myerson (1981).

A necessary condition for the optimality of SMP The profit function Π_n^i for each good i is trivially quasi-concave at its monopoly type n^{i*}. For separate sales to be optimal, it is necessary that the profit functions for all the other goods are also quasi-concave at the monopoly type n^{i*} for good i.

Definition 3 (Quasi-concavity at profit maxima) *For every pair of goods i and j, the profit function Π_n^i of good i is quasi-concave at every monopoly type $n^{j*} \in \arg\max_n \Pi_n^j$ of good j.*

Proposition 1 *Under monotone valuations, separate monopoly pricing is suboptimal if quasi-concavity at profit maxima (Definition 3) does not hold.*

Proof. See Appendix A.2 for a formal proof. ■

Under SMP, the allocation is type-increasing, and for each good j, there is a discrete jump in quantity q_n^j at the monopoly type n^{j*}. Since valuations are also type-increasing, the (strict) SCP implies that for every type other than the cutoff monopoly type n^{j*}, its IC constraints against any type with distinct consumption of good j are slack. Hence, the allocation of another good i can be locally perturbed within each of the two subsets of types divided by whether good j is consumed.[16] If quasi-concavity at profit maxima fails to hold, a profitable perturbation can be constructed.

To see this, note that under SMP, if some type above n^{j*} does not purchase good i, all types below must also be excluded. However, if the profit function of good i is not quasi-concave at the monopoly type n^{j*}, neither the local maximizer of Π^i_n for $n \le n^{j*}$ nor that for $n \ge n^{j*}$ coincides with n^{j*}; that is, $n^{j*} \notin \arg\max_{n \le n^{j*}} \Pi^i_n$ and $n^{j*} \notin \arg\max_{n \ge n^{j*}} \Pi^i_n$. In this case, due to the shift in demand from being elastic to being inelastic, the seller has an incentive to sell good i to some of the highest types below n^{j*}, even when some of the lowest types above n^{j*} do not purchase it.

Specifically, fix SMP defined by the monopoly type n^{j*} for good j, and let the corresponding monopoly type for good i be n^{i*}. Let $\tilde{n}^{i*}$ denote the local maximizer of Π^i_n on the opposite side of n^{i*} relative to n^{j*}:

$$\tilde{n}^{i*} \in \begin{cases} \arg\max_{n > n^{j*}} \Pi^i_n, & \text{if } n^{i*} < n^{j*}, \\ \arg\max_{n < n^{j*}} \Pi^i_n, & \text{if } n^{i*} > n^{j*}. \end{cases}$$

If $n^{i*} < n^{j*}$, uniformly decrease q^i_n by a small $\epsilon > 0$ for types n between the monopoly type n^{j*} and the local maximizer $\tilde{n}^{i*}$ ($n^{j*} \le n < \tilde{n}^{i*}$); if $n^{i*} > n^{j*}$, uniformly increase q^i_n by a small $\epsilon > 0$ for types n between $\tilde{n}^{i*}$ and n^{j*} ($\tilde{n}^{i*} \le n < n^{j*}$). Prices are then adjusted so that all the local downward IC constraints and the lowest type's IR constraint remain binding. In each case, revenue increases strictly by $\epsilon(\Pi^i_{\tilde{n}^{i*}} - \Pi^i_{n^{j*}}) > 0$.

The proposition and the "If" part of Theorem 1 together imply that quasi-concavity at profit maxima holds robustly under monotone MRS for all type distributions. Here, I first provide a direct proof to highlight the role of monotone MRS in ensuring quasi-concavity at profit maxima. The "Only If" part of the theorem is then established by proving the converse.

Lemma 3 *Under monotone valuations, quasi-concavity at profit maxima holds for any type distribution if monotone MRS holds.*

Proof. Monotone MRS implies the single-crossing property (SCP) for the profit functions across goods: if $\Pi^i_n \ge \Pi^i_{n'}$ (resp. $>$) for some $n > n'$, then $\Pi^j_n \ge \Pi^j_{n'}$ (resp. $>$) for every $j > i$.

[16] In response to this perturbation, the price of good j is adjusted so that the local downward IC constraint between types n^{j*} and $n^{j*} - 1$ remains binding.

To see this, fix $n > n'$ and $j > i$. Then,

$$\begin{aligned}\frac{\Pi_n^i - \Pi_{n'}^i}{v_{n'}^i} &= (1 - F_{n-1})\left(\frac{v_n^i}{v_{n'}^i} - 1\right) - (F_{n-1} - F_{n'-1}) \\ &\leq (1 - F_{n-1})\left(\frac{v_n^j}{v_{n'}^j} - 1\right) - (F_{n-1} - F_{n'-1}) = \frac{\Pi_n^j - \Pi_{n'}^j}{v_{n'}^j},\end{aligned}$$

where the inequality follows from monotone MRS.

That is, if type n is a left-dominant (resp. right-dominant) peak for good i, then it is also a left-dominant (resp. right-dominant) peak for every higher-indexed good $j > i$ (resp. lower-indexed good $j < i$):

$$\begin{aligned}\Pi_n^i \geq \Pi_{n'}^i, \quad \forall\, n' < n \implies \Pi_n^j \geq \Pi_{n'}^j, \quad \forall\, n' < n,\ \forall\, j > i, \\ \Pi_n^i \geq \Pi_{n'}^i, \quad \forall\, n' > n \implies \Pi_n^j \geq \Pi_{n'}^j, \quad \forall\, n' > n,\ \forall\, j < i.\end{aligned}$$

Since a monopoly type n^{i*} for good i is both a left- and right-dominant peak for that good, it must be either a left- or a right-dominant peak for any other good $j \neq i$. By Lemma 2, quasi-concavity at profit maxima follows. ■

Violation of the necessary condition Under non-monotone MRS, quasi-concavity at profit maxima may hold for some type distributions but not for all. It may still hold when the violation of monotone MRS is sufficiently limited so that the SCP of the profit functions across goods, as defined in the proof of Lemma 3, continues to hold.[17] However, in the extreme case where demand is uniformly unit-elastic for a good j (i.e., when Π^j is constant), even a slight violation of monotone MRS can lead to a failure of the quasi-concavity.

To see this, suppose that monotone MRS does not hold. Then, there exist two goods i and j such that the ratio $\frac{v_m^i}{v_m^j}$ is non-monotone in $m \in [N]$. Without loss of generality, assume that for some types $n' < n < n''$,

$$\frac{v_n^i}{v_n^j} < \min\left\{\frac{v_{n'}^i}{v_{n'}^j}, \frac{v_{n''}^i}{v_{n''}^j}\right\}.\text{[18]} \tag{5}$$

[17]Nevertheless, the SCP alone does not ensure the optimality of SMP. An illustrative example is provided in the final part of Appendix A.3.

[18]Otherwise, it must hold that $\frac{v_n^i}{v_n^j} > \max\left\{\frac{v_{n'}^i}{v_{n'}^j}, \frac{v_{n''}^i}{v_{n''}^j}\right\}$ for some types $n' < n < n''$. Switching goods i and j then yields (5).

Now, consider a type distribution that is uniformly unit-elastic for good j. For $m \in [N-1]$,

$$f_m = v_1^j \left(\frac{1}{v_m^j} - \frac{1}{v_{m+1}^j} \right),$$

and let $f_N = \frac{v_1^j}{v_N^j}$. Under this distribution, the profit function Π^j is constant across all types:

$$\Pi_1^j = \cdots = \Pi_{n'}^j = \cdots = \Pi_n^j = \cdots = \Pi_{n''}^j = \cdots = \Pi_N^j. \tag{6}$$

Since $\Pi_m^i = \frac{v_m^i}{v_m^j} \Pi_m^j$ for every $m \in [N]$, condition (5) implies

$$\Pi_n^i < \min\{\Pi_{n'}^i, \Pi_{n''}^i\}. \tag{7}$$

Consequently, quasi-concavity at profit maxima fails under this type distribution. By (6), n is a monopoly type for good j, while (7) shows that Π^i is not quasi-concave at n.[19]

4.4 Remarks

Relation to Bergemann, Bonatti, Haupt, and Smolin (2021) Under both monotone valuations and monotone MRS, Bergemann, Bonatti, Haupt, and Smolin (2021) show that SMP is optimal if type distribution is "mostly regular", that is, the intervals over which the profit functions $\{\Pi_n^i\}_{i \in [d]}$ require ironing to restore regularity are either disjoint or fully overlapping across goods. Theorem 1 strengthens this result by showing that optimality holds even without any assumption on the type distribution.

Under monotone valuations, BBHS also show that for any separate-sales mechanism, there exists an outcome-equivalent upgrade pricing scheme, or equivalently, a menu of nested bundles, and vice versa. In other words, the bundling induced by upgrade pricing yields outcomes that are essentially equivalent to those from separate sales. Therefore, Theorem 1 also implies that monotone MRS is both necessary and sufficient for the robust optimality of upgrade pricing; that is, for every type distribution consistent with the given type space, there exists an optimal upgrade pricing mechanism (which may be distinct).

Relation to McAfee, McMillan, and Whinston (1989). In a two-good monopoly with a continuous type distribution, McAfee, McMillan, and Whinston (1989) provide a general condition under which SMP is suboptimal. Their condition requires that a marginal

[19] This conclusion does not rely on a non-generic case: increasing f_n by a small $\epsilon > 0$ and reducing f_{n-1} by the same amount makes n the unique monopoly type for good j while preserving the strict inequality (7). Hence, the failure of quasi-concavity at profit maxima is robust to small perturbations of the type distribution.

perturbation of one price, either of an individual good or of the bundle, generates nonzero marginal revenue.[20] This condition can hold quite generally under full-support distributions over valuations, for example, under independently distributed values across the goods, since holding the other prices fixed breaks the relation $p^1 + p^2 = p^G$ that characterizes separate sales.[21]

By contrast, under the monotone valuations assumed here, once extended to the corresponding continuous-type setting, this condition cannot be satisfied. Due to the outcome equivalence between separate sales and upgrade pricing, any perturbation of a single price under SMP can be reinterpreted as a perturbation to another separate sales that preserves the relation $p^1 + p^2 = p^G$. Since SMP is optimal within the class of separate sales, such perturbations yield zero marginal revenue.

To see this, consider SMP with prices $(p^1, p^2, p^G) = (p^{1*}, p^{2*}, p^{1*} + p^{2*})$. Suppose that p^1 is increased marginally by $\epsilon > 0$ while keeping p^2 and p^G unchanged. Then, the effective price of good 2 conditional on purchasing good 1 becomes $p^G - (p^{1*} + \epsilon) = p^{2*} - \epsilon$. Under monotone valuations, however, at most one good, say good i, is sold alone to a positive measure of buyers, while all other goods are purchased only as add-ons. If $i = 1$, the perturbation is outcome-equivalent to a perturbation into separate sales with prices $(\tilde{p}^1, \tilde{p}^2) = (p^{1*} + \epsilon,\ p^{2*} - \epsilon)$, since the standalone price of good 2 has no effect due to the zero measure of types willing to purchase only good 2. If $i = 2$, or if no such good exists (so that SMP coincides with pure bundling), good 1 is sold only as an add-on, so perturbing its standalone price likewise has no first-order effect. Consequently, in all cases, such a perturbation yields zero marginal revenue.

5 Discussion

5.1 Additive vs. Non-additive Valuations: The Role of Ratio Monotonicity

Both this paper and Haghpanah and Hartline (2021) propose ratio monotonicity conditions that ensure the optimality of simple mechanisms under monotone valuations. However, the two studies consider opposite extremes of bundling: separate sales and pure bundling. This contrast arises because additivity determines the structure of the feasibility set of allocations, which in turn determines the subset of feasible prefix allocations.

Non-additive valuations can be equivalently represented as additive valuations over deter-

[20] The resulting marginal revenue is the same regardless of which price is perturbed.

[21] Recall that (p^1, p^2, p^G) denote the prices of the bundles $(1, 0)$, $(0, 1)$, and $(1, 1)$, respectively.

ministic, non-empty bundles $B = \{b \mid b \subseteq [d],\, b \neq \varnothing\}$. Normalizing the valuation of the empty bundle to zero, type n's valuation for any (possibly randomized) bundle $q \in \Delta(B \cup \{\varnothing\})$ is

$$\sum_{b \in B} q(b) v_n^b,$$

where v_n^b denotes type n's value for bundle b, and $q(b)$ is the probability that bundle b is assigned. The additional constraint $\sum_{b \in B} q(b) \leq 1$ restricts feasible allocations relative to the additive case, where quantities across goods can vary freely within $[0,1]^d$.

The extended Myersonian approach developed in this paper applies directly to such non-additive environments. In both cases, ratio monotonicity serves the same key function. It allows any feasible allocation to be effectively projected onto a valuation-equivalent allocation along a one-dimensional prefix structure, where goods or bundles less preferred by higher types are assigned first.

Under non-additive valuations, however, this adjustment must satisfy the additional feasibility constraint $\sum_{b \in B} q(b) \leq 1$; once a bundle is fully allocated, no other bundle can be assigned. That is, the prefix allocation spans only one non-empty bundle. Under free disposal, i.e., $v_n^{[d]} \geq v_n^b$ for all $n \in [N]$ and $b \in B$, the only bundle that permits a valuation-preserving projection is the grand bundle. As a result, the prefix allocation can span only the grand bundle, so ratio monotonicity needs to be imposed only relative to that bundle. Consequently, under the weaker ratio monotonicity condition with monotone valuations in Haghpanah and Hartline (2021), which requires both $\frac{v_n^b}{v_n^{[d]}}$ and $v_n^{[d]}$ to increase in n for every $b \in B$, the same logic as in Lemma 1 restricts the feasible allocations to those q satisfying $q(b) = 0$ for all $b \notin \{[d], \varnothing\}$, thereby establishing the optimality of pure bundling.[22]

Methodologically, this paper adopts a Myersonian approach, whereas Haghpanah and Hartline (2021) is duality-based. Although the duality approach is more general and allows for finer ironing, this additional generality does not permit weakening of monotone MRS to a weaker ratio monotonicity condition. Indeed, Theorem 1 shows that monotone MRS is not only sufficient, but also necessary for the robust optimality of SMP with respect to the type distribution.

[22]This result corresponds to the discrete-type result in Haghpanah and Hartline (2021), which is extended there to a general compact type space. Theorem 1 extends similarly: the only-if direction is immediate, and the if direction follows from the approximation argument of Madarász and Prat (2017), applied as in Carroll (2017) via a fine discretization of the type space under monotone valuations and monotone MRS.

5.2 Unique Optimality of SMP

Under a stronger preference ordering, SMP becomes uniquely optimal, albeit not for all type distributions. Here, I assume that the type distribution f has full support.

Definition 4 (Strictly Monotone MRS) *For all $i \in [d-1]$, the ratio $\frac{v_n^{i+1}}{v_n^i}$ is strictly increasing in n.*

Proposition 2 *SMP is uniquely optimal if strictly monotone MRS holds and the monopoly type for each good i is unique, that is, $\arg\max_{n\in[N]} \Pi_n^i$ is a singleton for every $i \in [d]$.*

To see this, suppose that strictly monotone MRS holds. Then, Lemma 1 can be strengthened to show that every solution to problem (2) satisfies both conditions stated in the lemma, the prefix structure and monotonicity of allocations. Under strictly monotone MRS, the corresponding prefix allocation (q_n, p_n) with equivalent price and valuation yields strictly lower payoffs to all higher types than the original allocation $(\tilde{q}_n, \tilde{p}_n)$ if the latter is non-prefix. Hence, all downward IC constraints against type n become slack, allowing the seller to increase revenue strictly by slightly increasing q_n and p_n accordingly to keep type n's utility unchanged.

Therefore, restricting attention to type-increasing allocations in problem (2) preserves not only the maximized revenue, but also the set of optimal mechanisms. Furthermore, if each good i has a unique monopoly type, maximizing total virtual surplus (problem (3)) yields SMP uniquely, since the posted-price mechanism is the unique optimum of each decomposed single-good problem (problem (4)), as shown in Myerson (1981).

5.3 Optimal SMP under Non-Monotone MRS

It is also possible to build on Theorem 1 and Proposition 1 to obtain regularity conditions on the type distribution for optimal SMP without monotone MRS. Although full monotonicity of allocations (Lemma 1(ii)) cannot be achieved under non-monotone MRS, it can still hold piecewise, since MRS is monotone within each interval between successive *local* maxima. Quasi-concavity at these local MRS maxima for each good's profit function compensates for the lack of global monotonicity.[23] It ensures that across each local MRS maximum, there is no shift from elastic to inelastic demand for any good.

While all downward IC constraints are essential under monotone MRS, under non-monotone MRS with the quasi-concavity condition, the essential downward IC constraints

[23] Local MRS maxima are the types n^{jk*} at which the MRS between some goods j and k attains a local maximum, that is, $\frac{v_n^k}{v_n^j} \geq \max\left\{\frac{v_{n-1}^k}{v_{n-1}^j}, \frac{v_{n+1}^k}{v_{n+1}^j}\right\}$ for $n = n^{jk*}$.

are those within each segment of types between successive local MRS maxima. Consider the relaxed problem that includes only the downward IC constraints within these segments. As in the monotone MRS case, downward IC combined with optimality implies piecewise monotonicity of allocations for each good, allowing the multi-product problem to be decomposed into single-good problems. Although ironing can no longer be applied over the entire type space, applying it separately within each segment still yields the quasi-concave closure of the profit function, thereby ensuring the optimality of separate sales. This illustrates how monotone MRS and regularity can substitute for each other in generating optimal SMP. For more detail, see the previous working paper (Kwak, 2025).

5.4 Multi-dimensional Types

The optimality of SMP under monotone MRS extends to multi-dimensional type spaces with imperfect positive correlation. This extension follows the decomposition approach of Haghpanah and Hartline (2021). If the type distribution is a mixture of distributions, each admitting the same SMP as optimal, then that SMP remains optimal under the mixture. This is because allowing mechanisms to vary across component distributions can only weakly improve revenue, leaving a uniformly applied SMP optimal.

Let the support of valuations for good $i \in [d]$ be $V^i := \{v^i_1, v^i_2, \ldots, v^i_{N^i}\}$ with cardinality $|V^i| = N^i > 0$, and let $N := N^1$ for $i = 1$. Under monotone valuations, $N^i = N$ for all $i \in [d]$, and once $n \in [N]$ (or equivalently, v^1_n) is chosen, v^i_n for each higher-indexed good $i > 1$ is fully determined. More generally, $N^i \neq N$ for some goods $i > 1$, and the buyer's type is represented by $(n, n^2, \ldots, n^d) \in \Theta := [N] \times [N^2] \times \cdots \times [N^d]$. The buyer's valuation for each good i is $v^i_{n^i}$ for $i > 1$ and v^1_n for $i = 1$.

Given a distribution f over the type space Θ, let $F^i_{n^i}$ denote the marginal CDF of the valuation for good $i \in [d]$, and let p^{i*} denote the monopoly price for good i corresponding to its marginal distribution. For each $i > 1$, let $F(\cdot \mid v^1_n, v^2_{n^2}, \ldots, v^{i-1}_{n^{i-1}})$ denote the conditional CDF of good i's valuation given the valuations of all lower-indexed goods $j < i$, $(v^1_n, v^2_{n^2}, \ldots, v^{i-1}_{n^{i-1}}) \in V^1 \times \cdots \times V^{i-1}$. Using the corresponding quantile function, denoted by $F^{-1}(\cdot \mid v^1_n, v^2_{n^2}, \ldots, v^{i-1}_{n^{i-1}})$ over $[0,1]$, define $(v^2_n(t), \ldots, v^d_n(t)) : [0,1]^{d-1} \to V^2 \times \cdots \times V^d$ for each $n \in [N]$ and $t := (t^2, \ldots, t^d) \in [0,1]^{d-1}$ as follows: for $i = 2, \ldots, d$,

$$v^i_n(t) = F^{-1}(t^i \mid v^1_n, v^2_n(t), \ldots, v^{i-1}_n(t)).$$

Then, the type distribution can be represented as a mixture with respect to the uniform distribution over $t \in [0,1]^{d-1}$, where each component distribution indexed by t is supported on

$$\Theta_t := \{(v_n^1, v_n^2(t), \ldots, v_n^d(t)) \mid n \in [N]\}$$

and preserves the marginal distribution F_n^1 over v_n^1.

Suppose that each component distribution satisfies monotone valuations and monotone MRS, with common monopoly prices. Specifically, for each $t \in [0,1]^{d-1}$, the following conditions hold:

- For each $i > 1$:
  - $v_n^i(t)$ is strictly increasing in n;
  - $p^{i*} \in \arg\max_{v_n^i(t)} v_n^i(t)(1 - F_{n-1}^1)$.
- Monotone MRS holds for Θ_t; that is, $\frac{v_n^2(t)}{v_n^1}$ and $\frac{v_n^{i+1}(t)}{v_n^i(t)}$ are increasing in n for all $i = 2, \ldots, d-1$.

Then, by the "if" part of Theorem 1, the optimal mechanism under each component distribution over Θ_t is SMP with the price p^{i*} for each $i \in [d]$.[24] Hence, the original multi-dimensional type distribution also admits the same SMP as optimal.

6 Conclusion

This paper derives conditions under which separate monopoly pricing (SMP) is optimal in a multi-product monopoly problem with additive and monotone valuations. Monotone MRS is shown to be both necessary and sufficient for SMP to be robustly optimal, regardless of the type distribution.

The result is obtained by extending the conventional Myersonian approach rather than relying on the more recent duality methods. While duality provides a powerful general framework for characterizing optimal screening, it does not offer a concrete procedure for constructing the dual variables, which often makes the implementation complex and context-specific. By contrast, the Myersonian approach employed here uses elementary modifications of mechanisms and transparent relaxations of constraints to uncover the underlying incentive structure. Its simplicity makes it more applicable.

The extended Myersonian approach recovers type-increasing allocations with a one-dimensional prefix structure or, at least, clarifies the incentive structure when allocations decrease. It shows that any decrease in quantities must be associated with essentially binding

[24] In fact, only finitely many distinct Θ_t appear (that is, $|\{\Theta_t \mid t \in [0,1]^{d-1}\}| < \infty$) because each set V^i for $i \in [d]$ is finite.

incentive constraints within the subset of types where monotone MRS fails. This piecewise monotonicity, whether increasing or decreasing, substantially restricts the set of optimal allocations in the multi-dimensional allocation space. These restrictions can become particularly powerful when combined with alternative preference orderings that further constrain incentive compatibility.

A natural and important direction for future research is to characterize the structure of optimal bundling when separate monopoly pricing is not optimal. This paper offers a characterization of the boundary between profitable and unprofitable bundling, and the methods developed here could be extended to identify conditions that help explain observed bundling practices.

A Proofs

A.1 Proof of Lemma 1

Fix a solution $\tilde{M} = \{(\tilde{q}_n, \tilde{p}_n)\}_{n\in[N]}$ to problem (2).

Part (i). Construct a new solution $M = \{(q_n, p_n)\}_{n\in[N]}$ as follows. For each type n, set $p_n = \tilde{p}_n$ and choose q_n to be a prefix allocation such that type n's valuation is preserved:

$$\sum_{i=1}^{d} v_n^i q_n^i = \sum_{i=1}^{d} v_n^i \tilde{q}_n^i.$$

Such a prefix exists by the intermediate value theorem, since the set of prefix allocations varies continuously from the empty bundle to the grand bundle, and type n's valuation from $\tilde{q}_n$ lies between the corresponding extremes. Moreover, the prefix is unique because the set of prefix allocations is totally ordered by the standard componentwise order $\geq$, and type n's valuation v_n^i for each good $i \in [d]$ is strictly positive.

Hence, type n's utility is unchanged under the two mechanisms:

$$U_n(q_n, p_n) = U_n(\tilde{q}_n, \tilde{p}_n) \quad \forall n \in [N]. \tag{8}$$

Since $p_n = \tilde{p}_n$ for all $n \in [N]$, it also follows that total revenue is the same in M as in $\tilde{M}$.

To verify feasibility, note first that the lowest type's IR constraint still holds by (8). It remains to check the downward IC constraints. Since $\tilde{M}$ satisfies them, and by (8) the utility of each higher type $n' \geq n$ from its own allocation is unchanged, it suffices to show that

$$U_{n'}(q_n, p_n) \leq U_{n'}(\tilde{q}_n, \tilde{p}_n).$$

By the prefix structure of q_n, there exists a cutoff index $\hat{i} \in [d]$ such that

$$q_n^i \geq \tilde{q}_n^i \quad \text{for } i \leq \hat{i}, \qquad q_n^i \leq \tilde{q}_n^i \quad \text{for } i > \hat{i}. \tag{9}$$

More precisely, if the cutoff good of q_n is i_n (that is, $q_n^i = 1$ for $i < i_n$ and $q_n^i = 0$ for $i > i_n$), then

$$\hat{i} = \begin{cases} i_n, & \text{if } q_n^{i_n} \geq \tilde{q}_n^{i_n}, \\ i_n - 1, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}
U_{n'}(q_n, p_n) - U_{n'}(\tilde{q}_n, \tilde{p}_n) &= \sum_{i=1}^{\hat{i}} v_{n'}^i (q_n^i - \tilde{q}_n^i) - \sum_{i=\hat{i}+1}^{d} v_{n'}^i (\tilde{q}_n^i - q_n^i) \\
&\leq \max_{i \leq \hat{i}} \left\{ \frac{v_{n'}^i}{v_n^i} \right\} \left(\sum_{i=1}^{\hat{i}} v_n^i (q_n^i - \tilde{q}_n^i) \right) - \min_{i \geq \hat{i}+1} \left\{ \frac{v_{n'}^i}{v_n^i} \right\} \left(\sum_{i=\hat{i}+1}^{d} v_n^i (\tilde{q}_n^i - q_n^i) \right) \\
&= \frac{v_{n'}^{\hat{i}}}{v_n^{\hat{i}}} \left(\sum_{i=1}^{\hat{i}} v_n^i (q_n^i - \tilde{q}_n^i) \right) - \frac{v_{n'}^{\hat{i}+1}}{v_n^{\hat{i}+1}} \left(\sum_{i=\hat{i}+1}^{d} v_n^i (\tilde{q}_n^i - q_n^i) \right) \\
&\leq \frac{v_{n'}^{\hat{i}}}{v_n^{\hat{i}}} \big(U_n(q_n, p_n) - U_n(\tilde{q}_n, \tilde{p}_n) \big) \\
&= 0.
\end{aligned}$$

The first inequality follows from (9), the second equality and the second inequality from monotone MRS, and the last equality from (8).

Thus, M yields the same revenue as the solution $\tilde{M}$, satisfies the lowest type's IR constraint and all downward IC constraints, and is therefore also a solution to (2).

Part (ii). Suppose that the solution $M = \{(q_n, p_n)\}_{n \in [N]}$ constructed in part (i) does not satisfy type-increasing allocations. Let n be the smallest index such that $q_n \leq q_{n+1}$ does not hold. By the prefix structure, this implies $q_n > q_{n+1}$ under the strict component-wise order, that is, $q_n \geq q_{n+1}$ and $q_n \neq q_{n+1}$.

Because each $q_{\tilde{n}}^i$ is non-decreasing in $\tilde{n} \leq n$ for all $i \in [d]$, the (strict) SCP holds between (q_n, p_n) and $(q_{\tilde{n}}, p_{\tilde{n}})$ under monotone valuations. Therefore, the downward IC constraint of type n against type $\tilde{n}$,

$$U_n(q_n, p_n) \geq U_n(q_{\tilde{n}}, p_{\tilde{n}}),$$

implies that type $n+1$ also prefers type n's allocation to type $\tilde{n}$'s:

$$U_{n+1}(q_n, p_n) \geq U_{n+1}(q_{\tilde{n}}, p_{\tilde{n}}).$$

In other words, type $n+1$ prefers type n's allocation the most among all lower-type allocations. Thus, it must hold that

$$U_{n+1}(q_{n+1}, p_{n+1}) = U_{n+1}(q_n, p_n).$$

Otherwise, $U_{n+1}(q_{n+1}, p_{n+1}) > U_{n+1}(q_n, p_n)$. Then, by slightly increasing p_{n+1}, the seller could strictly raise revenue without violating any downward IC constraints or the lowest type's IR constraint, contradicting the optimality of M.

But then, since $q_n > q_{n+1}$, the strict SCP again implies that each higher type $n' > n+1$ strictly prefers (q_n, p_n) to (q_{n+1}, p_{n+1}). Together with the downward IC constraint between types n' and n, this yields

$$U_{n'}(q_{n'}, p_{n'}) \;\geq\; U_{n'}(q_n, p_n) \;>\; U_{n'}(q_{n+1}, p_{n+1}).$$

Hence, there is no binding downward IC constraint against type $n+1$. Because $q_{n+1} < q_n$, the prefix allocation q_{n+1} must have $q^d_{n+1} < 1$. Now, raise q^d_{n+1} by a small $\epsilon > 0$ and increase p_{n+1} by $v^d_{n+1}\epsilon$ so that type $n+1$'s utility is preserved. Because there are only finitely many downward IC constraints against type $n+1$, and each is slack, ϵ can be chosen sufficiently small so that this perturbation strictly increases revenue without violating any downward IC constraint or the lowest type's IR constraint, contradicting the optimality of M.

A.2 Proof of Proposition 1

Suppose that Π^i_n for some good i is not quasi-concave at a monopoly type $n^{j*} \in \arg\max_{n\in[N]} \Pi^j_n$ for another good j, meaning that

$$\Pi^i_{n^{j*}} < \min\left\{\max_{n<n^{j*}} \Pi^i_n,\; \max_{n>n^{j*}} \Pi^i_n\right\}. \tag{10}$$

Hence, n^{j*} cannot be a monopoly type n^{i*} for good i, implying that one of the following must hold: (i) $\max_{n\in[N]} \Pi^i_n = \max_{n<n^{j*}} \Pi^i_n$, or (ii) $\max_{n\in[N]} \Pi^i_n = \max_{n>n^{j*}} \Pi^i_n$.

Case (i): $\max_{n\in[N]} \Pi^i_n = \max_{n<n^{j*}} \Pi^i_n$ In this case, there exists a monopoly type $n^{i*} \in \arg\max_{n\in[N]} \Pi^i_n$ for good i such that $n^{i*} < n^{j*}$. Let $\tilde{n}^{i*}$ denote the local maximizer of Π^i_n on the opposite side of n^{i*} relative to n^{j*}:

$$\tilde{n}^{i*} \in \arg\max_{n>n^{j*}} \Pi^i_n.$$

Then, condition (10) is equivalent to

$$\Pi^i_{n^{j*}} < \Pi^i_{\tilde{n}^{i*}}. \tag{11}$$

When multiple monopoly types exist for a good, there may be multiple separate monopoly pricing mechanisms yielding the same revenue. However, to establish the suboptimality of SMP, it suffices to show that one such mechanism is suboptimal. Fix an SMP $\{(q_n, p_n)\}_{n\in[N]}$ in which the allocations for goods i and j are determined by the monopoly types n^{i*} and n^{j*}, respectively:

$$q^i_n = \begin{cases} 1, & \text{if } n \geq n^{i*}, \\ 0, & \text{otherwise,} \end{cases} \qquad q^j_n = \begin{cases} 1, & \text{if } n \geq n^{j*}, \\ 0, & \text{otherwise.} \end{cases}$$

Now, consider the following perturbation: decrease q^i_n by a sufficiently small $\epsilon > 0$ for all $n = n^{j*}, \ldots, \tilde{n}^{i*} - 1$, while adjusting prices so that all local downward IC constraints and the lowest type's IR constraint remain binding. More precisely, the perturbed mechanism $\{(\tilde{q}_n, \tilde{p}_n)\}_{n\in[N]}$ satisfies $\tilde{q}^\ell_n = q^\ell_n$ for all $\ell \neq i$ and $n \in [N]$, and

$$\tilde{q}^i_n = \begin{cases} q^i_n - \epsilon, & \text{if } n^{j*} \leq n < \tilde{n}^{i*}, \\ q^i_n, & \text{otherwise,} \end{cases} \qquad \tilde{p}_n = \begin{cases} p_n - \epsilon v^i_{n^{j*}}, & \text{if } n^{j*} \leq n < \tilde{n}^{i*}, \\ p_n + \epsilon\left(v^i_{\tilde{n}^{i*}} - v^i_{n^{j*}}\right), & \text{if } n \geq \tilde{n}^{i*}, \\ p_n, & \text{otherwise.} \end{cases} \tag{12}$$

Hence, this perturbation strictly increases revenue by

$$\sum_{n=1}^{N} f_n \tilde{p}_n - \sum_{n=1}^{N} f_n p_n = \epsilon\left(v^i_{\tilde{n}^{i*}}(1 - F_{\tilde{n}^{i*}-1}) - v^i_{n^{j*}}(1 - F_{n^{j*}-1})\right) = \epsilon\left(\Pi^i_{\tilde{n}^{i*}} - \Pi^i_{n^{j*}}\right) > 0.$$

Equivalently, because revenue can be expressed as total virtual surplus when local downward IC constraints and the lowest-type IR constraint bind, the same increase can be derived as

$$\sum_{\ell=1}^{d} \sum_{n=1}^{N} V^\ell_n (\tilde{q}^\ell_n - q^\ell_n) = -\epsilon \sum_{n=n^{j*}}^{\tilde{n}^{i*}-1} V^i_n = \epsilon\left(\Pi^i_{\tilde{n}^{i*}} - \Pi^i_{n^{j*}}\right) > 0.$$

It remains to show that the perturbed mechanism is feasible in problem (1). Once all IC constraints are satisfied, the IR constraint for the lowest type, ensured by construction, guarantees IR for all higher types under monotone valuations. Therefore, it suffices to check that all IC constraints remain satisfied, which will be demonstrated in the following steps.

Step 1: The perturbed mechanism $\{(\tilde{q}_n, \tilde{p}_n)\}_{n\in[N]}$ satisfies all IC constraints between types n and n' with $n < n' \leq n^{j*} - 1$ or $n' > n \geq n^{j*}$. In addition, the downward IC between types n^{j*} and $n < n^{j*}$ also holds.

Proof. Within each of the two type subsets, $[n^{j*} - 1]$ and $[N] \backslash [n^{j*} - 1]$, the allocations remain type-increasing. Therefore, the SCP continues to hold among allocations within each subset. Hence, the SCP, together with the binding local downward IC constraints, implies that all IC constraints hold within each subset of types.

The downward IC constraints between type n^{j*} and any type $n < n^{j*}$ also hold due to the SCP between the allocations to types $n^{j*} - 1$ and n. Since $\tilde{q}_{n^{j*}-1} \geq \tilde{q}_n$, the downward IC between types $n^{j*} - 1$ and n,

$$U_{n^{j*}-1}(\tilde{q}_{n^{j*}-1}, \tilde{p}_{n^{j*}-1}) \geq U_{n^{j*}-1}(\tilde{q}_n, \tilde{p}_n),$$

together with the SCP, implies that

$$U_{n^{j*}}(\tilde{q}_{n^{j*}}, \tilde{p}_{n^{j*}}) = U_{n^{j*}}(\tilde{q}_{n^{j*}-1}, \tilde{p}_{n^{j*}-1}) \geq U_{n^{j*}}(\tilde{q}_n, \tilde{p}_n),$$

where the equality follows from the binding local downward IC between types n^{j*} and $n^{j*} - 1$. □

Step 2: All other IC constraints, namely those between types n and n' with $n' > n^{j*} > n$, as well as the upward IC constraints of each type $n < n^{j*}$ against n^{j*}, are slack under the original SMP $\{(q_n, p_n)\}_{n \in [N]}$. Hence, they remain satisfied under the perturbed mechanism $\{(\tilde{q}_n, \tilde{p}_n)\}_{n \in [N]}$ for sufficiently small $\epsilon > 0$.

Proof. Under the SMP, $q_{n^{j*}} > q_{n^{j*}-1}$ since $q^j_{n^{j*}} > q^j_{n^{j*}-1}$. Thus, under monotone valuations, the strict SCP holds between the allocations of types n^{j*} and $n^{j*} - 1$. The binding local downward IC between these two types,

$$U_{n^{j*}}(q_{n^{j*}}, p_{n^{j*}}) = U_{n^{j*}}(q_{n^{j*}-1}, p_{n^{j*}-1}),$$

then implies that for every pair $n < n^{j*} < n'$,

$$U_n(q_{n^{j*}}, p_{n^{j*}}) < U_n(q_{n^{j*}-1}, p_{n^{j*}-1}) \leq U_n(q_n, p_n), \tag{13}$$

$$U_{n'}(q_{n^{j*}-1}, p_{n^{j*}-1}) < U_{n'}(q_{n^{j*}}, p_{n^{j*}}) \leq U_{n'}(q_{n'}, p_{n'}), \tag{14}$$

The weak inequalities follow from the upward IC between types n and $n^{j*} - 1$ and the downward IC between types n' and n^{j*}, respectively.

By the SCP and the type-increasing allocation under SMP, type n prefers type n^{j*}'s allocation most among all allocations for types above n^{j*}, including type n''s. Hence, by

(13), the upward IC constraints of type n against both n' and n^{j*} are slack:

$$U_n(q_{n'}, p_{n'}) \leq U_n(q_{n^{j*}}, p_{n^{j*}}) < U_n(q_n, p_n).$$

Similarly, by the SCP, type n' prefers type $n^{j*}-1$'s allocation most among those of all types below $n^{j*}-1$, including type n's. Thus, by (14), the downward IC between types n' and n is slack:

$$U_{n'}(q_n, p_n) \leq U_{n'}(q_{n^{j*}-1}, p_{n^{j*}-1}) < U_{n'}(q_{n'}, p_{n'}).$$

Moreover, by (12), the changes in quantities and prices induced by the perturbation are of order ϵ (also linear in ϵ). Accordingly, the induced changes in the utilities entering the incentive constraints are also of order ϵ, and thus vanish as $\epsilon \to 0$. Consequently, all the (finitely many) slack incentive constraints remain satisfied for sufficiently small $\epsilon > 0$. □

Case (ii): $\max_{n\in[N]} \Pi^i_n = \max_{n>n^{j*}} \Pi^i_n$ By a similar argument, the suboptimality of SMP can be shown with a slight modification of the perturbation. In this case, there exists a monopoly type n^{i*} for good i such that $n^{i*} > n^{j*}$. The local maximizer $\tilde{n}^{i*}$ of Π^i_n on the opposite side of n^{i*} relative to n^{j*} is

$$\tilde{n}^{i*} \in \arg\max_{n<n^{j*}} \Pi^i_n,$$

and condition (11) still holds. Hence, increase q^i_n by a sufficiently small $\epsilon > 0$ for all $n = \tilde{n}^{i*}, \dots, n^{j*}-1$, while adjusting prices so that all local downward IC constraints and the lowest type's IR constraint remain binding. More precisely, consider a perturbed mechanism $\{(\tilde{q}_n, \tilde{p}_n)\}_{n\in[N]}$ satisfying $\tilde{q}^\ell_n = q^\ell_n$ for all $\ell \neq i$ and $n \in [N]$, and

$$\tilde{q}^i_n = \begin{cases} q^i_n + \epsilon, & \text{if } \tilde{n}^{i*} \leq n < n^{j*}, \\ q^i_n, & \text{otherwise,} \end{cases} \qquad \tilde{p}_n = \begin{cases} p_n + \epsilon v^i_{\tilde{n}^{i*}}, & \text{if } \tilde{n}^{i*} \leq n < n^{j*}, \\ p_n + \epsilon\left(v^i_{\tilde{n}^{i*}} - v^i_{n^{j*}}\right), & \text{if } n \geq n^{j*}, \\ p_n, & \text{otherwise.} \end{cases}$$

Then, revenue again increases strictly by

$$\sum_{n=1}^{N} f_n \tilde{p}_n - \sum_{n=1}^{N} f_n p_n = \epsilon\left(v^i_{\tilde{n}^{i*}}(1 - F_{\tilde{n}^{i*}-1}) - v^i_{n^{j*}}(1 - F_{n^{j*}-1})\right) = \epsilon\left(\Pi^i_{\tilde{n}^{i*}} - \Pi^i_{n^{j*}}\right) > 0.$$

Feasibility is preserved, since Steps 1 and 2 still apply under this modification.

A.3 An example: non-monotone MRS

The continuation of the example in Section 3 also illustrates why SMP can be suboptimal with non-monotone MRS. Suppose now that $v_3^2 < v_3^1$, implying that the MRS $\frac{v_n^2}{v_n^1}$ is single-peaked at type 2, as illustrated in Figure 4.

In contrast to the monotone MRS case in Section 3, mixed bundling is no longer dominated by separate sales with $p^1 = v_1^1$ and $p^2 = v_3^2$. Specifically, the revenue-maximizing mixed bundling assigns good 1 to type 1, good 2 to type 2, and the grand bundle to type 3, with prices

$$(p^1, p^2, p^G) = (v_1^1,\ v_1^1,\ v_1^1 + v_3^2).$$

The mechanism follows directly from the IC constraints and type 1's IR constraint stated in Section 3.[25] Type 2 remains indifferent between $(1,0)$ and $(0,1)$, but type 3 is now indifferent between $(1,1)$ and $(1,0)$, rather than $(0,1)$. This occurs because the relative attractiveness of good 2 peaks at type 2. Type 3 obtains an information rent equal to $v_3^1 - v_1^1$, which coincides with the rent under separate sales with $p^1 = v_1^1$ and $p^2 = v_3^2$. Since types 1 and 2 also pay the same under the two mechanisms, both mechanisms generate the same expected revenue regardless of the type distribution.

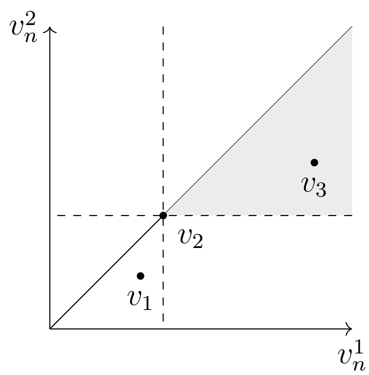

Figure 4: Non-monotone MRS

In fact, a more complex bundling mechanism yields strictly higher revenue than both mixed bundling and separate sales for any full-support type distribution. Consider perturbing the mixed bundling mechanism by replacing the bundle $(0,1)$ at price p^2 with $(q,1)$ at price $p^2 + qv_2^1$ for small $q > 0$. Under the mixed bundling, both types 1 and 3 strictly prefer their own allocations, $(1,0)$ and $(1,1)$, to that of type 2, $(0,1)$. As a result, this perturbation strictly increases the revenue from type 2 without incurring any incentive compatibility issues.

[25] The IC constraints rule out a mixed bundling that assigns good 2 to type 1, good 1 to type 2, and the grand bundle to type 3, since $v_1^1 - v_1^2 > 0 = v_2^1 - v_2^2$.

The upper bound on such q, denoted by $\bar{q}$, is reached when type 3 becomes indifferent between $(q, 1)$ and $(1, 1)$, i.e.,

$$v_3^1 + v_3^2 - p^G = v_3^2 - p^2 + (v_3^1 - v_2^1)\bar{q},$$

which yields

$$\bar{q} := \frac{v_3^1 - v_3^2}{v_3^1 - v_2^1} < 1.$$

For $q > \bar{q}$, selling $(q, 1)$ at price $p^2 + qv_2^1$ requires lowering p^G by $(v_3^1 - v_2^1)(q - \bar{q})$ to keep type 3 purchasing $(1, 1)$. The trade-off is unprofitable if and only if the revenue from selling good 1 at price v_2^1 is lower than that from selling it at price v_3^1, that is,

$$\Pi_2^1 < \Pi_3^1. \tag{15}$$

This implies that the demand for good 1 is inelastic above type 2.

Consequently, under any full-support type distribution satisfying (15), the revenue is therefore uniquely maximized at $q = \bar{q}$. Moreover, if $q = 1$, the mechanism coincides with separate sales with $p^1 = v_1^1$ and $p^2 = v_2^2$. In particular, if v_1^1 and v_2^2 are the respective monopoly prices of goods 1 and 2 when sold individually, i.e.,

$$\begin{aligned} &\Pi_2^1 < \Pi_3^1 \leq \Pi_1^1, \\ &\Pi_2^2 \geq \max\{\Pi_3^2, \Pi_1^2\}, \end{aligned}$$

the separate sales mechanism coincides with separate monopoly pricing, which in turn is strictly dominated by the modified mixed bundling mechanism with $q = \bar{q}$.

This suboptimality of SMP is consistent with Proposition 1. The inequalities above imply that the corresponding type distribution violates quasi-concavity at profit maxima in Definition 3: Π_n^1 attains a unique minimum at type 2, which coincides with a monopoly type for good 2. As noted in the final part of Section 4.3, such inequalities arise, for example, under the type distribution inducing unit-elastic demand for good 2 (i.e., Π_n^2 is constant) when $\frac{v_2^2}{v_2^1} > \frac{v_3^2}{v_3^1} \geq \frac{v_1^2}{v_1^1}$, since $\Pi_n^2 = \frac{v_n^2}{v_n^1}\Pi_n^1$ for every $n \in [N]$.

Lastly, under non-monotone MRS, SMP can be suboptimal even when the SCP of the profit functions across goods holds and hence quasi-concavity at profit maxima is satisfied, as shown in the proof of Lemma 3. To illustrate this point, suppose in addition to the single-peaked MRS at type 2 that the remaining ratios satisfy

$$\frac{v_3^2}{v_3^1} > \frac{v_1^2}{v_1^1}.$$

Consider the following type distribution, obtained as a perturbation of the unit-elastic distribution for good 2 for sufficiently small $\epsilon > 0$:

$$(f_2 + f_3, f_3) = \left(\frac{v_1^2}{v_2^2} - \epsilon,\ \frac{v_1^2}{v_3^2} + \epsilon \right).$$

Under this distribution, the profit functions satisfy the SCP across goods:

$$\Pi_1^1 > \Pi_3^1 > \Pi_2^1,$$
$$\Pi_3^2 > \Pi_1^2 > \Pi_2^2.$$

Nevertheless, the corresponding SMP with $p^1 = v_1^1$ and $p^2 = v_3^2$ is suboptimal for any full-support type distribution, as shown above.

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