Open Rule Legislative Bargaining

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Abstract

The seminal paper by Baron and Ferejohn (1989) leaves significant gaps in our understanding of open rule bargaining. We aim to fill these gaps by providing a fresh analysis of open rule bargaining. Our approach relies on an appealing class of stationary equilibria. In this class, we show that delays tend to be longer and allocations tend to be less egalitarian than originally predicted by Baron and Ferejohn. Our results shed new light on the efficiency and fairness implications of using an open vs. closed rule in legislatures and of bargaining processes in general.

Keywords: Bargaining, Legislatures, Open Rules, Baron and Ferejohn, Stationary Equilibrium

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1 Introduction

We consider open rule bargaining on the division of a surplus under a simple majority rule. There is a vast game-theoretic literature on the resolution of surplus division problems through bargaining (see for instance the seminal papers by Rubinstein (1982) as well as Baron and Ferejohn (1989)). The bargaining approach to surplus division problems has important applications in political economy: The classical example is a parliament that negotiates about the allocation of funds from the government budget, while each member of the legislature wishes to obtain funds for projects in his/her own district. Such *legislative bargaining* models have been studied, among others, by Banks and Duggan (2000), Eraslan (2002), Battaglini and Coate (2007), and Battaglini et al. (2014). On the notion and current state of legislative bargaining, see Eraslan and Evdokimov (2019).

The game-theoretic bargaining literature emphasizes the importance of formal rules that structure the bargaining process. In particular, it assumes that negotiations consist of several rounds, where the rejection of a proposal triggers the next round. In these models, equilibrium outcomes depend crucially on who has the right to make a proposal at what point in time, and on how costly the delay in moving from one round to the next is.

Moreover, it matters how each round is structured. One important distinction can be made between the "open rule" and "closed rule" bargaining. Under an *open rule,* a proposal may be amended once or several times before it is put to vote. In contrast, under a *closed rule,* each proposal is immediately voted upon, allowing only Yes-or-No answers. A new proposal is only made when the previous one has been rejected.

In practice, open rules are very common in legislative decision-making. For instance, open rule procedures are an important part of legislative bargaining in the U.S. Congress (Oleszek (2011)). In particular, the House of Representatives makes use of a variety of rules, including open rules, structured rules, i.e. rules that allow only specified amendments rather than a fully open amendment process, and closed rules.[1] Open rules are not only prevalent at federal level in the United States, but also at state level (Primo (2003)).

However, most of the bargaining-theoretic literature has focused on closed rules. In

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[1] Detailed information on the legislative processes in the House of Representatives can be found on the website of its Committee on Rules: https://rules.house.gov (retrieved on July 1, 2024).

particular, this is true for most of the contributions following Baron and Ferejohn's seminal paper. The theoretical understanding of closed rule bargaining has been furthered considerably by the work of Ansolabehere et al. (2005), Banks and Duggan (2000, 2006), Diermeier and Feddersen (1998), Eraslan (2002), and McCarty (2000), among others.

Compared to bargaining games with a closed rule, Baron and Ferejohn's model of open rule legislative bargaining has received less attention. Primo (2007) shows how a proposer can randomize between different coalitions in open rule legislative bargaining. His findings indicate that the equilibrium found by Baron and Ferejohn cannot be unique. Baron and Ferejohn (1989) offer a comparison of open and closed rule bargaining. In a nutshell, Baron and Ferejohn assume that a player who finds it optimal to amend a proposal will always want to vote against that same proposal once it is put to a vote. In this paper, we study open rule bargaining from scratch, taking into account the distinction between *voting in favor of a proposal* and *not amending the proposal.* In our bargaining model, proposals can be amended before a vote takes place, and voting operates at two levels: between a proposal and amendment and between an endorsed proposal and the continuation of the bargaining game. In the latter case, the approval ends the game. Thus, for example, it can happen that an agent may want to amend the proposal if he has the option, but nevertheless will vote in favor of the proposal to prevent a delay and end the game. Putting this distinction at the center of our analysis, we propose a new stylized approach to Baron and Ferejohn's model of the open rule bargaining game. We introduce an equilibrium refinement and proceed from a simplified to an extended version to ensure a certain tractability. We obtain a sharp equilibrium prediction in the limit as the discount factor goes to one.

While immediate agreement is a standard result for closed rule bargaining, equilibrium delays typically occur under open rules. The tendency to delay agreement under an open rule can be understood as follows: Ideally, a proposer would want to ensure that his/her proposal is not amended before a vote. To this end, s/he would have to propose a surplus allocation that is sufficiently attractive for other players to discourage them from making any amendment. However, the proposer is uncertain about *which* player will have a chance to make an amendment. Hence, s/he has to weigh the risk of an amendment against the cost of discouraging other players from amendments. Typically, the optimal proposal gives sufficiently generous offers to some, but not all, players. As a result, there remains a positive probability of an amendment, leading to delay.

More specifically, with our stylized model of open rule legislative bargaining and the particular equilibrium concept, we make the following contributions:

- We provide a definition of stationary strategies in the open rule legislative bargaining model. We define a class of equilibrium candidates that consists of relatively simple stationary strategies. We derive necessary and sufficient conditions under which such an equilibrium candidate is indeed a stationary subgame–perfect Nash equilibrium of the game.

- For the limit case, as the discount factor converges to one, we compute an explicit equilibrium prediction of the proposals, the payoffs, and the expected length of delay before an agreement.

- We compare the equilibrium outcome of open rule bargaining to that of the canonical closed rule bargaining process. While closed rule bargaining leads to immediate agreement, open rule bargaining typically leads to delays on the equilibrium path of play. We find that these equilibrium delays can be longer than predicted by Baron and Ferejohn. We also show that the inefficiency inherent to an open rule can be so great that *all* players would be ex ante better off with a closed rule. Importantly, as shown in Section 4, the outcomes with an open rule are quite distinct from the outcomes under a closed rule, even if the discount factor is arbitrarily close to 1.

- Smaller legislatures yield less delay and a more egalitarian allocation than larger legislatures under open rules. These properties could be preserved in large legislatures by the following protocol: Surplus division is first delegated to a smaller committee that itself is representative of the legislature. The committee uses the open rule and the committee decision is put to a final vote in the legislature under a closed rule.

Our work is complementary to a stream of literature that analyzes bargaining with an endogenous status quo, see Anesi (2010), Diermeier and Fong (2011, 2012), and Bowen and Zahran (2012). In their models, negotiations can continue even after an agreement has been reached. In each round, the status quo is given by the most recent agreement. In our model, when an endorsed proposal is accepted by a majority, this approval ends the game. As a result, we obtain, for instance, equilibria in which only a minority of amenders are willing to endorse an original proposal. A different protocol for proposals with amendment procedures with an evolving status quo has been developed and examined by Anesi and Seidmann (2014).

Fréchette et al. (2003) provide a theoretical and experimental investigation comparing closed and open rules, and identify crucial differences between closed and open rules. Fréchette et al. (2003) show in their experiment that we may not always have minimal winning coalitions in the open rule bargaining. We recover such results in our model when the group is not too large. Our analysis suggests a variety of further hypotheses for larger groups and other discount factors than in Fréchette et al. (2003). This could be taken up in future experimental work, and we summarize some of these hypotheses at the end of the concluding section. Falconieri (2004) gives a comparative analysis of open and closed rules in the context of lobbying and delegation.

The remainder of the paper is organized as follows: Section 2 contains the formal description of the open rule legislative bargaining game. In Section 3, we provide a rigorous definition of stationary strategies which is suitable for the analysis of this game and conduct an equilibrium analysis of a slightly simplified version of the open rule bargaining game. In Section 3.3, we consider the limit as players are sufficiently patient and explicitly compute the equilibrium predictions. Afterwards, we use some numerical examples to illustrate our findings in Section 4. We provide a more detailed account of the relation between our present paper and Baron and Ferejohn's work in Section 5. We show that the main results and conclusions we have reached for the simplified open rule bargaining game carry over to the original game under some assumptions. We make some concluding remarks and discuss possible directions of future research in Section 6. Most of the proofs are relegated to Appendix B.

2 The open rule legislative bargaining game

2.1 The set-up

We consider open rule bargaining with an odd[2] number of players, $n \geq 3$. The set of players is denoted by N, and we will frequently use i or j to index its members. There is a surplus of unit size to be divided among the players. Thus, the space of possible agreements is $\Delta^n := \{\theta \in \mathbb{R}^n_{\geq 0} | \sum_{i \in N} \theta_i \leq 1\}$.[3] The decision to implement a particular surplus division is taken by simple majority voting, that is, it requires the approval of at least $(n+1)/2$ players. The bargaining process is structured in rounds $t = 0, 1, \ldots$. The number of rounds is potentially infinite.

Baron and Ferejohn's open rule bargaining process involves a complex chain of events in which players can make proposals, suggest amendments, choose between a proposal and an amendment, and eventually vote on the implementation of a proposal.

We approach this complexity in steps. We start with a simplified version of the bargaining process. In the simplified version, an amendment immediately replaces the proposal on the floor. In two extended versions, there is a vote on whether the amendment should replace the proposal on the floor, which is the bargaining protocol in Baron and Ferejohn (1989). There, we will argue that the main results and conclusions derived for the simplified version also hold in the extended versions.

We start with the simplified version. In order to make this open rule bargaining process clear, we divide the description into the following three steps:

Step 1: Proposal on the floor

Consider any bargaining round t. Two cases must be distinguished: Either, bargaining round t begins with a *proposal on the floor,* or it begins *without a proposal on the floor.*

- *Proposal on the floor:* If round t begins with a proposal on the floor, then round t of the game directly proceeds to Step 2 below.
- *No proposal on the floor:* If round t begins without a proposal on the floor, then a *proposer* is randomly chosen from N with equal probability. Let us say that player

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[2] We follow Baron and Ferejohn (1989) in assuming an odd number of players in order to avoid ties.

[3] Without loss of generality, we will focus on feasible agreements that satisfy $\sum_{i \in N} \theta_i = 1$. In particular, a proposer never finds it optimal to make a proposal that does not fully divide the available surplus.

i is chosen as the proposer. Then, player i chooses some proposal $\overline{\theta} \in \Delta^n$, which thereby becomes the proposal on the floor.[4] Now the game proceeds to Step 2.

For rounds $t > 0$, the game may begin either with a proposal on the floor or without one, depending on the outcome of the previous round. Bargaining round t can only begin with a proposal on the floor if that proposal has been made in a previous round. Therefore, the initial bargaining round $t = 0$ begins without a proposal on the floor.

Step 2: Amendment or endorsement

Suppose that the proposal $\overline{\theta} \in \Delta^n$ made by some player $i \in N$ is on the floor in round t. Now, a new proposer is randomly chosen with equal probability from $N \setminus \{i\}$. Let us say that player j has been chosen. Then, player j decides whether to *endorse* or *amend* the proposal on the floor.[5]

- *Endorsement:* If player j *endorses* the proposal on the floor $\overline{\theta}$, then round t of the game proceeds directly to Step 3.
- *Amendment:* Suppose that player j chooses to amend the proposal $\overline{\theta}$ on the floor. Then, in a second step, s/he has to decide on an *amendment* $\theta' \in \Delta^n$ that will be announced. The amendment θ' immediately replaces the proposal on the floor and the game moves to a new bargaining round $t + 1$. The new round starts at Step 1 with the amended proposal θ' as proposal on the floor. Players can keep making amendments every round, and thus repeating Step 2, indefinitely.

Step 3: Voting on an endorsed proposal

Now suppose that in some bargaining round t, a proposal on the floor $\overline{\theta}$ is endorsed by player j. Then, all players simultaneously cast votes in favor of or against the endorsed proposal $\overline{\theta}$. Again, there are two cases:

- *Majority approval:* If at least $(n + 1)/2$ players accept the endorsed proposal $\overline{\theta}$, then the game ends and $\overline{\theta}$ is implemented.

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[4] Here, we renounce indexing $\overline{\theta}$ by i to ease presentation.

[5] The assumption is that the original proposer cannot be chosen to endorse or amend his/her own proposal. The underlying idea is that a non-trivial endorsement is required for one's proposal before it can be voted on.

- *No majority approval:* If strictly less than $(n+1)/2$ players accept $\overline{\theta}$, then the game moves to round $t+1$. That bargaining round begins again in Step 1, without a proposal on the floor.

Note that a new bargaining round begins whenever either *(i)* an amendment is made (in Step 2), or *(ii)* an endorsed proposal is not approved by the majority (in Step 3). Every time a new bargaining round starts, a discount factor $\delta \in (0,1)$ is applied. This discount factor can be suitably interpreted as a measure for the players' impatience.

Our modeling of open rule legislative decision-making captures the fact that the act of voting is short and that proposal-making and amendments take time, as documents need to be produced that are legally sound and not in outright contrast to other laws and rules. An example is the formation of the government in Germany. Coalition agreements typically involve determining the federal budget to some extent. There is no time limit for the formation of the government, and the process has historically taken a month up to half a year since the German reunification in 1990.[6] In this context, the cost can be seen as the inability to take action until an agreement is reached, the risk of being excluded from the government, or the possibility of reelections.

Yet, we also capture instances when amendment processes can be shorter, which can happen in the US Senate,[7] as we allow δ to be converging towards 1. Importantly, even if δ is close to one, it turns out that the open rule differs significantly from the closed rule, since a proposer may only want to give a small minority a sufficiently large share to motivate it to endorse the proposal under an open rule to maximize his/her share. This is demonstrated in Example 2 in Section 4.

We assume that players are risk-neutral. Thus, if a proposal $\theta \in \Delta^n$ is implemented in bargaining round t, then player i receives a payoff of $\delta^t \theta_i$. If no proposal is ever endorsed, or if no endorsed proposal is ever approved by the majority, then bargaining is trapped in *perpetual disagreement,* which gives all players zero payoff. This completes the formal description of the simple open rule legislative bargaining game (henceforth SORBG) $\hat{G}(\delta, n)$.

Throughout most of this paper, we are going to analyze SORBG. We will prove that the main results and conclusions derived for the simplified version also hold in the extended

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[6] See `https://www.bundestag.de/datenhandbuch` Chapter 6.7, accessed March 21, 2025

[7] See, `https://crsreports.congress.gov/product/pdf/RL/98-853`, accessed on May 6, 2024.

versions, which corresponds to the open rule bargaining game originally proposed by Baron and Ferejohn (1989). Every equilibrium found for the simplified version corresponds to a generalized equilibrium in the extended version of the model, i.e. in the original ORBG of Baron and Ferejohn (1989), where the reverse holds under certain conditions. The simplified ORBG differs from the original version as follows: Whenever a player makes an amendment to a proposal on the floor in Step 2, the amendment immediately replaces the proposal on the floor, without a vote being held. For completeness, we give the definition of the history.

Definition 1 (History)**.** *Let $t \in \mathbb{N}$. The history is the sequence that records all past and current states as well as strategies played in the order in which they have occurred. For example after Step* 3*, if the game has not ended yet, the history is given by*

$$h^t = (\theta^t, \sigma^{i_t}, i_t, \theta^{t-1}, \sigma^{i_{t-1}}, i_{t-1}, \dots, \theta^0),$$

where $\theta^t \in \Delta^n$ is the amendment made by player i_t in round t, playing the strategy σ^{i_t}. Furthermore, denote by

$$H_j^{i,\theta} := \left\{h^t = (j_t, \theta^t, \sigma^{i_t}, i_t, \dots) \mid j_t = j, \theta^t = \theta, \text{and } i_t = i\right\}$$

the set of histories where it is player j's turn and the proposal θ is on the floor made by player i. By omitting one or more indices, we denote larger subsets, e.g. H^θ is the set of histories at which the proposal θ is on the floor.

The time index t is suppressed because $H_j^{i,\theta}$ collects all histories, across all rounds, with the same current local state: player j moves, proposal θ is on the floor, and i is the player who made it.

We make two remarks for a better understanding of bargaining power in this game: First, consider a history of this game where players are in Step 3 and thus vote on an endorsed proposal. Their choice is either to stop bargaining and implement the proposal now, or to move back to Step 1 and start bargaining from scratch in the next round. This is similar to the choice that players make when responding to proposals in a closed rule bargaining game. In such games, the prospect of discounting discourages players from rejecting a proposal, which leads to a bargaining advantage for the proposer, often called the *proposer premium.* With an open rule, it seems intuitive that this proposer premium is shared between the player who has originally made the proposal, and the one who has endorsed it. This intuition is confirmed formally below. One crucial question in this paper will be how many players share the proposer premium, and how it is divided.

Second, consider a history of the game where players vote between a proposal on the floor and an amendment. They decide whether the current proposal on the floor remains the proposal on the floor in the next round, or whether the amendment becomes the new proposal on the floor. Regardless of the outcome of such a vote, another round of bargaining is required to reach an agreement, and thus another round of discounting will occur either way. Loosely speaking, while players are under "time pressure" when they vote on an endorsed proposal, this time pressure does not affect them when choosing between a proposal on the floor and an amendment.

2.2 Stationary strategies

It is well-known that non-cooperative bargaining games with more than two players admit a wide multiplicity of subgame-perfect equilibrium allocations. The bargaining literature has focused on analyzing subgame-perfect equilibria in stationary strategies (SSPE), e.g. in the tradition of Rubinstein (1982). In this paper, we consider an open rule and employ the same spirit to define stationary strategies, namely that with stationary strategies, the same actions are chosen in all structurally equivalent subgames. We also focus from the outset on strategies that depend only on payoff-relevant information and thus, our equilibria are Markov-perfect equilibria.

In addition, we assume that a proposer does treat other members equally if the allocation of payoffs does not matter for him/her. This means that s/he has no preference regarding which particular members obtain higher shares than others.

That is, a proposer can choose a configuration of payoffs that s/he wants to offer to the other players, while leaving it to chance which payoff is offered to which player. This equal treatment is achieved by an anonymity requirement. Hence, we focus on anonymous stationary strategies and simply call them *stationary strategies.*

In order to make this latter idea of anonymity more precise, let us define an *anonymous proposal* as a vector $\eta \in \Delta^n$ such that $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n$. Furthermore, let $\mathcal{M}(i)$ be the collection of $(n \times n)$-permutation matrices M such that $m_{1,i} = 1$. For any anonymous proposal $\eta \in \Delta^n$, let $\Theta^i(\eta) := \{\theta \in \Delta^n | \ \theta = M^\top \eta \text{ for some } M \in \mathcal{M}(i)\}$. The ordering is chosen so that, after the permutation placing η_1 on player i, the proposer receives the highest share while the remaining components are ranked from high to low. Each of the proposals in $\Theta^i(\eta)$ assigns η_1 to player i, and the payoffs $\eta_2, \ldots, \eta_n$ to the remaining players. Of course, the assignment of the payoffs to the individual players $N \setminus \{i\}$ differs across the different elements of $\Theta^i(\eta)$.[8] Note that η is an ordered payoff vector, not a player-indexed allocation. Player labels are assigned only after applying the permutation in $\Theta_i(\eta)$.

Formally, in the open rule legislative bargaining game $G(\delta, n)$, a *stationary strategy* for player i consists of the following three elements:

1. An *anonymous proposal* $\eta^i \in \Delta^n$, such that at every history at which there is no proposal on the floor and player i is the proposer, s/he randomizes uniformly among all the elements of $\Theta^i(\eta^i)$.

2. Let $\mathcal{Q}(\Delta^n)$ be the space of probability measures on Δ^n with the Borel σ-algebra. An *amendment rule* is a map $\psi^i : \Delta^n \to \mathcal{Q}(\Delta^n)$ which prescribes how player i should behave when s/he is the proposer and proposal θ is on the floor. If $\psi^i(\theta) = \delta_\theta$,[9] player i endorses the proposal θ. If $\psi^i(\theta) = \delta_{\theta'}$ for some $\theta' \neq \theta$, player i makes an amendment θ' when θ is on the floor. If $\psi^i(\theta)$ does not give mass 1 to a single proposal and $\psi^i(\theta)[\{\theta\}] > 0$,[10] then player i endorses the proposal θ with probability $\psi^i(\theta)[\{\theta\}]$, and chooses every other element as an amendment with probability given by $\psi^i(\theta)$. If $\psi^i(\theta)[\{\theta\}] = 0$, then player i amends the proposal θ.

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[8] The requirement that player i randomizes uniformly among all the elements of $\Theta^i(\eta^i)$ adds an anonymity requirement to the stationarity requirement. Strategies in which proposals are made in a "stationary but not anonymous" way play no role in our paper. They would complicate the analysis without offering new insights. Therefore, it seems convenient to include the anonymity requirement in the definition of stationarity.

[9] For any $\theta \in \Delta^n$, let δ_θ denote the Dirac measure at θ, defined by

$$\delta_\theta(A) = \begin{cases} 1 & \text{if } \theta \in A, \\ 0 & \text{if } \theta \notin A, \end{cases}$$

for every measurable set $A \subseteq \Delta^n$.

[10] Recall that $\psi^i(\theta)$ is a probability measure on Δ^n, and hence for $A \subseteq \Delta^n$ measurable $\psi^i(\theta)[A]$ denotes the probability that player i endorses/makes a new proposal that lies in A.

3. An *acceptance rule* $A^i \subseteq \Delta^n$ describes player i's voting decisions at histories where s/he votes on an endorsed proposal. More precisely, player i votes in favor of an endorsed proposal θ if and only if $\theta \in A^i$. Of course, the set A^i is specified independently of the history of play and depends only on the i-th component θ_i, the proposed payoff for the player i. A player accepts his/her own proposals, hence, $\Theta^i(\eta^i) \subseteq A^i$.

We use $\sigma^i = (\eta^i, \psi^i, A^i)$ to describe the *stationary strategy* for player i and we write $\sigma = (\sigma^1, \ldots, \sigma^n)$ for a profile of stationary strategies. We note that our definition of stationary strategies implies the following: When there is no proposal on the floor, a proposer can only make an anonymous proposal. Amendments, of course, are not anonymous. We will focus on amendments for which all proposers assign themselves the same amount depending only on the public information, δ and n, while distributing the $n-1$ remaining components arbitrarily. In particular, the amendment rule ψ^i is independent of the previous proposal θ if the proposer does not endorse it. Given this, the equilibrium concept is perfectly standard: Indeed, a *stationary subgame-perfect equilibrium (SSPE)* is a profile of stationary strategies that is a subgame-perfect equilibrium. In the SORBG, a stationary strategy consists of an anonymous proposal, an amendment rule, and an acceptance rule.[11]

2.3 Relation to Baron and Ferejohn

Although the present paper deals with the open rule bargaining model proposed by Baron and Ferejohn (1989), our analysis and results differ from theirs in two respects. In this section, we discuss the reasons for these differences:

1. Baron and Ferejohn impose that a player votes in favor of the amendment if s/he is "indifferent" between the proposal on the floor and the amendment. Unfortunately, it is not straightforward what it means to be indifferent between the proposal on the floor and the amendment: Player i's preferences over the proposal on the floor, say $\overline{\theta}$, and the amendment, say θ', do not only depend on the components $\overline{\theta}_i$ and θ'_i,

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[11] Mutatis mutandis, the definition of an SSPE in the SORBG corresponds to that in the extended versions (ORBG), to which we will return in Section 5.

but also on the probabilities with which either $\overline{\theta}$ or θ' will be endorsed or amended in the future. For instance, even if $\bar{\theta}_i < \theta'_i$, player i may want to vote in favor of $\bar{\theta}$ because s/he believes that $\bar{\theta}$ will be endorsed with a higher probability than θ'. Along a path of play of Baron and Ferejohn's supposed SSPE, a proposal on the floor and an amendment always have the same probability of being endorsed. However, this may not be true anymore off the equilibrium path. In this paper, we work around this problem in two different ways: First, we analyze the SORBG in which the problem is redundant. Second, in Section 5, we return to the original model and show that players' best-responses to deviations from SSPE must have a certain recursive structure. Therefore, we can do an equilibrium analysis without explicitly determining the optimal voting behavior for each player, for each proposal on the floor, and for each amendment. This analysis confirms that the results and conclusions obtained in the SORBG carry over to the ORBG itself.

2. Baron and Ferejohn tacitly assume that a player who is willing to vote for a given proposal is also willing to endorse it. However, we will demonstrate that this need not always be true: We will see that there may be players who would want to amend a proposal if they had a chance, but who would nevertheless want to vote in favor of that same proposal once it had been endorsed. As a result, we find longer equilibrium delays and less egalitarian allocations than Baron and Ferejohn.

3 Analysis

3.1 Equilibrium candidates for the SORBG

In this section, we entirely focus on the SORBG, and discuss a particular family of stationary strategy profiles that we call *k-supporter strategy with randomization* (henceforth called *k-SSR*). These equilibrium candidates are particularly appealing by their tractability, as this produces a system of equations for the equilibrium values with the least number of equations, as we will illustrate in Section 5 (first paragraph).

More formally, the stationary strategy profiles we consider have the following properties:

- On the path of play induced by a k-SSR, whenever a player amends a proposal on the floor, s/he does so by simply assigning to his/her component the largest component of the proposal on that floor and by permuting the remaining $n-1$ components arbitrarily.

- Every proposal and every amendment made on a path of play of a k-SSR has the following structure: The proposer offers k players a payoff that makes them willing to endorse the proposal, and to vote in its favor. The proposer does not belong to the chosen k players and the proposal is accepted once it is endorsed (because $k+m+1 \geq (n+1)/2$). If $k \leq \frac{n-1}{2}$, the proposer offers an additional $\frac{n-1}{2}-k$ players a payoff that makes them willing to vote in favor of the proposal once it has been endorsed, but not to endorse it themselves.

In the open rule legislative bargaining game, one obtains a multitude of equilibria by changing the way in which amendments reshuffle the surplus allocation relative to proposals, see Primo (2003). To the best of our knowledge, we are the first to use simple swaps and randomization to reshuffle allocations to characterize a suitable class of subgame perfect equilibria in stationary strategies. A player who makes an amendment is not interested in the other players' payoff, so that it seems natural to assume that s/he would choose a simple swap and then arbitrarily randomizes the remaining payoff offerings. This is analogous to the kind of anonymous subgame perfect equilibria in stationary strategies in a symmetric closed rule bargaining when the proposers keep a share for themselves and buy off a random minimal winning coalition. It will also turn out that simple swaps and randomization yield a simple and tractable system of equations for the equilibria.

Take the number of players n and the discount factor $\delta \in (0,1)$ as given. For any $k = 1, \ldots, n-1$, let the variables X_k, Y_k, and Z_k represent the amount allocated to the proposer, to the k players who will endorse the proposal immediately, and to the pro-voters, respectively. Furthermore, let V_k and W_k be the expected payoffs for the proposer and a non-proposer, respectively, and call the total expected payoff E_k. For determining the payoffs, we do not have to distinguish between the different groups that do not propose, because the randomization step in the proposal selection by the proposer equalizes their expected payoffs. Expectations are taken with respect to the probabilities of the random choices taken in the game.

Then the following are part of the equilibrium equations, setting $m := \max\{0, \frac{n-1}{2} - k\}$:

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\begin{aligned}
V_k &= \frac{k}{n-1} X_k + \frac{n-k-1}{n-1} \delta W_k && (1) \\
W_k &= \frac{k}{n-1} \left(\frac{k}{n-1} Y_k + \frac{n-k-1}{n-1} \delta W_k \right) && (2) \\
&\quad + \frac{m}{n-1} \left(\frac{k}{n-1} Z_k + \frac{1}{n-1} \delta V_k + \frac{n-k-2}{n-1} \delta W_k \right) \\
&\quad + \frac{n-k-m-1}{n-1} \left(\frac{k}{n-1} \cdot 0 + \frac{1}{n-1} \delta V_k + \frac{n-k-2}{n-1} \delta W_k \right) \\
E_k &= V_k + (n-1) W_k && (3) \\
1 &= X_k + k Y_k + m Z_k. && (4)
\end{aligned}
$$

Here, the first term in V_k is the payoff the proposer P obtains in case an endorsing player is selected for endorsement. In the other case, the proposal is not endorsed and a randomly selected other player J will make a new proposal, whence P will become a non-proposer. The expression of W_k consists of three main terms. The first summand of (2) describes the payoff if the player P considered is selected to be one of those k players who would immediately endorse. This happens with probability $k/(n-1)$. Then there is a probability of $k/(n-1)$ that J, one of those players who will immediately endorse, is asked for his/her endorsement. This will end the game and provide P with payoff Y_k. Otherwise, it will not be endorsed, a new proposal will be selected, and P will not be the proposer for another round because J will be a proposer and J does not belong to the chosen k players. Here, P belongs to the remaining $n-k-m-1$ players, i.e. to those players who are neither among the k immediate endorsers nor among the $m = \max\{0, (n-1)/2 - k\}$ additional pro-voters, and therefore receive payoff 0 if the current proposal is endorsed. The second term of (2) describes the situation when P is one of the pro-voters, who will not immediately endorse. This will happen with probability $m/(n-1)$. Then three different things can happen when player J is asked whether s/he will endorse the proposal on the table. With probability $k/(n-1)$, player J is one of those who endorse immediately, and so P will receive payoff Z_k. $J = P$ will happen with probability $1/(n-1)$ and P will become the new proposer in the next round because P will not endorse. Finally, with probability $(n-k-2)/(n-1)$ player J will be another player who does not endorse, and P will not be the proposer in the next round. The final term of Equation (2) describes the situation when P does not receive any payoff in the current proposal. Again, the next player J is chosen. With probability $k/(n-1)$, immediate endorsement happens, and P receives no payoff when the game ends. Again with probability $1/(n-1)$, $J = P$, and in the remaining case, P will remain in the group of non-proposers.

If a solution of this system of equations is an equilibrium, an endorser must be indifferent between endorsing and amending the proposal. If s/he amends, the expected payoff is δV_k. Hence, we obtain

$$Y_k = \delta V_k. \tag{5}$$

At the time of the vote, $k+1$ votes will be secured. So in the case $k < \frac{n-1}{2}$, $\frac{n-1}{2} - k$ additional votes are required. Therefore, an additional $\frac{n-1}{2} - k$ players will receive Z_k if they vote for the proposal. If the proposal is rejected, the expected payoff for every player will be $\frac{1}{n}\delta V_k + \frac{n-1}{n}\delta W_k = \frac{\delta}{n}E_k$, since a new proposer is randomly chosen from N. Hence,

$$Z_k = \frac{\delta}{n}E_k. \tag{6}$$

We can then derive the following simplified system of equations:

Lemma 1. *Combining the linear equations* (1)-(6) *yields*

$$\begin{aligned}
E_k &= \frac{k}{k\delta + (n-1)(1-\delta)} \\
V_k &= \frac{k(n-1) + \left(n-k-1-\frac{km(n-1)}{n}\right)\delta E_k}{(n-1)^2 + (n-k-1+k^2(n-1))\delta} \\
X_k &= 1 - k\delta V_k - \frac{m\delta}{n}E_k \\
W_k &= \frac{E_k - V_k}{n-1}.
\end{aligned} \tag{7}$$

The proof is given in Appendix B.

For the analysis in the remainder of the paper, an important auxiliary result is that the variables (V_k, W_k, X_k, E_k) as defined in Equation (7) are strictly positive. The formal claim is stated in Lemma 2 below. The proof is provided in Appendix B.

Lemma 2. *For every $k = 1, \dots, n-1$, the system of Equations (7) has a unique solution. Furthermore:*

(i) If $k = n-1$ and $\delta = 1$, then $V_k = W_k > 0$.

(ii) For any other choices of $k = 1, \dots, n-1$ and $\delta \in (0, 1]$, it holds that $V_k > W_k > 0$.

(iii) For any $k = 1, \dots, n-1$, all components of solutions (V_k, W_k, X_k, E_k) to the system of equations (7) are strictly positive.

Moreover:

(a) If $k = n-1$, then $X_k = V_k$.

(b) For any other choices of $k = 1, \dots, n-2$ and $\delta \leq 1$, it holds that $X_k > \delta V_k$.

Given the uniqueness by Lemma 2 as well as the fact that $X_k \geq Y_k$, we can now state a more formal definition of the k-SSR. For this, we introduce the proposal

$$\eta_k = (X_k, \underbrace{Y_k, \dots, Y_k}_{k \text{ timess}}, \underbrace{Z_k, \dots, Z_k}_{m \text{ times}}, 0, \dots, 0) \in \Delta^n \tag{8}$$

with X_k, Y_k and Z_k the solutions of the Equations (5), (6) and (7) and $m = \max\{0, \frac{n-1}{2} - k\}$. Note that this η_k is public information and therefore known by all players. Let $H^{\emptyset}$ be the set of histories at which there is no proposal on the floor.

Definition 2. *Consider the SORBG $\widehat{G}(\delta, n)$. Let (V_k, W_k, X_k, E_k) be defined as solutions to Equations (7). For every $k = 1, \dots, n-1$, a profile of stationary strategies is a k-SSR if the following holds:*

1. *At every history $h \in H^{\emptyset}$, the proposer makes an anonymous proposal η which gives him/herself X_k, gives δV_k to k other players, gives $\frac{\delta}{n}((n-1)W_k + V_k)$ to $\max\{0, \frac{n-1}{2} - k\}$ more players, and zero to all remaining players.*

2. *Consider a history $h \in H_j^{i,\theta}$. At such a history, player $j \in N$ is the responding player, and the proposal on the floor θ was made by player $i \in N$. Suppose that $\theta \in \Theta^i(\eta_k)$. Player j endorses the proposal on the floor θ if and only if $\theta_j \geq \delta V_k$.*

Otherwise, s/he randomly chooses an amendment from $\Theta^j(\eta_k)$ *with equal probability. Now suppose that* $\theta \notin \Theta^i(\eta_k)$*. Player* j *endorses* θ *if and only if there is a subset* L *of players,* $L \subseteq N \setminus \{i\}$ *with* $|L| \geq (n-1)/2$ *players and* $\theta_l \geq \frac{\delta}{n}((n-1)W_k + V_k)$ $\forall l \in L$ *and, moreover, it holds that* $\theta_j \geq \delta V_k$*. Otherwise, player* j *randomly chooses an amendment from* $\Theta^j(\eta_k)$ *with equal probability.*

3. *Whenever player* i *votes on an endorsed proposal* θ*, s/he votes in favor if and only if* $\theta_i \geq \frac{\delta}{n}((n-1)W_k + V_k)$*.*

Definition 2 is related to the informal description given at the beginning of the section in the following way: Point 1 describes the *anonymous proposal,* Point 2 specifies the *amendment rule,* and the *acceptance rule* is spelled out in Point 3. Recall that, since we are considering the SORBG, it is redundant to specify a voting decision between a proposal on the table and the amendment. Since $V_k \geq W_k$, according to Lemma 2, we have $\frac{\delta}{n}E_k \leq \delta V_k$, and thus a player who is willing to endorse a proposal will also vote in its favor.

By restricting attention to the SORBG and to k-SSRs, the analysis of stationary strategy profiles becomes more tractable than in any previous work on open rule bargaining that we know of. There are two reasons for this:

- In a SORBG, there is a strategic equivalence between subgames that start at a node where a proposal on the floor can be amended or endorsed and subgames that start at a node where no proposal is on the floor.
- If a k-SSR is played, the actions taken after a history $H_j^{i,\theta}$ do not depend on whether the proposal θ was originally made as an amendment to some other proposal, or whether it was made at a history without a proposal on the floor.

Intuitively, in a k-SSR played in a SORBG, a player who can make an amendment to a proposal on the floor can achieve the same payoff (up to discounting) that s/he could also achieve if s/he were the proposer at a history without a proposal on the floor.

3.2 Testing equilibrium candidates

In this section, we introduce a test to verify whether a k-SSR is an SSPE of a SORBG.

Lemma 3. *A k-SSR is an SSPE of the SORBG if and only if there is no profitable unilateral deviation from it at any history $h \in H^{\emptyset}$, where $H^{\emptyset}$ is the set of histories at which there is no proposal on the floor.*

Proof. It is easily verified that a profitable unilateral deviation from a k-SSR is impossible at histories where players vote on an endorsed proposal. Thus, we have to focus on the possibility of profitable unilateral deviations from k-SSR at histories where a proposal can be made. Recall that a proposal can be made at histories in $H^{\emptyset}$ or at histories in H^{θ} through an amendment. Consider a history $h \in H^{\theta}$ at which player i chooses to endorse or amend the proposal on the floor θ. Suppose that player i obtains an expected payoff of $\delta\widetilde{V}$ if s/he makes the amendment $\widetilde{\theta}$. Now consider a history in $H^{\emptyset}$ where player i is the proposer. At that history, player i can obtain a payoff of $\widetilde{V}$ by proposing $\widetilde{\theta}$. When player i proposes at a history in $H^{\emptyset}$ using a k-supporter strategy, his/her expected payoff is V_k, and when s/he proposes at a history in H^{θ}, his/her expected payoff is δV_k. Thus, if player i has a profitable deviation at a history in H^{θ}, then s/he also has a profitable deviation at a history in $H^{\emptyset}$. $\square$

Lemma 3 shows that, in order to test whether a k-SSR is an SSPE in the SORBG, we only have to consider profitable unilateral deviations at histories in $H^{\emptyset}$. In the SORBG, all histories at which a particular player can make a proposal or an amendment are "equivalent" in the sense that the continuation game is the same. This is a manifestation of the one-shot deviation principle, applied in the subsequent propositions.

Next, we consider deviations from the k-SSR. Suppose that player i makes a unilateral one-shot deviation from the k-SSR by proposing the amount δV_k to $k+1$ instead of k players, proposing $\left(\frac{\delta}{n}\right) E_k$ to $\max\left\{0, \frac{n-1}{2} - (k+1)\right\}$ players, and proposing to take the remainder for him/herself. We denote the proposer's expected gain from such a deviation by λ_k^+. Similarly, let λ_k^- denote the proposer's expected gain from a unilateral one-shot deviation under which the proposer offers the amount δV_k only to $k-1$ instead of to k players, and offers $\left(\frac{\delta}{n}\right) E_k$ to $\max\left\{0, \frac{n-1}{2} - (k-1)\right\}$ players. In order to understand the expressions below, recall that Point 2 in Definition 2 says that if after the deviation, the proposal is amended, then that amendment is again based on the anonymous proposal associated with the k-SSR.

$$\lambda_k^+ = \begin{cases} 0 & \text{if } k = n-1, \\ -\left(\frac{k+1}{n-1}\right)\delta V_k + \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{(n-1)/2, \ldots, n-2\}, \\ -\left(\frac{k+1}{n-1}\right)\left(\delta V_k - \frac{\delta}{n}E_k\right) + \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{1, \ldots, (n-3)/2\}. \end{cases} \tag{9}$$

$$\lambda_k^- = \begin{cases} \left(\frac{k-1}{n-1}\right)\delta V_k - \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{(n+1)/2, \ldots, n-1\}, \\ \left(\frac{k-1}{n-1}\right)\left(\delta V_k - \frac{\delta}{n}E_k\right) - \left(\frac{1}{n-1}\right)(X_k - \delta W_k) & \text{if } k \in \{2, \ldots, (n-1)/2\}, \\ 0 & \text{if } k = 1. \end{cases} \tag{10}$$

It is straightforward that the k-SSR can only be an SSPE if λ_k^+ and λ_k^- are non-positive. The next proposition implies the converse: If the proposer has any profitable deviation, then either λ_k^+ or λ_k^- must be strictly positive. In particular, if the proposer cannot gain by offering δV_k to one additional player, or to one player less, then s/he cannot gain either by offering δV_k to any number of players other than k.

Lemma 4. *(i) If there exists $\widehat{\theta} \in \Delta^n$ such that proposing $\widehat{\theta}$ instead of the proposal prescribed by the k-SSR is a profitable deviation for the proposer, then either $\lambda_k^+ > 0$ or $\lambda_k^- > 0$.*

(ii) A k-SSR is an SSPE of the SORBG if and only if $\lambda_k^+ \leq 0$ and $\lambda_k^- \leq 0$.

The proof of point (i) is relegated to Appendix B. Point (ii) follows from point (i) and Lemma 3. Baron and Ferejohn (1989) have already argued that a proposer should choose a large k when δ is small. The intuition is as follows: For small δ, any delay is very costly. Hence, it seems intuitive that in equilibrium, the proposer selects proposals that lead to immediate endorsement and acceptance of his/her proposal. In order to ensure that his/her proposal is endorsed immediately with probability one, s/he needs to make all other players willing to endorse it. The corollary below formalizes this argument.

Corollary 1. *If $\delta \leq \sqrt{\left(\frac{n-2}{2}\right)^2 + 1} - \left(\frac{n-2}{2}\right)$, then the $(n-1)$-SSR is an SSPE.*

In an $(n-1)$-SSR, immediate agreement is reached on an allocation which gives $\frac{1}{1+\delta(n-1)}$ to the proposer and $\frac{\delta}{1+\delta(n-1)}$ to each of the other players. This corresponds exactly to the payoff division that one would expect under closed rule unanimity bargaining.[12]

It is important to emphasize that our analysis so far does not yield results on the "existence" or "uniqueness" of k-SSR's that are SSPE in the SORBG. Without any restrictions on the parameters δ and n, we do not claim that there must be a k, such that the k-SSR is an SSPE. We do not show either that there is at most one k, such that the k-SSR is an SSPE. In Section 4, however, we do consider some numerical examples. In each of the examples, it does turn out that exactly one k-SSR is an SSPE.

In the next section, we consider k-SSRs that are SSPE in the limit as $\delta \to 1$ and $n \geq 9$. In that case, we do obtain results which show that there is a unique k such that the k-SSR is an SSPE.

3.3 Stationary equilibrium with patient players

So far, we have defined a family of equilibrium candidates in the SORBG, and we have introduced a test to verify which of these candidates are indeed SSPE of the simplified ORBG. In the present section, we will focus on the case where the discount factor is sufficiently close to one. In that case, we will explicitly compute the limit of SSPE payoffs.

As a first step, we show that for sufficiently large δ (i.e., $\delta > \overline{\delta}$ for some $\overline{\delta}$ close to but below 1) and n, a k-SSR can only be an SSPE if $k \leq (n-3)/2$. Consequently, for sufficiently large δ and n, a k-SSR can only be an SSPE if there are players who are willing to vote in favor of proposals that they would not be willing to endorse.

We say that a k-SSR involves *majority endorsement* if $k \geq (n-1)/2$, and it involves *super-majority endorsement* if $k \geq (n+1)/2$. Intuitively, (super-)majority endorsement means that the proposer and the players who are willing to endorse his/her proposal form a (super-)majority.

Proposition 1. *An SSPE can be characterized as follows:*

1. *If a k-SSR with super-majority endorsement is an SSPE of the SORBG, it holds that $\delta(n+1) \leq 4$.*

---

[12] A closed rule bargaining game with linear utility functions and equal recognition probabilities is a special case of the games studied in Laruelle and Valenciano (2008) and Britz et al. (2014).

2. *For any $\delta \in (0,1)$, there exists an odd integer n_δ sufficiently large, so that a k-SSR with majority endorsement cannot be an SSPE of the SORBG if $n \geq n_\delta$.*

One implication is that a k-SSR with super-majority endorsement cannot be an SSPE if δ is sufficiently close to one.[13] Another implication is that, for any given $\delta > 0$, a k-SSR with super-majority endorsement cannot be an SSPE if the number of players satisfies $n > \frac{4}{\delta} - 1$, and thus if the number of players is too large.

The proof of Proposition 1 can be found in Appendix B. Intuitively, the argument runs as follows: Consider an $((n-1)/2)$-SSR. Suppose that a proposer makes a unilateral deviation under which s/he offers one player only $\frac{\delta}{n} E_k$ instead of δV_k. This player would no longer be willing to endorse the proposal. However, s/he would still be willing to vote in favor of the proposal once it has been endorsed. In the formal proof of Proposition 1, we derive a parameter condition under which this deviation is profitable for the proposer, and we show that this condition boils down to an upper bound on n. In particular, the result below follows from the proof of Proposition 1.

Proposition 2. *1. Suppose that $n \leq 7$. If δ is sufficiently close to one, then the $((n-1)/2)$-SSR is an SSPE of the SORBG.*

2. *Suppose that $n \geq 9$. If δ is sufficiently close to one, a k-SSR with majority endorsement cannot be an SSPE of the SORBG.*

The proof of Proposition 2 can be found in Appendix B.

The two propositions above reflect the gist of how our results differ from those in Baron and Ferejohn (1989): Their findings suggest that, for δ sufficiently high, equilibrium proposals are endorsed by $(n-1)/2$ players. Hence, the probability of an endorsement is one half for each proposal on the equilibrium path. Since an endorsed proposal is implemented in equilibrium, this further implies that the game ends in bargaining round t with probability $\frac{1}{2^{t+1}}$, which corresponds to an expected equilibrium delay of length one. Baron and Ferejohn's supposed equilibrium is based on strategies in which amendments are made in a more complicated way than with the simple swaps and randomization used

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[13] This follows from Proposition 1 for $n \geq 5$. For the special case with $n = 3$, it can be verified by direct computation: Plugging in $n = 3$ and $k = 2$ as well as $\delta = 1$ into Equations (7) yields $X_2 = V_2 = W_2 = 1/3$. Plugging into Equations (9)-(10), we see that $\lambda_2^+ = 0$ and $\lambda_2^- = 1/6 > 0$. Indeed, the 2-SSR is not an SSPE.

here. We conclude that k-SSRs that involve majority endorsement are not SSPE when $n \geq 9$ and δ is close to one. In Section 4, we provide an example with $n = 51$ and δ close to one in which only 7 (rather than 25) of the 50 responding players endorse the proposal in equilibrium. In that example, the probability that any particular proposal is endorsed on the equilibrium path is only $7/50 = 0.14$ (instead of $1/2$). As a result, the expected length of equilibrium delay is more than six times as long as it would be with $k = 25$.[14]

So far, we have shown that for $n \geq 9$ and sufficiently large δ, a k-SSR can only be an SSPE if $k \leq (n-3)/2$. Since the expressions in Equations (7)–(10) are continuous in δ, computing the limit behavior of the variables $V_k, W_k, X_k, E_k, \lambda_k^+$, and λ_k^- when δ converges to one is equivalent to computing them while setting δ equal to one.

Indeed, let us restate Equations (7)–(10) for $\delta = 1$ and $k \leq (n-3)/2$:

$$\begin{aligned}
\overline{E}_k &= 1 && (11)\\
\overline{V}_k &= \frac{(k+1)(n-1) - \frac{k}{2n}\big(n(n-2k)+1+2k\big)}{(n+k^2)(n-1)-k}, && (12)\\
\overline{W}_k &= \frac{1-\overline{V}_k}{n-1}, && (13)\\
\overline{X}_k &= 1 - k\overline{V}_k - \left(\frac{n-1-2k}{2n}\right), && (14)\\
\overline{\lambda}_k^+ &= -\left(\frac{k+1}{n-1}\right)\left(\overline{V}_k - \frac{1}{n}\right) + \left(\frac{1}{n-1}\right)(\overline{X}_k - \overline{W}_k), && (15)\\
\overline{\lambda}_k^- &= \left(\frac{k-1}{n-1}\right)\left(\overline{V}_k - \frac{1}{n}\right) - \left(\frac{1}{n-1}\right)(\overline{X}_k - \overline{W}_k). && (16)
\end{aligned}$$

Equations (12)–(14) are a system of three independent linear equations in three unknowns. We can solve this system for the variables $\overline{V}_k$, $\overline{W}_k$, and $\overline{X}_k$, and substitute the

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[14] On the path of play of a k-SSR, the probability that the proposal on the floor is endorsed (and then approved by majority voting) is $\frac{k}{n-1}$ in every round. Thus, the expected length of equilibrium delay can be written as $\frac{k}{n-1}\sum_{t=0}^{\infty}(1-\frac{k}{n-1})^t\ t = \frac{n-1-k}{k}$. For any n, if $k = \frac{n-1}{2}$, the expected length of delay is always one. In our example with $n = 51$ and $k = 7$, however, it is $\frac{51-1-7}{7} = \frac{43}{7} \approx 6.14$.

resulting expressions into Equations (15)–(16) to obtain:

$$\overline{\lambda}_k^+ = (n-k-k^2)\left(\frac{n-1}{2n}\right)\left(\frac{1}{(n+k^2)(n-1)-k}\right), \tag{17}$$

$$\overline{\lambda}_k^- = (k^2-k-n)\left(\frac{n-1}{2n}\right)\left(\frac{1}{(n+k^2)(n-1)-k}\right). \tag{18}$$

Recalling that $n \geq 3$ and $1 \leq k \leq n-1$, it is easily verified that

$$\left(\frac{n-1}{2n}\right)\left(\frac{1}{(n+k^2)(n-1)-k}\right) > 0.$$

Therefore, $\overline{\lambda}_k^+ > 0$ if and only if $n-k-k^2 > 0$, and $\overline{\lambda}_k^- > 0$ if and only if $k^2-k-n > 0$. Combined with Lemma 4, this implies Theorem 1.

Theorem 1. *Suppose that $n \geq 9$ and δ is sufficiently close to one. A k-SSR is an SSPE of the SORBG if and only if the inequalities $k \leq (n-3)/2$ and $k^2-k \leq n \leq k^2+k$ are satisfied.*[15]

It follows that, for δ sufficiently close to one, the k-SSR is an SSPE if $k \leq (n-3)/2$ and

$$k \in \left[-\frac{1}{2}+\sqrt{n+\frac{1}{4}},\ \frac{1}{2}+\sqrt{n+\frac{1}{4}}\right].$$

Corollary 2. *Suppose that $n \geq 9$ and δ is sufficiently close to one. There exists a unique $k = 1, \ldots, (n-3)/2$ such that the k-SSR is an SSPE of the SORBG. This k is the unique integer contained in the interval $\left[-\frac{1}{2}+\sqrt{n+\frac{1}{4}},\ \frac{1}{2}+\sqrt{n+\frac{1}{4}}\right]$.*

The proof of Corollary 2 can be found in Appendix B. This corollary has important implications for the expected length of the equilibrium delay given by

$$\mathbb{E}(delay) = \frac{n-1-k}{k},$$

with n the total number of players and k the number of endorsers.[14] For δ close to one

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[15] Recall that n is odd, and so the inequalities will always be strict.

and $n \geq 9$, by Corollary 2 it holds that $k \in \Theta(\sqrt{n})$.[16] Thus,

$$\mathbb{E}(delay) = \Omega\left(\sqrt{n}\right),$$

and in particular the delay can get arbitrarily large for n large enough, leading to a "never ending" game. This constitutes one of several possible inefficiencis that are further discussed in Section 4.2.

Moreover, Proposition 2 and Corollary 2 readily imply together the following existence result, which no longer has any restriction on the number of agents n.

Corollary 3. *Suppose that δ is sufficiently close to one. There exists $k \in \{1, \ldots, n-1\}$ such that the k-SSR is an SSPE of the SORBG.*

4 Numerical illustration

4.1 Equilibrium values of k

In this section, we give some numerical examples for our findings.[17]

For $n = 51$ and various values of δ, Table 1 shows the unique value of k such that the k-SSR is an SSPE of the SORBG.[18] Recall that the payoffs induced by a k-SSR are given as the solutions to a system of equations which is continuous at $\delta = 1$. Therefore, we can find the limit values as δ converges to one by considering the relevant equations for $\delta = 1$.

For values of δ close to zero, we see that $k = n - 1 = 50$. This exemplifies the finding in Proposition 1 that a proposal is endorsed by all players in an SSPE when discounting is sufficiently severe. As δ increases, the equilibrium value of k decreases, which is in line with Baron and Ferejohn's findings. However, Baron and Ferejohn predict that the equilibrium value of k falls only until it reaches $(n-1)/2 = 25$. Again, this is because they do not take into account that players who do not endorse a proposal may still vote

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[16] For two functions $f, g : \mathbb{N} \to \mathbb{R}$, the big omega notation $f \in \Omega(g)$ indicates that f is bounded asymptotically from below by g, meaning that $\liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0$. We say that $f \in \Theta(g)$ if f is of the same order as g, i.e. $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(f)$.

[17] The code used to simulate these examples is available from the authors upon request.

[18] We note that in all the numerical examples listed in the table, there is exactly one $k = 1, \ldots, n-1$, so that the k-SSR is an SSPE.

for it. In our model, however, the equilibrium value of k continues to fall. For δ close to one, it eventually reaches 7, which is indeed the integer close to $\sqrt{n} = \sqrt{51} \approx 7.1$ (see Proposition 3 and Theorem 2). The relationship found in Corollary 2 between n and the equilibrium value of k for δ close to one is displayed in Figure 1.

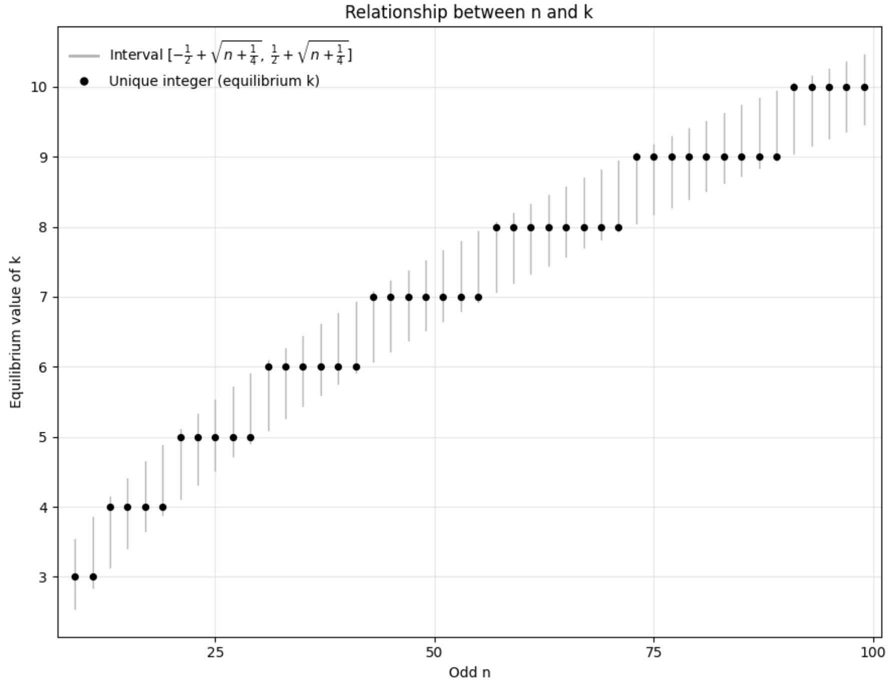

Figure 1: The intervals $\left[-\frac{1}{2} + \sqrt{n + \frac{1}{4}},\ \frac{1}{2} + \sqrt{n + \frac{1}{4}}\right]$ for each odd n on the x-axis are plotted in gray. The unique integer in each interval, which is the equilibrium value of k, is displayed as a black dot.

4.2 Efficiency and equity for varying discount factors

One purpose of the original analysis by Baron and Ferejohn was to compare closed rule and open rule bargaining procedures with regard to the efficiency and equity of equilibrium outcomes. While open rules tend to lead to a more egalitarian distribution of the surplus—in the sense that is illustrated by the following three examples—, they entail inefficiencies. The reason is that the option of making amendments tends to lead to equilibrium delays, while closed rule bargaining models always predict immediate agreement. One important

δ	Equilibrium value of k
0.02	50
0.03	41
0.08	25
0.09	24
0.5	10
1	7

Table 1: Equilibrium values of k for $n = 51$ and various values of δ.

question is how one could weigh the efficiency loss against the equity gain.

We consider the example with $n = 51$ for different values of the discount factor. The resulting equilibria, including the amendment structure, the expected payoffs, and delays, are summarized in Table 2.

δ	rule	k	X_k	Y_k	Z_k	V_k	W_k	E_k	delay
$\delta = 0.02$	closed	25	0.9902	0.0004	–	0.9902	0.0002	1	0
	open	50	0.5	0.01	0	0.5	0.01	1	0
$\delta = 0.5$	closed	25	0.7550	0.0098	–	0.7550	0.0049	1	0
	open	10	0.4707	0.0480	0.0032	0.0960	0.0047	1/3	4
$\delta \to 1$	closed	25	0.5098	0.0196	–	0.5098	0.0098	1	0
	open	7	0.2693	0.0540	0.0196	0.0540	0.0189	1	6.14

Table 2: Comparison of the closed and open rule bargaining for different discount factors δ. There are large gains in fairness and no loss in efficiency when δ is either close to zero or one, however, for intermediate values the opposite can occure.

The cases of $\delta = 0.02$ and $\delta \to 1$ suggest that there is a large gain in fairness and no loss in efficiency when δ either is close to zero or close to one. However, the case $\delta = 0.5$ shows that for intermediate values of δ, the efficiency loss from open rules can be so large that even the responding players are ex ante better off under a closed rule.

We take a closer look at the case $\delta = 0.5$. With closed rule bargaining, 25 players would obtain the reservation payoff $\delta/n = 0.5/51 \approx 0.0098$, the proposer would keep the remaining 0.755. Under open rule bargaining, we find that $X_{10} \approx 0.4707$ and ten players to receive 0.048. Another 15 players would receive $0.5/51 \approx 0.0098$ and the remaining 25 players would receive nothing. However, with open rule, the expected delay is 4 and the expected surplus is 1/3. So while the outcome under open rule is certainly more equitable than under closed rule, it is much less efficient. Recall that V_{10} and W_{10} are the ex ante expected payoffs of the proposer and any player other than the proposer, respectively. We see that $V_{10} \approx 0.096$ and $W_{10} = 0.0047$. In a closed rule bargaining game, the analogous ex ante payoffs would be 0.755 for the proposer (since agreement is immediate) and $0.5\frac{\delta}{n} = 0.0049$ for any other player. Thus ex ante, all players are better off with closed rule bargaining than with open rule bargaining for $\delta = 0.5$. The efficiency loss from delay is so great that even the gain in fairness cannot compensate the responders for it.

5 ORBG

5.1 The set-up and issues

In previous sections, we focused on the SORBG. In the present section, we return to the original ORBG. Recall the crucial difference between both games: In the original ORBG, whenever a player makes an amendment to a proposal on the floor, a vote determines whether or not the amendment replaces the proposal on the floor. We will show in this section that the main results and conclusions from our analysis of the SORBG carry over to the ORBG itself. The definition of the ORBG can be obtained by adapting Step 2 of the SORBG, where Steps 1 and 3 remain the same. Specifically, Step 2 now reads as follows:

Step 2: Amendment or endorsement

- *Endorsement:* If player j *endorses* the proposal on the floor $\overline{\theta}$, then round t of the game proceeds directly to Step 3.
- *Amendment:* Suppose that player j chooses to amend the proposal $\overline{\theta}$ on the floor. S/he does so by announcing an *amendment* $\theta' \in \Delta^n$. Then, all players simultaneously cast votes in favor of the proposal on the floor $\overline{\theta}$ or in favor of the amendment θ'. If at least $(n+1)/2$ players vote in favor of θ', then bargaining round $t+1$ begins with the amendment θ' as the new proposal on the floor. If at least $(n+1)/2$ players

vote in favor of $\overline{\theta}$, then bargaining round $t+1$ begins with the proposal $\overline{\theta}$ on the floor.

Players can keep making amendments, and thus repeating Step 2, indefinitely. However, a new bargaining round begins (and thus discounting takes place) every time a new amendment is made.

In contrast to the SORBG, when a player makes an amendment, the proposal on the floor remains payoff relevant since an additional voting round decides which of them will be considered in the next round. We first define a set of equilibrium candidates which are analogous to the k-SSR in the SORBG. The game is denoted by ORBG, $G(\delta, n)$. For this definition, we first recall the system of Equations (7)

$$\begin{aligned}
E_k &= \frac{k}{k\delta + (n-1)(1-\delta)} \\
V_k &= \frac{k(n-1) + \left(n-k-1-\frac{k(n-1)}{n}m\right)\delta E_k}{(n-1)^2 + (n-k-1+k^2(n-1))\delta} \\
X_k &= 1 - k\delta V_k - \frac{\delta m}{n} E_k \\
W_k &= \frac{E_k - V_k}{n-1},
\end{aligned}$$

where $m = \max\{0, \frac{n-1}{2} - k\}$. Next, we introduce the following notation: For any proposal $\theta \in \Delta^n$ and any two players $i, j \in N$, let $\Pi_{i\to j}(\theta)$ be the set of permutations of θ whose jth component equals θ_i, i.e.,

$$\Pi_{i\to j}(\theta) := \{\theta' \in \Delta^n \mid \theta' = M^\top \theta, M \in \mathcal{M}(i,j)\},$$

where $\mathcal{M}(i,j)$ is the set of all $n \times n$ permutation matrices M with $m_{ij} = 1$.

Definition 3. *Consider the ORBG $G(\delta, n)$. For every $k = 1, \dots, n-1$, a profile of stationary strategies σ is a* generalized k-SSR *if the following holds:*

1. *At every history $h \in H^\emptyset$, the proposer makes an anonymous proposal η^k which gives him/herself X_k, gives δV_k to k other players, gives $\frac{\delta}{n}((n-1)W_k + V_k)$ to* $\max\{0, \frac{n-1}{2} - k\}$ *more players, and zero to all remaining players.*

2. *At any history* $h \in H_j^{i,\theta}$, *player* i *expects the same payoff, say* $p_{ij}(\theta, \sigma)$.

3. *Let* $p_i(\theta, \sigma) = \frac{1}{n-1} \sum_{j \in N \setminus \{i\}} p_{ij}(\theta, \sigma)$. *For any* $i_1, i_2 \in N$ *and every* $\theta' \in \Pi_{i_1 \to i_2}(\theta)$, *it holds that* $p_{i_2}(\theta', \sigma) = p_{i_1}(\theta, \sigma)$.

4. *For every* $\theta \in \Delta^n$, *there is a set* $T(\sigma, \theta) \subset \Delta^n$ *such that the following holds: Whenever players vote between the proposal on the floor* θ *and some amendment* θ', *then the majority votes in favor of* θ' *if and only if* $\theta' \in T(\sigma, \theta)$. *Moreover, for every* $\theta \in \Delta^n$, *the set* $T(\sigma, \theta)$ *has the following properties:* $\Pi_{i \to j}(\theta) \subseteq T(\sigma, \theta)$ *and* $\Pi_{i \to j}(\widetilde{\theta}) \subseteq T(\sigma, \widehat{\theta})$ *for all* $\widehat{\theta} \in \Pi_{i \to j}(\theta)$ *if* $\widetilde{\theta} \in T(\sigma, \theta)$ *for any* $i, j \in N$.

5. *Whenever player* i *votes on an endorsed proposal* θ, *s/he votes in favor of* θ *if and only if* $\theta_i \geq \frac{\delta}{n}((n-1)W_k + V_k)$.

Let us compare the generalized k-supporters to the k-supporter with SSaR(Simple Swaps and Randomization) in Definition 2:

Points 1 and 5 in Definition 3 are familiar from Definition 2. In contrast to that earlier definition, however, we now also have to specify a *selection rule.* We impose on that selection rule similar stationarity and anonymity requirements as the ones spelled out in Points 2, 3, and 4 of Definition 3. We verbally discuss these points in turn:

- Point 2 in Definition 3 imposes a *stationarity requirement* for histories at which an amendment can be made: Whenever player j can endorse or amend the same proposal θ made by the same player i, s/he acts in the same way. One implication is that player i has the same expected payoff, independently of the amendment, whenever his/her proposal θ is on the floor.

- Point 3 in Definition 3 adds an *anonymity requirement* to the previous point: The expected payoff of player i_1 when his/her proposal θ is on the floor is the same as the expected payoff of player i_2 when any of his/her proposals from $\Pi_{i_1 \to i_2}(\theta)$ is on the floor.

- Point 4 in Definition 3 puts *stationarity and anonymity restrictions* on the voting behavior when a proposal on the floor is pitted against an amendment: First, whenever the same proposal on the floor and the same amendment are pitted against each other, the winner is the same. Second, if the amendment is a simple swap with randomization of the proposal on the floor, then the amendment wins. Third, the majority's decision for an amendment or a proposal on the floor is unresponsive to a change in the players' "labels."

Consider a k-SSR which is an SSPE in the SORBG. From that k-SSR, let us construct a generalized k-SSR by preserving the same *anonymous proposal,* the same *amendment rule,* and the same *voting rule* but adding the following *selection rule*: Whenever players choose between a proposal on the floor and an amendment, all players vote in favor of the amendment. It is trivially true that no unilateral deviation from this selection rule can improve a player's payoff. Crucial is that a majority is willing to vote for the amendment over the proposal. This is shown in the next subsection 5.2 if individuals could be pivotal. Hence, it is intuitive that the generalized k-SSR so constructed is an SSPE in the ORBG. This is formally stated in the next proposition.

Proposition 3. *If the k-SSR is an SSPE of the SORBG, then there is a generalized k-SSR that is an SSPE of the ORBG.*

The proof of Proposition 3 can be found Appendix B. Proposition 3 tells us that the SSPE found by studying the SORBG corresponds to an SSPE of the original ORBG. The insight of Proposition 3 is admittedly simple and two crucial questions follow, which are dealt with in the next two subsections.

5.2 The art of randomization

First, is it guaranteed that by randomization, there is a majority of agents who receive higher or equal shares in the amendment and are willing to vote for the amendment if there is a chance that they could be pivotal (or if voting occurred sequentially)?[19]

For this first question, we obtain:

Proposition 4. *Let $n \geq 5$ and suppose that there is a k-SSR in the SORBG. Then, randomization can always be performed such that a majority receives higher or equal shares in the amendment than in the original proposal and all agents, except the amender, expect the same expected payoff before an amendment is made.*

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[19] We assume that ties are broken in favor of the amendment.

The proof of Proposition 4 is given in Appendix B. Randomization is simple in some cases, but requires careful approaches in other cases.

5.3 The reverse for ORBG

Second, we ask whether the converse is also true: Can a generalized k-SSR be an SSPE of the ORBG without corresponding to a k-SSR that is an SSPE in the SORBG? Loosely speaking, the issue in this section is what we have missed by restricting attention to the SORBG.

This second question is also far from trivial. To see what the problem is, note that a strategy profile has to specify for each pair proposal/amendment how each player would choose between the two. A player's choice is *not* merely determined by the amount of surplus which s/he receives under the proposal on the floor and the amendment. Rather, it depends on the *entire* proposal and the *entire* amendment through the expected length of further delays after one or the other option wins. We will make this issue tractable by focusing on properties of the sequences of amendments which occur after a deviation from the supposed equilibrium path of play.

More concretely, let us consider the generalized $\widehat{k}$-SSR for some $\widehat{k}$. Suppose that it is optimal for the initial proposer to deviate unilaterally from the generalized $\widehat{k}$-SSR by making the proposal that s/he would make when playing according to the generalized k-SSR for some $k \neq \widehat{k}$. We show that, if this were indeed optimal, then any amendment to the proposal would be a simple swap with randomization of it. As a result, the initial proposer's unilateral one-shot deviation leads to a path of play that resembles the generalized k-SSR until a proposal is endorsed and voted upon. This is shown in Propositions 6 and 7 in the Appendix A1.

Next, one can compute the payoff which the initial proposer could achieve by the aforementioned deviation. Due to the premise that this deviation is optimal for the initial proposer, it follows that the payoff we compute must not be less than the initial proposer's payoff from the generalized $\widehat{k}$-SSR. This yields the following necessary condition for the generalized $\widehat{k}$-SSR to be an SSPE.

Proposition 5. *If the generalized $\widehat{k}$-SSR is an SSPE of the open rule legislative bargaining game $G(\delta, n)$, then $\widehat{k} \in \arg\max_{k\in\{1,\ldots,n-1\}} V_k$.*

A proof of Proposition 5 can be found in Appendix A1, where we discuss the derivation in detail. For the case with δ sufficiently close to one, one can show that this necessary condition puts $\widehat{k}$ in a neighborhood around $\sqrt{n}$. Based on this insight, we can argue that (except in a knife–edge case) there is only one $\widehat{k} \in \{1, \ldots, n-1\}$ so that the generalized $\widehat{k}$-SSR satisfies the necessary condition for an SSPE.

Theorem 2. *Suppose that $n \geq 15$ and δ is sufficiently close to one. If a generalized k-SSR is an SSPE of the ORBG, then it holds that*

$$k \in \{1, \ldots, n-1\} \cap (\sqrt{n}-1, \sqrt{n}+1).$$

Combining the findings in Proposition 5 and Theorem 2, we get the following corollary.

Corollary 4. *Suppose that $n \geq 15$ and δ is sufficiently close to one. Suppose that the k^*-SSR is an SSPE of the ORBG. Moreover, suppose that there is some $k^{**} \neq k^*$ such that the generalized k^{**}-SSR is an SSPE. Then, k^* and k^{**} are successive integers and it holds that $\overline{V}_{k^*} = \overline{V}_{k^{**}}$.*

The interpretation is as follows: Consider the case where $n \geq 15$ and δ is close enough to one. In our analysis of the SORBG, we have found one k such that the k-SSR is an SSPE. In Proposition 5, we have shown that this SSPE of the SORBG easily extends to an SSPE of the ORBG. Hence, we already have found one particular k such that the generalized k-SSR of the ORBG is an SSPE. Theorem 2 and Corollary 4 tell us that the SSPE which we have already found is (except in a knife–edge case) the *only* generalized k-SSR that is an SSPE. In addition, even in that knife–edge case, there can at most be two values of k such that the generalized k-SSR that is an SSPE, these two values of k must be successive integers, and the proposer's payoffs in both potential equilibria are equal. The conclusion is that our main results from the SORBG carry over to the ORBG.

We relinquish from providing additional numerical examples because Proposition 3 and Corollary 4 the numerical illustration of Section 4.2 for the SORBG carry over to the generalized k-supporter equilibrium. Moreover, for δ close to one, due to Theorem 2, the number of supporters, k, is again in a neighborhood around $\sqrt{n}$.

5.4 No Randomization

In Appendix A2, we provide a second version of the original ORBG. As in the previous section, when a player proposes an amendment to the current proposal on the floor, a vote determines whether the amendment replaces it. However, we now focus on a simpler class of amendments that exclude randomization. These deterministic amendments give rise to equilibrium candidates analogous to the k-SSR in both the SORBG and ORBG, preserving the same qualitative properties. Nonetheless, the analysis becomes considerably more intricate.

6 Conclusion

We have introduced a stylized model of bargaining with open rules and we have introduced particular equilibrium concepts in simple and extended versions. We find that outcomes under these particular strategies may be even less efficient and also less egalitarian than suggested by Baron and Ferejohn. The analysis does not rule out the possibility of other stationary equilibria with other properties.

Moreover, our analysis may have important implications for the design of legislatures and their committees. For instance, the tendency of open rules to produce egalitarian outcomes, even after the proposer has been selected at the cost of significant delays, opens up a more detailed comparison of the egalitarian efficiency trade-offs between closed and open rules. Smaller legislatures yield less delay and a more egalitarian allocation than larger legislatures under open rules.

The size of the legislature may need to be quite large for other reasons than the ones examined in this paper, e.g. to be sufficiently representative of the underlying electorate. Then, surplus division could be first delegated to a smaller committee that itself is representative of the legislature. If the committee uses the open rule and the committee decision is put to a final vote in the legislature under a closed rule, the efficiency and equality advantages of open rules could be preserved.

Another finding is that patient players induce more delays, even to the extreme that the expected delay becomes arbitrarily large, leading to “never ending” games.

Many closed rule bargaining models allow players to differ in their probabilities of being recognized as an agenda setter or for a general acceptance rule rather than simple majority. These extensions would further complicate the analysis, although we would expect a general acceptance rule to be more easily feasible than an extension to heterogeneous recognition probabilities.

Finally, one potential avenue for future research would be to investigate experimentally the size of "coalitions" that emerge in open vs. closed rule bargaining. In particular, a variety of hypotheses could be tested. Here are some examples. If agents are sufficiently patient, and the size of the legislature is not very small, proposers allocate different shares to agents who are supposed to endorse the proposal and to agents who are supposed to vote for a proposal. Moreover, the probability that a proposal is approved is well below $\frac{1}{2}$. The expected delay increases with the size of the legislature and with the patience of the agents. If agents are not too impatient, the proposer has a considerably larger share than agents who are supposed to endorse the proposal but less than in a closed-rule game. Another open question is whether or not randomized strategies lead to similar outcomes. For example, do strategies only using simple swaps as outlined in the appendix also yield more egalitarian allocations? However, we suspect that this could be difficult to solve analytically, and numerical methods are possibly inevitable.

Appendix A1: Details for ORBG

In this appendix, we present some details of ORBG and provide a full derivation of the results. We introduce the following notation: Let σ be a generalized k-SSR. Take any sequence $(\theta^0, \theta^1, \dots, \theta^T)$ such that with strictly positive probability, σ induces a path of play along which the proposal θ^0 is made at some history $h^0 \in H^{\emptyset}$ by player i^0. Then, the game reaches history h^1, where player i^1 makes the amendment θ^1, and it reaches history h^2, where player i^2 makes the amendment θ^2, and so on, until eventually player i^{T+1} endorses θ^T at history h^{T+1}.

Proposition 6. *Let $h \in H^{\emptyset}$ and $h' \in H^{i,\theta}$, for some $\theta \in \Delta^n$. Let player i be the proposer at h, and player j be the proposer at h'. Suppose that it is optimal for player i to propose θ at history h, provided that σ is played at all histories following h. Consider the choice of player j at history h'. Provided that σ is played at all histories following h', either it is optimal for player j to endorse the proposal θ at history h', or it is optimal to make any amendment from $\Pi_{i\to j}(\theta)$.*

Proof. Before giving the proof, we introduce the following notation to keep it concise. Let $\pi_{i\leftrightarrow j}(\theta) \in \Pi_{i\to j}(\theta)$ be the proposal that coincides with θ in all entries except the ith and the jth, i.e., the jth and ith entries are swapped. Suppose that, at history h', it is strictly better for player j to make an amendment $\widetilde{\theta} \notin \Pi_{i\to j}(\theta)$, instead of all amendments from $\Pi_{i\to j}(\theta)$. Thus $p_j(\widetilde{\theta}, \sigma) > p_j(\widehat{\theta}, \sigma)$ for all $\widehat{\theta} \in \Pi_{i\to j}(\theta)$. By definition of a generalized k-supporter (see Definition 3), it follows that

$$p_i(\pi_{j\leftrightarrow i}(\widetilde{\theta}), \sigma) = p_j(\widetilde{\theta}, \sigma) > p_j(\widehat{\theta}, \sigma) = p_i(\theta, \sigma).$$

This implies that it is not optimal for player i to propose θ at history h, and the proof of the proposition is complete. □

Repeating the same line of argument, we can also show the next proposition:

Proposition 7. *Let $h \in H^{i,\theta}$ and $h' \in H^{j_1,\widehat{\theta}}$ for $\widehat{\theta} \in \Pi_{i\to j_1}(\theta)$. Let player j_1 be the proposer at h, and player j_2 be the proposer at h'. Suppose that it is optimal for player j_1 to make the amendment $\widehat{\theta}$. Consider the choice of player j_2 at history h'. Provided that*

σ is played at all histories following h', either it is optimal for player j_2 to endorse $\widehat{\theta}$, or it is optimal to make an amendment from $\Pi_{j_1 \to j_2}(\widehat{\theta})$.

The two propositions above lead to the following conclusions: Suppose that players $i^0, i^1, \ldots, i^T$ choose the initial proposal and the amendments optimally. Then, θ^t can be described as a simple swap with randomization of θ^{t-1} for any $t = 1, \ldots, T$. Moreover, if player i^0 at history h^0 has an expected payoff of V, then any player i^t who makes an amendment at history h^t with $t = 1, \ldots, T$, has an expected payoff of $\delta^t V$. Finally, player i^{T+1} endorses θ^T because that proposal gives him/her δV.

We assume that it is optimal for the initial proposer to deviate from the generalized $\widehat{k}$-SSR by making the anonymous proposal associated with the generalized k-SSR for some $k \neq \widehat{k}$. Since we have already proved that $Y_k > Z_k$, the only possibility for a change from k to $\widehat{k}$ is a no-vote of at least one player that would receive $Z_k = \delta E_k / n$. By

$$E_{k+1} - E_k = \frac{(n-1)(1-\delta)}{\big((k+1)\delta + (n-1)(1-\delta)\big)\big(k\delta + (n-1)(1-\delta)\big)} \geq 0,$$

with equality if and only if $\delta = 1$, E_k is monotonically increasing. After rejection of a proposal and switch to a $\widehat{k}$-supporter with swaps and randomization, the expected payoff for every player is $\delta E_{\widehat{k}}/n$. Since the payoff for a prospective pro-voter in the k-supporter is $\delta E_k/n$, a switch from k to $\widehat{k}$ is only possible for $k < \min\{\widehat{k}, (n-1)/2\}$. Moreover, whether a no-vote is advantageous is independent of the round in which the vote is cast.

Once a proposal is voted down, say in round l, there is no longer a proposal on the floor, and at the beginning of the next round $l+1$, a random player is picked to choose a new proposal. By assumption, this new proposal will be a generalized $\widehat{k}$-supporter, and by Propositions 6 and 7, this will automatically be a $\widehat{k}$-supporter with simple swaps and randomization. Therefore, every player will have the same expected payoff of $\delta^l E_{\widehat{k}}/n$, no matter who had initially proposed in the first round. The probability p_l that the proposal is endorsed and subsequently voted down in round l is

$$p_l = \left(\frac{n-k-1}{n-1}\right)^{l-1} \frac{k}{n-1},$$

and so the expected payoff for every player is

$$P_{k}^{\widehat{k}} = \sum_{l=1}^{\infty} p_l \delta^l \frac{E_{\widehat{k}}}{n} = E_{\widehat{k}} \frac{\delta}{n} \frac{k}{n-1} \sum_{l=1}^{\infty} \left(\delta \frac{n-k-1}{n-1} \right)^{l-1} = E_k E_{\widehat{k}} \frac{\delta}{n}.$$

Therefore, initially changing the $\widehat{k}$-supporter to a k-supporter is never advantageous for the first player.

As a consequence, we see that a generalized $\widehat{k}$-supporter can only be an SSPE if $V_{\widehat{k}} \geq V_k$ for all $k \in \{1, \ldots, n-1\} \setminus \{\widehat{k}\}$. Hence, we obtain

Proposition 5. *If the generalized $\widehat{k}$-supporter is an SSPE of the open rule legislative bargaining game $G(\delta, n)$, then $\widehat{k} \in \arg\max_{k \in \{1,\ldots,n-1\}} V_k$.*[20]

It is instructive to focus on the case with patient players. For each $k = 1, \ldots, n-1$, let $\overline{V}_k$, $\overline{W}_k$, and $\overline{X}_k$ be the solution to the following system of equations:

$$\overline{V}_k = \frac{(k+1)(n-1) - k - \max\left\{0, \frac{n-1}{2} - k\right\} \frac{k(n-1)}{n}}{(n-1)(n+k^2) - k}, \tag{19}$$

$$\overline{W}_k = \frac{1 - \overline{V}_k}{n-1} \tag{20}$$

$$\overline{X}_k = 1 - k\overline{V}_k - \frac{1}{n} \max\left\{0, \frac{n-1}{2} - k\right\}. \tag{21}$$

There, Equations (19)–(21) are the same system of linear equations as in (7), now applied at $\delta = 1$.

Now we are ready to state the main result of this section.

Theorem 2. *Suppose that $n \geq 15$ and δ is sufficiently close to one. If a generalized*

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[20] Note that the one-shot deviation principle is now applied differently than in previous sections of this paper. In the present section, the choice between a proposal on the floor and an amendment is no longer rendered trivial. Therefore, it is no longer true that all histories at which a particular player is the proposer are "equivalent." It is true, however, that all histories in the set $H^{\emptyset}$ at which the same player proposes are followed by the same continuation game. Hence, we think of the open rule legislative bargaining game as a stochastic game. It moves to a new state whenever a history in $H^{\emptyset}$ is reached. If the proposer at that history is player i, then the game is in state i.

k-SSR is an SSPE of the ORBG, then it holds that

$$k \in \{1, \dots, n-1\} \cap (\sqrt{n} - 1, \sqrt{n} + 1).$$

The proof of Theorem 2 is relegated to Appendix B.

In order to assess the implications of Theorem 2, let us first consider the case where n is such that $\sqrt{n}$ is an integer. In that case, $\sqrt{n}$ is the only integer contained in the interval $(\sqrt{n} - 1, \sqrt{n} + 1)$. Hence, the generalized $\sqrt{n}$-SSR is the only generalized k-SSR that can be an SSPE.

Now consider the case where n is such that $\sqrt{n}$ is not an integer. In that case, the interval $(\sqrt{n} - 1, \sqrt{n} + 1)$ contains two integers. Let us denote them by k^* and $k^* + 1$. Now we use Equations (19)–(21) to compute $\overline{V}_{k^*}$ and $\overline{V}_{k^*+1}$. There is no reason to expect that these two amounts are generally equal. If $\overline{V}_{k^*} > \overline{V}_{k^*+1}$, then Proposition 5 implies that the generalized $(k^* + 1)$-SSR cannot be an SSPE. Similarly, if $\overline{V}_{k^*} < \overline{V}_{k^*+1}$, then Proposition 5 implies that the generalized k^*-SSR cannot be an SSPE.

From these observations, we obtain the following corollary:

Corollary 4. *Suppose that $n \geq 15$ and δ is sufficiently close to one. Suppose that the k^*-SSR is an SSPE of the ORBG. Moreover, suppose that there is some $k^{**} \neq k^*$ such that the generalized k^{**}-SSR is an SSPE. Then, k^* and k^{**} are successive integers and it holds that $\overline{V}_{k^*} = \overline{V}_{k^{**}}$.*

Appendix A2: Strategies without randomization

In this section, we introduce a second strategy profile for the original ORBG and outline how the results obtained for the k-SSR can possibly be extended. As in the previous section, whenever a player makes an amendment to a proposal on the floor, a vote determines whether or not the amendment replaces the proposal on the floor. However, we consider an even simpler type of amendments that renounce randomization, as we had in the previous two versions. These new amendments will yield equilibrium candidates that are analogous to the k-SSR in the SORBG and ORBG with the same qualitative properties. Unfortunately, the analysis becomes significantly more complex. The following analysis also illustrates that the previous strategies, (generalized) k-SSR, produce systems of equations with the smallest numbers of equations, as any deviation from these concepts produces more heterogeneity among continuation values. Nevertheless, this new version turns out to be tractable, too.

We focus on stationary strategy profiles that we call *k-supporter strategy with simple swaps.* These equilibrium candidates are particularly appealing for their simplicity: Intuitively, they are those equilibrium candidates which lead to the lowest possible number of permutations along an equilibrium path of play. More formally, the stationary strategy profiles that we consider have the following properties:

- On the path of play induced by a k-supporter strategy with simple swaps, whenever a player amends a proposal on the floor, s/he does so by simply swapping his/her component of the proposal on the floor with that of the player who has made the proposal on the floor, i.e. with the highest component.
- Every proposal and every amendment made on a path of play of a k-supporter strategy with simple swaps has the following structure: The proposer offers k players a payoff that makes them willing to endorse the proposal, and to vote in its favor. If $k \leq \frac{n-1}{2}$, the proposer offers an additional $\frac{n-1}{2} - k$ players a payoff that makes them willing to vote in favor of the proposal once it has been endorsed, but not to endorse it themselves.

This amendment process has appeal, as it requires the smallest change of the original proposal—it is only a "simple swap". Yet, despite the simplicity of the amendment process, we now have to take into account that when a proposal is on the table, the continuation values depend on the offered share of the surplus, as agents anticipate that they will receive the same surplus in an amendment if they are not themselves chosen to amend the proposal. To take these complications into account, we start from the original system of equations:

Take the odd number of players n and the discount factor δ as given. For any $k = 1, \ldots, n-1$, let the quadruple (V_k, W_k, X_k, E_k) be defined as the solution of the earlier system of equations (7):

$$
\begin{aligned}
E_k &= \frac{k}{k\delta + (n-1)(1-\delta)} \\
V_k &= \frac{k(n-1) + \left(n-k-1-\frac{km(n-1)}{n}\right)\delta E_k}{(n-1)^2 + (n-k-1+k^2(n-1))\delta} \\
X_k &= 1 - k\delta V_k - \frac{m\delta}{n} E_k \\
W_k &= \frac{E_k - V_k}{n-1},
\end{aligned}
$$

where $m = \max\left\{0, \frac{n-1}{2} - k\right\}$. The meaning of (V_k, W_k, X_k, E_k) has been described in Section 3.

Since there is no randomization, we now have to distinguish three different types of non-proposers with reference to a particular proposal on the floor. In this section, we also follow the description of the family of strategy profiles k-supporter strategy with simple swaps and distinguish three types:[21]

- Non-proposers of **type 1** would endorse the current proposal on the floor immediately and would vote in its favor. In a k-supporter strategy with simple swaps strategy profile, there are exactly k such non-proposers. We denote the expected payoff for a type 1 non-proposer by W_k^1.

- Second, non-proposers of **type 2** would not endorse the current proposal on the floor, but would vote in its favor. In a k-supporter strategy with simple swaps, there are m non-proposers of type 2. Their expected individual payoff is denoted by W_k^2.

- Third, non-proposers of **type 3** would not endorse the current proposal on the floor, and would not vote in its favor. In a k-supporter strategy with simple swaps, these are the players who receive nothing if the proposal on the floor were to be implemented. There are $n - k - m - 1$ non-proposers of type 3. Accordingly, we denote their expected payoff by W_k^3.

The values of W_k^1, W_k^2 and W_k^3 can be different. We start with the formula of a non-proposer of type 1 who obtains $Y_k = \delta V_k$ (see Equation (5)) in case the proposal on the

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[21] We are grateful to a referee for suggesting this approach.

floor is endorsed and stays a non-proposer of type 1 in a complementary case of a simple swap amendment, since in k-supporter strategies with simple swaps, only the payoffs for the proposer and the amender are swapped. We therefore obtain

$$W_k^1 = \frac{k}{n-1}\delta V_k + \frac{n-1-k}{n-1}\delta W_k^1. \tag{22}$$

Next, a non-proposer of type 2 receives Z_k in case the proposal on the floor is endorsed, becomes a proposer (with discounted expected payoff δV_k) in case s/he gets to make a simple swap amendment, and stays a non-proposer of type 2 in case another non-proposer of type 2 or 3 gets to make a simple swap amendment. Hence, we obtain:

$$W_k^2 = \frac{k}{n-1}Z_k + \frac{1}{n-1}\delta V_k + \frac{n-2-k}{n-1}\delta W_k^2. \tag{23}$$

Similarly, we can write down the expected payoff of a non-proposer of type 3 which is given by:

$$W_k^3 = \frac{k}{n-1} \cdot 0 + \frac{1}{n-1}\delta V_k + \frac{n-2-k}{n-1}\delta W_k^3. \tag{24}$$

We now set up an analogous system of equations for this setting as in (7), taking into account the different types of non-proposers. First, the expected payoff for the proposer satisfies

$$V_k = \frac{k}{n-1}X_k + \frac{m}{n-1}\delta W_k^2 + \frac{n-k-m-1}{n-1}\delta W_k^3. \tag{25}$$

The first summand for V_k is the probability that a non-proposer of type 1 is selected for either endorsing or amending times the corresponding payoff for the proposer. Since a type 1 non-proposer endorses immediately, and the k-supporter strategy with simple swaps is constructed such that an endorsed proposal is also accepted in a vote, the proposer obtains X_k in this case. The second and the third summand correspond to the case where a non-proposer of type 2 or 3, respectively, is selected to either endorse or amend. Since they would both make a swap amendment, the proposer will become a non-proposer of type 2 or 3, resp., in the next round, obtaining δW_k^2 or δW_k^3.

Next, the total expected payoff E_k can be expressed as

$$E_k = V_k + kW_k^1 + mW_k^2 + (n-k-m-1)W_k^3. \tag{26}$$

We can now combine the new expressions (22)-(26) with $1 = X_k + kY_k + mZ_k$ from (4), $Y_k = \delta V_k$ from (5), $Z_k = \frac{\delta}{n} E_k$ from (6), and obtain the following system of equations:

$$\begin{aligned}
V_k &= \frac{k}{n-1} X_k + \frac{m}{n-1} \delta W_k^2 + \frac{n-k-m-1}{n-1} \delta W_k^3 \\
W_k^1 &= \frac{k}{n-1} \delta V_k + \frac{n-1-k}{n-1} \delta W_k^1 \\
W_k^2 &= \frac{k}{n-1} Z_k + \frac{1}{n-1} \delta V_k + \frac{n-2-k}{n-1} \delta W_k^2 \\
W_k^3 &= \frac{1}{n-1} \delta V_k + \frac{n-2-k}{n-1} \delta W_k^3 \\
E_k &= V_k + kW_k^1 + mW_k^2 + (n-k-m-1) W_k^3 \\
1 &= X_k + kY_k + mZ_k \\
Y_k &= \delta V_k \\
Z_k &= \frac{\delta}{n} E_k.
\end{aligned}$$

Simplifying the above equations then yields

$$\begin{aligned}
V_k &= \frac{k}{n-1} X_k + \frac{m}{n-1} \delta W_k^2 + \frac{n-k-m-1}{n-1} \delta W_k^3 \\
W_k^1 &= \frac{k\delta}{n-1-\delta(n-1-k)} V_k \\
W_k^2 &= \frac{k\delta}{n(n-1-\delta(n-2-k))} E_k + \frac{\delta}{n-1-\delta(n-2-k)} V_k \\
W_k^3 &= \frac{\delta}{n-1-\delta(n-2-k)} V_k \\
X_k &= 1 - k\delta V_k - \frac{m\delta}{n} E_k \\
E_k &= V_k + kW_k^1 + mW_k^2 + (n-k-m-1) W_k^3.
\end{aligned} \tag{27}$$

We observe that W_k^1 and W_k^3 can be expressed by V_k alone. Hence, the system of equations (27) can be reduced to a system of equations with two variables (V_k, E_k). One can then proceed in the same way as for the generalized k-SSR and show that there is a unique solution and that all components of the solution are strictly positive.

This opens the door to a detailed equilibrium analysis, which is left for future research.

Appendix B: Proofs

Proof of Lemma 1. Simplification of (1)-(4) yields

$$\begin{aligned}
V_k &= \frac{k}{n-1}X_k + \frac{n-k-1}{n-1}\delta W_k \\
W_k &= \frac{k}{(n-1)^2}\big(kY_k + mZ_k\big) + \delta\frac{n-k-1}{(n-1)^2}(V_k + (n-2)W_k) \\
E_k &= V_k + (n-1)W_k \\
1 &= X_k + k\delta V_k + m\frac{\delta}{n}E_k.
\end{aligned} \tag{28}$$

Substituting Equation (4) into the second equation of (28) yields

$$\begin{aligned}
(n-1)V_k &= kX_k + (n-k-1)\delta W_k \\
(n-1)^2 W_k &= k(1-X_k) + \delta(n-k-1)(V_k + (n-2)W_k) \\
E_k &= V_k + (n-1)W_k \\
1 &= X_k + k\delta V_k + m\frac{\delta}{n}E_k.
\end{aligned} \tag{29}$$

Now we can multiply the third equation of (29) by $(n-1)$ and substitute the first and second equations, obtaining

$$\begin{aligned}
(n-1)V_k &= kX_k + (n-k-1)\delta W_k \\
(n-1)^2 W_k &= k(1-X_k) + \delta(n-k-1)(V_k + (n-2)W_k) \\
(n-1)E_k &= k + (n-k-1)\delta E_k \\
1 &= X_k + k\delta V_k + m\frac{\delta}{n}E_k.
\end{aligned} \tag{30}$$

Solving the third equation of (30) for E_k yields

$$E_k = \frac{k}{k\delta + (n-1)(1-\delta)}. \tag{31}$$

Furthermore, we obtain:

$$W_k = \frac{E_k - V_k}{n-1} \tag{32}$$

$$(n-1)V_k = kX_k + (n-k-1)\delta\frac{E_k - V_k}{n-1} \tag{33}$$

$$1 = X_k + k\delta V_k + m\frac{\delta}{n}E_k. \tag{34}$$

Simplification of (32)-(34) finally yields

$$\begin{aligned}
E_k &= \frac{k}{k\delta + (n-1)(1-\delta)} \\
V_k &= \frac{k(n-1) + \big(n-k-1-\frac{km(n-1)}{n}\big)\delta E_k}{(n-1)^2 + (n-k-1+k^2(n-1))\delta} \\
X_k &= 1 - k\delta V_k - \frac{m\delta}{n}E_k \\
W_k &= \frac{E_k - V_k}{n-1}.
\end{aligned} \tag{35}$$

□

Proof of Lemma 2. Since the system of equations (7) is triangular with nonzero diagonal entries, it has a unique solution. We have $0 < E_k \leq 1$, since its denominator is a convex combination of k and $n-1$. Furthermore, $E_k = 1$ if and only if $k = n-1$ or $\delta = 1$. The denominator of V_k is obviously positive, and since $k(n-1) \geq k(n-1)\frac{m}{n}$ also the numerator is positive, so $V_k > 0$. Solving (7) for W_k yields

$$W_k = \frac{k\delta\big(n(n-1) + k(m + (k-1)n)\big)}{n\big(k\delta + (n-1)(1-\delta)\big)\big((n-1)^2 + (n-k-1+k^2(n-1))\delta\big)}, \tag{36}$$

which is obviously positive. Furthermore, we can calculate

$$V_k - W_k = \frac{kn\big((n-1)^2 - ((n-1)(n-k-1) + k(k+m))\delta\big)}{n\big(k\delta + (n-1)(1-\delta)\big)\big((n-1)^2 + (n-k-1+k^2(n-1))\delta\big)}. \tag{37}$$

The denominator in (37) is positive, and

$$\begin{aligned}(n-1)^2 &- ((n-1)(n-k-1)+k(k+m))\delta\\ &\geq (n-1)^2-((n-1)(n-k-1)+k(k+m))\\ &= (n-1)\big(n-1-(n-k-1)\big)-k(k+m)\\ &= (n-1)k-k(k+m)\\ &= k\big(n-1-k-\max(0,\tfrac{n-1}{2}-k)\big)\geq 0,\end{aligned}$$

so $V_k \geq W_k > 0$, and $V_k = W_k$ if and only if $k=n-1$ and $\delta = 1$, since only then both inequalities hold with equality. This proves (ii).

We also find from (7)

$$X_k = \frac{\begin{aligned}n(n-1)^3-\delta(n-1)\big((n-1)(n(n-2)-k(n-m))+kn\big)\\ -\delta^2(n-k-1)\big(k(m+(k-1)n)+n(n-1)\big)\end{aligned}}{n\big(k\delta+(n-1)(1-\delta)\big)\big((n-1)^2+(n-k-1+k^2(n-1))\delta\big)} \tag{38}$$

and again the denominator is positive. Using that $\delta \in (0,1]$ and therefore also $\delta^2 \in (0,1]$, so we find for the numerator in (38)

$$\begin{aligned}n(n-1)^3 &- \delta(n-1)\big((n-1)(n(n-2)-k(n-m))+kn\big)\\ &\quad -\delta^2(n-k-1)\big(k(m+(k-1)n)+n(n-1)\big)\\ &\geq n(n-1)^3-(n-1)\big((n-1)(n(n-2)-k(n-m))+kn\big)\\ &\quad -(n-k-1)\big(k(m+(k-1)n)+n(n-1)\big)\\ &= k(n-1)^2(n-m)-(n-k-1)(k(m+(k-1)n))\\ &= k(n^2(n-1-k-m)+km+k^2n+mn)>0.\end{aligned}$$

Therefore, $X_k > 0$, proving (iii).

Plugging $k=n-1$ and $\delta=1$ into (7), we obtain

$$\begin{aligned}E_{n-1} &= 1, & V_{n-1} &= \frac{1}{n},\\ X_{n-1} &= 1-(n-1)V_{n-1}, & W_{n-1} &= \frac{1-V_{n-1}}{n-1},\end{aligned} \tag{39}$$

hence, $V_{n-1} = W_{n-1} = X_{n-1} = Y_{n-1} = \frac{1}{n}$, proving (i).

To prove the additional Points (a) and (b), we start with (a).

Setting $k = n-1$ in (7), we obtain

$$\begin{aligned} &E_{n-1} = 1, \qquad V_{n-1} = \frac{1}{1+(n-1)\delta}, \qquad W_{n-1} = \frac{1-V_{n-1}}{n-1} = \frac{\delta}{1+(n-1)\delta}, \\ &X_{n-1} = 1-(n-1)\delta V_{n-1} = \frac{1}{1+(n-1)\delta} = V_{n-1}, \end{aligned} \tag{40}$$

proving (a).

In order to see why (b) is true, recall from (a) that for any choices of k and δ other than $k = n-1$ and $\delta = 1$, we have $V_k > W_k$. Thus, Equation (30) implies the inequality $(n-1)V_k < kX_k + (n-1-k)\delta V_k$. Rearranging and using (7) yields $X_k E_k > V_k \geq \delta V_k$, which implies the required inequality, since $E_k \leq 1$.

□

Proof of Lemma 4. Proof of (i). Suppose that there is a vector $\widehat{\theta} \in \Delta^n$ such that proposing $\widehat{\theta}$ instead of the proposal prescribed by the k-SSR is a profitable deviation for the proposer, say player i. Suppose that there is a player $j \in N \setminus \{i\}$ such that $0 < \widehat{\theta}_j < \frac{\delta}{n}((n-1)W_k + V_k)$. Player j neither endorses the proposal $\widehat{\theta}$, nor does s/he vote in its favor. Consequently, it would also be a profitable deviation for player i to offer zero to player j, and offer $\widehat{\theta}_l$ to all players $l \in N \setminus \{i, j\}$. By the same token, suppose that there is a player $j \in N \setminus \{i\}$ such that $\frac{\delta}{n}((n-1)W_k + V_k) < \widehat{\theta}_j < \delta V_k$. In that case, player j is willing to vote in favor of $\widehat{\theta}$, but not willing to endorse it. This would not change if s/he was offered $\frac{\delta}{n}((n-1)W_k + V_k)$ instead of $\widehat{\theta}_j$. Thus, it would also be a profitable deviation for player i to offer player j only $\frac{\delta}{n}((n-1)W_k + V_k)$, and offer each player $l \in N \setminus \{j\}$ the amount $\widehat{\theta}_l$. Repeating the same argument, we see that if the proposer has any profitable deviation $\widehat{\theta}$, then s/he has a profitable deviation to a proposal $\widetilde{\theta}$ which gives each player other than the proposer either zero, or $\frac{\delta}{n}((n-1)W_k + V_k)$, or δV_k.

Let $\mathcal{P}_k$ be the set of vectors $\theta \in \Delta^n$ that, for some $k' \in \{1, \ldots, n-1\}$, contain k' components equal to δV_k and $\max\{0, \frac{n-1}{2} - k'\}$ components equal to $\frac{\delta}{n} E_k$, where V_k and E_k are as defined in Equations (7). Moreover, a vector $\theta \in \mathcal{P}_k$ has one component equal to $1 - k'\delta V_k - \max\{0, \frac{n-1}{2} - k'\}\frac{\delta}{n}E_k$. Any remaining components are equal to zero.[22]

Let λ_k^{+m} be the gain which the proposer can make by offering δV_k to $k+m$ players instead of to k players, starting from a proposal $\theta \in \mathcal{P}_k$, where $m \geq 2$. We want to show that $\lambda_k^{+m} > 0$ implies $\lambda_k^{+} > 0$. Suppose that $k \geq (n-1)/2$. Then we have

$$\lambda_k^{+m} = -\left(\frac{k}{n-1}\right) m\delta V_k + \left(\frac{m}{n-1}\right)(X_k - \delta W_k - m\delta V_k),$$

since in case the proposal is endorsed, the payoff reduces by $m\delta V_k$, the additional amount spent for the m extra players, and the payoff increases, since m more players will endorse the proposal, by the payoff for the proposer reduced by the additional spendings multiplied by the probability that one of the new m players is being chosen for endorsement.

If λ_k^{+m} is strictly positive, then dividing by m yields

$$-\left(\frac{k}{n-1}\right)\delta V_k + \left(\frac{1}{n-1}\right)(X_k - \delta W_k) - \left(\frac{m}{n-1}\right)\delta V_k > 0,$$

which can be rewritten equivalently as

$$-\left(\frac{k}{n-1}\right)\delta V_k + \left(\frac{1}{n-1}\right)(X_k - \delta W_k - \delta V_k) > \left(\frac{m-1}{n-1}\right)\delta V_k,$$

hence

$$\lambda_k^{+} > \left(\frac{m-1}{n-1}\right)\delta V_k.$$

We have shown earlier that $V_k > 0$. Thus, it follows that $\lambda_k^{+} > 0$, as desired. Now consider the case where $k \leq (n-3)/2$. We distinguish two subcases. If $k + m < (n-1)/2$, then the deviating proposal still contains $(n-1)/2 - (k+m)$ players who receive $\frac{\delta}{n}E_k$. Hence

$$\begin{aligned}\lambda_k^{+m} &= \frac{k+m}{n-1}\Big(X_k - m\Big(\delta V_k - \frac{\delta}{n}E_k\Big)\Big) + \frac{n-1-k-m}{n-1}\delta W_k - \Big[\frac{k}{n-1}X_k + \frac{n-1-k}{n-1}\delta W_k\Big] \\ &= \frac{m}{n-1}\Big(X_k - \delta W_k - (k+m)\Big(\delta V_k - \frac{\delta}{n}E_k\Big)\Big).\end{aligned}$$

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[22] Recall that the set Δ^n consists of vectors that are non-negative in all components and sum up to (at most) one. It follows that the number k' satisfies the inequality $1 - k'\delta V_k - \max\{0, \frac{n-1}{2} - k'\}\frac{\delta}{n}E_k \geq 0$. In other words, it is ensured that for any proposal in $\mathcal{P}_k$, the share of surplus for the proposer remains non-negative. We also show in Lemma 2 that the proposer's share is larger than that of any other player.

If $k + m \geq (n-1)/2$, then there are no players who receive $\frac{\delta}{n}E_k$ under the deviation. Hence

$$\lambda_k^{+m} = \frac{k+m}{n-1}\Big(X_k + \Big(\frac{n-1}{2} - k\Big)\frac{\delta}{n}E_k - m\delta V_k\Big) + \frac{n-1-k-m}{n-1}\delta W_k - \Big[\frac{k}{n-1}X_k + \frac{n-1-k}{n-1}\delta W_k\Big]$$

$$= \frac{m}{n-1}\Big(X_k - \delta W_k - (k+m)\Big(\delta V_k - \frac{\delta}{n}E_k\Big)\Big) + \frac{k+m}{n-1}\Big(\frac{n-1}{2} - k - m\Big)\frac{\delta}{n}E_k$$

$$\leq \frac{m}{n-1}\Big(X_k - \delta W_k - (k+m)\Big(\delta V_k - \frac{\delta}{n}E_k\Big)\Big).$$

Therefore, in both subcases, $\lambda_k^{+m} > 0$ implies

$$X_k - \delta W_k - (k+m)\Big(\delta V_k - \frac{\delta}{n}E_k\Big) > 0.$$

Since $m \geq 2$ and $V_k > E_k/n$, we can conclude that also

$$X_k - \delta W_k - (k+1)\Big(\delta V_k - \frac{\delta}{n}E_k\Big) > 0,$$

and hence $\lambda_k^{+} > 0$, as desired.

An analogous argument can be used to show that $\lambda_k^{-m} > 0$ implies $\lambda_k^{-} > 0$. □

Proof of Corollary 1. By definition, $\lambda_{n-1}^{+} = 0$ so it remains to show that for δ sufficiently small, $\lambda_{n-1}^{-} \leq 0$. Substituting Equation (40), we obtain

$$\lambda_{n-1}^{-} = \frac{\delta(n-2) - 1 + \delta^2}{(n-1)(1 + \delta(n-1))}.$$

Indeed, it follows that $\lambda_{n-1}^{-} \leq 0$ if and only if $\delta^2 + \delta(n-2) - 1 \leq 0$. This inequality is satisfied when

$$\delta \leq \sqrt{\Big(\frac{n-2}{2}\Big)^2 + 1} - \Big(\frac{n-2}{2}\Big).$$

□

Proof of Proposition 1. We start by showing statement (i). Suppose that there is a $k \geq (n+1)/2$ such that some k-SSR is an SSPE. Now consider a deviation by the initial proposer from the supposed SSPE. Under the deviation, the proposer offers δV_k to

$k-1$ instead of to k players, and offers zero to the remaining $n-k$ players, where V_k is as before. This deviation gives the proposer an expected payoff of

$$\left(\frac{k-1}{n-1}\right)(1-k\delta V_k+\delta V_k)+\left(\frac{n-k}{n-1}\right)\delta W_k,$$

while the proposer's expected payoff when playing according to the supposed SSPE is

$$V_k=\left(\frac{k}{n-1}\right)(1-k\delta V_k)+\left(\frac{n-1-k}{n-1}\right)\delta W_k.$$

Clearly, a necessary condition for the k-SSR to be an SSPE is the inequality

$$\left(\frac{k-1}{n-1}\right)\delta V_k-\left(\frac{1}{n-1}\right)(1-k\delta V_k)+\delta\left(\frac{1}{n-1}\right)W_k\leq 0.$$

Since $\delta\left(\frac{1}{n-1}\right)W_k\geq 0$, it is necessary that

$$\left(\frac{k-1}{n-1}\right)\delta V_k-\left(\frac{1}{n-1}\right)(1-k\delta V_k)\leq 0. \tag{41}$$

The inequality can be rearranged to

$$\delta V_k\leq\frac{1}{2k-1}.$$

It follows that

$$1-k\delta V_k\geq\frac{k-1}{2k-1}.$$

Recall that V_k is the expected payoff of the proposer induced by the supposed SSPE. Under that strategy profile, the proposer offers him/herself $1-k\delta V_k$, and the proposal is endorsed (and then implemented) with probability $k/(n-1)$. Thus, we obtain

$$V_k\geq\left(\frac{k}{n-1}\right)(1-k\delta V_k)\geq\left(\frac{k}{n-1}\right)\left(\frac{k-1}{2k-1}\right).$$

Combining the above inequalities, we obtain

$$\frac{1/\delta}{2k-1} \geq V_k \geq \left(\frac{k}{n-1}\right)\left(\frac{k-1}{2k-1}\right).$$

This leads to the condition

$$\delta \leq \frac{n-1}{k^2-k} \leq \frac{4(n-1)}{(n-1)(n+1)} = \frac{4}{n+1}.$$

The last inequality follows from the premise that $k \geq (n+1)/2$. Canceling $(n-1)$, we obtain $\delta(n+1) \leq 4$, as desired.

To prove part (*ii*), note that in view of statement (*i*), we only have to consider the case where $k = (n-1)/2$. Indeed, fix some value of $\delta \in (0,1)$ and a number n of players, and suppose that the $\frac{n-1}{2}$-SSR is an SSPE. From these premises, we are going to derive an implicit upper bound on n. On the path of play of the supposed SSPE, every proposal is endorsed with probability 1/2, and is accepted with certainty once it is endorsed. Thus, by Equation (7) we find the following expression for the expected size of the total surplus divided: $V_k + (n-1)W_k = E_k = \frac{1}{2-\delta}$. Due to sincere voting (Point 3 in Definition 2), a player accepts an endorsed proposal if and only if it gives him/her at least $Z_k = \left(\frac{\delta}{n}\right)\left(\frac{1}{2-\delta}\right)$. Consider a deviation from the supposed SSPE by the current proposer. This deviation consists of changing the offer to one player from $Y_k = \delta V_k$ to Z_k. Consequently, that player will no longer endorse the proposal but will still vote for it once it has been endorsed by some other player.

This deviation gives the proposer an expected payoff of

$$\left(\frac{n-3}{2(n-1)}\right)\left(X_k + \delta V_k - \frac{\delta}{n}\frac{1}{2-\delta}\right) + \left(\frac{1}{2} + \frac{1}{n-1}\right)\delta W_k,$$

since only $\frac{n-3}{2}$ players remain to endorse the proposal, but the payoff after endorsement increases by $Y_k - Z_k = \delta V_k - \frac{\delta}{n}\frac{1}{2-\delta}$. Also, the chance for another iteration increases by $\frac{1}{n-1}$ since the number of players who do not endorse increases by 1. Recall that playing according to the supposed SSPE gives a proposer an expected payoff of

$$V_k = \frac{1}{2} X_k + \frac{1}{2} \delta W_k.$$

Consequently, the proposer's gain from the deviation can be written as

$$\left(\frac{n-3}{2(n-1)}\right)\left(\delta V_k - \left(\frac{\delta}{n}\right)\left(\frac{1}{2-\delta}\right)\right) + \left(\frac{1}{n-1}\right)\delta W_k - \left(\frac{1}{n-1}\right) X_k.$$

Taking into account that $X_k = 1 - \frac{n-1}{2}\delta V_k$, we find the following necessary condition for the $\frac{n-1}{2}$-SSR with SSaR to be an SSPE:

$$\begin{aligned}&\left(\frac{n-3}{2(n-1)}\right)\left(\delta V_k - \left(\frac{\delta}{n}\right)\left(\frac{1}{2-\delta}\right)\right)\\ &\qquad - \left(\frac{1}{n-1}\right)\left(1 - \left(\frac{n-1}{2}\right)\delta V_k\right) + \left(\frac{1}{n-1}\right)\delta W_k \leq 0.\end{aligned}$$

In view of the fact that $W_k \geq 0$, this implies the necessary condition

$$\left(\frac{n-3}{2(n-1)}\right)\left(\delta V_k - \frac{\delta}{n}\,\frac{1}{2-\delta}\right) - \left(\frac{1}{n-1}\right)\left(1 - \left(\frac{n-1}{2}\right)\delta V_k\right) \leq 0.$$

This yields

$$\delta V_k \leq \frac{2n(2-\delta) + \delta(n-3)}{2n(2-\delta)(n-2)}.$$

In the supposed SSPE, any proposal is endorsed with probability 1/2. If a player's proposal is endorsed, then it is also accepted, and so the expected payoff of a proposer can be bounded as follows:

$$V_k \geq \left(\frac{1}{2}\right)\left(1 - \left(\frac{n-1}{2}\right)\delta V_k\right).$$

Using the bound previously derived for δV_k, we can write

$$V_k \geq \frac{1}{2} - \left(\frac{n-1}{4}\right)\left(\frac{2n(2-\delta) + \delta(n-3)}{2n(2-\delta)(n-2)}\right).$$

Now we have bounded V_k both from above and below, as follows:

$$\frac{1}{\delta}\left(\frac{2n(2-\delta)+\delta(n-3)}{2n(2-\delta)(n-2)}\right) \geq V_k \geq \frac{1}{2} - \left(\frac{n-1}{4}\right)\left(\frac{2n(2-\delta)+\delta(n-3)}{2n(2-\delta)(n-2)}\right).$$

This readily implies the inequality

$$\left(\frac{2n(2-\delta)+\delta(n-3)}{2n(2-\delta)(n-2)}\right)\left(\frac{1}{\delta}+\frac{n-1}{4}\right) \geq 1/2.$$

Arranging this inequality in the quadratic form yields

$$\delta^2\left(3n^2-10n+3\right) - \delta\left(4n^2-8n+12\right) + 16n \geq 0. \tag{42}$$

As we are only interested in the limit as $n \to \infty$, we can assume that $n \geq 5$, which implies that $n^2 - 2n > 0$ and $3n^2 - 10n + 3 > 0$. Therefore, it follows that

$$\delta^2 - \delta\left(\frac{4n^2-8n+12}{3n^2-10n+3}\right) + \left(\frac{16n}{3n^2-10n+3}\right) \geq 0. \tag{43}$$

It is easily verified that for any $n \geq 13$,[23] we have

$$\left(\frac{4n^2-8n+12}{6n^2-20n+6}\right)^2 - \left(\frac{16n}{3n^2-10n+3}\right) > 0, \tag{44}$$

and consequently, the quadratic Inequality (42) has two distinct real roots

$$\overline{\delta}(n) = \frac{4n^2-8n+12}{6n^2-20n+6} + \sqrt{\left(\frac{4n^2-8n+12}{6n^2-20n+6}\right)^2 - \frac{16n}{3n^2-10n+3}}, \tag{45}$$

$$\underline{\delta}(n) = \frac{4n^2-8n+12}{6n^2-20n+6} - \sqrt{\left(\frac{4n^2-8n+12}{6n^2-20n+6}\right)^2 - \frac{16n}{3n^2-10n+3}}. \tag{46}$$

Existence of the supposed SSPE requires that either $0 < \delta \leq \underline{\delta}(n)$ or $\overline{\delta}(n) \leq \delta \leq 1$. However, in the limit, as $n \to \infty$, we find that $\overline{\delta}(n)$ converges to $4/3 > 1$, while $\underline{\delta}(n)$

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[23] In fact, if we consider the left-hand side of Ineq. (44) as a continuous function of n, then we find that it is equal to zero for $n \approx 3.52$ and $n \approx 11.99$, negative for values of n between these two roots, and positive otherwise. Recall that in our model, we assume that n is an odd integer and that $n \geq 5$.

converges to zero. This implies that Inequality (42) cannot be satisfied for n sufficiently large. In turn, this implies that a k-SSR with $k \geq (n-1)/2$ ("majority endorsement") is not an SSPE if n is sufficiently large. □

Proof of Proposition 2. It is already established that a k-SSR with super-majority endorsement cannot be an SSPE if δ is sufficiently close to one. Hence, we only have to deal with the case where $k = (n-1)/2$.

Due to continuity in δ, we can just substitute $\delta = 1$ and $k = (n-1)/2$ into Equations (7) and obtain

$$E_k = 1, \qquad V_k = \frac{2n}{n^2+2n-1},$$
$$X_k = \frac{3n-1}{n^2+2n-1}, \qquad W_k = \frac{n+1}{n^2+2n-1}.$$

Now we can use Equations (9)–(10) to check whether the candidate under consideration is an SSPE. We find

$$\begin{aligned} \lambda^+_{(n-1)/2} &= \frac{-n^2+n-2}{(n-1)(n^2+2n-1)}, \\ \lambda^-_{(n-1)/2} &= \frac{(n-1)(n-3)-4n}{2n(n^2+2n-1)}. \end{aligned}$$

Clearly, the denominators of these expressions are strictly positive, so that we only need to determine the sign of the numerators. For $n \leq 7$, it is easily verified that the numerators are strictly negative, and hence we have an equilibrium; this shows the first part of the proposition. For $n \geq 9$, however, $(n-1)(n-3)-4n > 0$, and so we have found a profitable deviation. This shows the second part of the proposition.

□

Proof of Corollary 2. We need to show that the interval $\left[-\frac{1}{2}+\sqrt{n+\frac{1}{4}},\ \frac{1}{2}+\sqrt{n+\frac{1}{4}}\right]$ contains exactly one integer. Since this interval is closed and of unit length, it can at most contain two integers. It is sufficient to show that it does not contain two integers. Indeed, suppose towards a contradiction that the endpoints of the interval are integers. In particular, let $z = -\frac{1}{2}+\sqrt{n+\frac{1}{4}}$ be an integer. Solving for n yields $n+\frac{1}{4} = (z+\frac{1}{2})^2$. This can be rewritten as $n+\frac{1}{4} = z^2+z+\frac{1}{4}$ or as $n = z(z+1)$. Either z is even, or $z+1$

is even. Therefore, the product $z(z+1)$ is even, and so n is even. This contradicts our assumption that n is odd.

$\square$

Proof of Proposition 3. The proof of this proposition consists of two steps:

We construct a generalized k-SSR in the following way: Suppose that the stationary strategy profile σ^* is a k-SSR with SSaR and that it is an SSPE of the SORBG. The profile σ^* consists of anonymous proposals η^{i*} for every $i \in N$, amendment rules ψ^{i*} for every $i \in N$, and acceptance rules A^{i*} for every $i \in N$. Let σ^{**} be a stationary strategy profile for the ORBG, which consists of these same anonymous proposals η^{i*} for every $i \in N$, amendment rules ψ^{i*} for every $i \in N$, and acceptance rules A^{i*} for every $i \in N$, and, in addition, features the following voting rule used by every player $i \in N$: *"Whenever a vote takes place between an amendment and a proposal on the floor, every player votes in favor of the amendment."* It is trivially true that a unilateral deviation by any player from this voting rule does not change the outcome of a vote, and therefore cannot be profitable for the deviating player. If there were profitable unilateral deviations at histories other than the ones governed by the selection rule, then the k-SSR would not be an SSPE of the SORBG. Hence, we have shown that the stationary strategy profile σ^{**} is an SSPE of the ORBG.

Finally, we observe that the stationary strategy profile σ^{**} is a generalized k-SSR: The anonymous proposal and the acceptance rule under σ^{**} satisfy Points 1 and 5, respectively, of Definition 3. Note that they correspond exactly to Points 1 and 3 of Definition 2. Under σ^{**}, every player always votes in favor of any amendment against the proposal on the floor, which trivially satisfies Point 4 of Definition 3. The amendment rule under σ^{**} corresponds to that in a k-SSR as defined in Point 2 of Definition 2. This amendment rule, combined with the trivial selection rule, ensures that Points 2, 3, and 4 of Definition 3 are satisfied.

$\square$

Proof of Proposition 4. Let $n \geq 3$ be odd and $x > y > z > 0$. Let $k_1, k_2 \in \mathbb{N}_0$ be given such that $k_1 + k_2 = \frac{n-1}{2}$. Suppose there is an original proposal $O =$

$(x, y, \ldots, y, z, \ldots, z, 0, \ldots, 0) \in \mathbb{R}^n$. We introduce the following notation

$$\begin{aligned} K_1 &= \{i \in [n] : O_i = y\} \quad \text{and} \quad k_1 = |K_1|, \\ K_2 &= \{i \in [n] : O_i = z\} \quad \text{and} \quad k_2 = |K_2|, \\ K_3 &= \{i \in [n] : O_i = 0\} \quad \text{and} \quad k_3 = |K_3|, \end{aligned}$$

where we used the shorthand notation $[n] := \{1, \ldots, n\}$ and $|\cdot|$ denotes the cardinality. Fix an amendment maker $1 \leq k_a \leq n$. We are going to derive necessary and sufficient conditions for the existence of random variables $A_i : \Omega \to \{0, x, y, z\}$ for $i \in [n]$ which fulfill the following:

(i) $A_{k_a} = x$. (Swap)

(ii) $|\{i \in [n] : A_i \geq O_i\}| \geq \frac{n+1}{2}$ for all $\omega \in \Omega$. (Majority)

(iii) $\mathbb{E}[A_i] = \mathbb{E}[A_j]$ for all $i, j \in \{1, \ldots, k_a - 1, k_a + 1, \ldots, n\}$. (Fairness)

In the following Lemma we show that there is always a random amendment that fulfills all three conditions.

Lemma 5 (Existence)**.** *Assume that $n \geq 5$ and $y \geq 3z$. For all $k_1, k_2 \in \mathbb{N}_0$ such that $k_1 + k_2 = \frac{n-1}{2}$, $k_1 + k_2 \geq 1$ there exists a random amendment $\{A_i\}_{i=1}^n$ such that the conditions (i)-(iii) are satisfied.*

Before proving the lemma, we show how Proposition 4 follows from it. Recall from Equation 5 and 6 that $Y_k = \delta V_k$ and $Z_k = \frac{\delta}{n} E_k$. Since E_k is the average expected payoff and V_k the expected payoff of the proposer, it follows that $E_k \leq V_k$. Thus, we have $Y_k \geq 3 Z_k$ because we assume throughout the paper that $n \geq 3$. Therefore, the conditions of Lemma 5 are fulfilled and Proposition 4 follows.

Proof of Lemma 5. We provide a constructive proof of Lemma 5. For this end we distinguish the following cases:

Case I: $\left(k_a \leq \frac{n+1}{2}\right)$ We are first going to consider the easier case and assume that $k_a \leq$

$\frac{n+1}{2}$. We claim that the following random variables satisfy the conditions (i) - (iii):

$$A_{k_a} = x$$

$$\text{and } A_i \text{ i.i.d. such that}$$

$$\mathbb{P}[A_i = 0] = \frac{k_3}{n-1},\ \mathbb{P}[A_i = y] = \frac{k_1}{n-1} \text{ and } \mathbb{P}[A_i = z] = \frac{k_2}{n-1}.$$

Such random variables can easily be constructed by using independent and identically multinomially distributed random variables. The fairness condition (iii) and the swap condition (i) are fulfilled by construction. Observe that $k_3 = \frac{n-1}{2}$, and for all $i \in K_3$ we have

$$A_i \geq 0 = O_i.$$

Moreover,

$$A_{k_a} = x \geq O_{k_a},$$

with strict inequality unless $k_a = 1$. Hence at least

$$|K_3| + 1 = \frac{n+1}{2}$$

players weakly prefer the amendment, so the majority condition (ii) is satisfied and we are done. Note that we did not impose any condition on k_1, k_2 or x, y and z.

Case II: $\left(k_a > \frac{n+1}{2}\right)$ We set $A_n = x$. Observe that $k_a > \frac{n+1}{2}$ implies that $O_{k_a} = 0$. Hence we can assume w.l.o.g. that $k_a = n$. We are going to consider four subcases and construct for each of them a valid amendment.

Case (i) **:** $(k_1 \geq 2)$ Assume that $k_1 \geq 2$. We construct the amendment $(A_i)_{1 \leq i \leq n}$ in the following way: First choose an agent i_1 uniformly in $K_1 \cup K_2$ and assign to this agent the payoff y. This guarantees that the majority condition (ii) is satisfied. Then independently choose an agent i_2 uniformly at random from $(\{1\} \cup K_3) \setminus \{n\}$ and assign to this agent the payoff y. Finally assign the remaining values $\{\underbrace{y, \ldots, y}_{k_1-2}, \underbrace{z, \ldots, z}_{k_2}, \underbrace{0, \ldots, 0}_{k_3-1}\}$ uniformly and independently to the remaining agents. Thus, by construction it is enough to check that in the first two assignment rounds, all agents have the same probability to obtain y.

Using independence we have that for $i \in K_1 \cup K_2$

$$\mathbb{P}[i \in \{i_1, i_2\}] = \frac{1}{k_1 + k_2} = \frac{2}{n-1}.$$

For an agent $i \in \{1\} \cup K_3 \setminus \{n\}$ it holds that

$$\mathbb{P}[i \in \{i_1, i_2\}] = \frac{1}{k_3} = \frac{2}{n-1}.$$

Thus, the fairness condition is satisfied and therefore we can construct an amendment which satisfies the conditions (i)-(iii) if $k_1 \geq 2$.

Case (ii)**:** $(k_1 = 0)$ Suppose that $k_1 = 0$ and observe that this implies $k_2 \geq 2$. Thus we can use the same construction as in the case $k_1 \geq 2$ by changing the role of y and z.

Case (iii)**:** $(k_1 = 1, k_2 \geq 2)$ Assume that $k_1 = 1$ and $k_2 \geq 2$. We set $A_n = x$ and choose a $p \in [0,1]$ which we are going to fix later. Take a Bernoulli distributed random variable B with $\mathbb{P}[B = 1] = p$ and $\mathbb{P}[B = 0] = 1 - p$ for some $p \in [0,1]$. If $B = 1$, we are going to distribute independently of B the value y among the agents K_2. This ensures that the majority condition is satisfied. Denote the agent which obtains y by i_y. Then distribute independently the remaining z's among the remaining agents $\{1, \ldots, n-1\} \setminus \{i_y\}$. If $B = 0$, we distribute one y and one z uniformly at random and independently among the agents of $\{1\} \cup K_1 \cup K_3 \setminus \{n\}$. To ensure that the majority condition is satisfied, distribute another z among the agents of K_2. We denote indices of the agents in K_2 which obtained z by i_{z_2} and the one in $\{1\} \cup K_1 \cup K_3 \setminus \{n\}$ by i_{z_1}. Then distribute the remaining payoffs among all other agents uniformly and independently. Using the linearity of the expectation we can write the expected payoff for each agent $i \in (K_1 \cup K_2 \cup K_3) \setminus \{n\}$ as

$$\begin{aligned}\mathbb{E}[A_j] =& \mathbb{P}[B = 1] \left(\mathbb{E}\left[A_j \mathbb{1}_{\{i_y = j\}} \mid B = 1\right] + \mathbb{E}\left[A_j \mathbb{1}_{\{i_y \neq j\}} \mid B = 1\right] \right) \\ &+ \mathbb{P}[B = 0] \left(\mathbb{E}\left[A_j \mathbb{1}_{j \in \{i_y, i_{z_1}, i_{z_2}\}} \mid B = 0\right] + \mathbb{E}\left[A_j \mathbb{1}_{j \notin \{i_y, i_{z_1}, i_{z_2}\}} \mid B = 0\right] \right).\end{aligned}$$

Hence, we can compute the expectation explicitly. If $i \in K_2$, we obtain

$$\mathbb{E}[A_i] = p\left(\frac{1}{k_2}y + \frac{k_2-1}{n-2}z\right) + (1-p)\left(\frac{1}{k_2}z + \frac{(k_2-1)(k_2-2)}{k_2(n-4)}z\right),$$

and for $i \in (\{1\} \cup K_1 \cup K_3) \setminus \{n\}$

$$\mathbb{E}[A_i] = p\frac{k_2}{n-2}z + (1-p)\left(\frac{1}{k_3+1}(y+z) + \frac{(k_3-1)(k_2-2)}{(k_3+1)(n-4)}z\right).$$

Note that since $k_1 = 1$, it holds that $k_2 = k_3 - 1$. To satisfy the fairness condition *(iii)*, we therefore need to choose p such that

$$\begin{aligned} p\left(\frac{1}{k_2}y - \frac{1}{n-2}z\right) &+ (1-p)\left(\frac{1}{k_2}z + \frac{(k_2-1)(k_2-2)}{k_2(n-4)}z\right) \\ &= (1-p)\left(\frac{1}{k_2+2}(y+z) + \frac{k_2(k_2-2)}{(k_2+2)(n-4)}z\right). \end{aligned}$$

This is equivalent to

$$\begin{aligned} &\underbrace{p\left(\left(\frac{k_2(k_2-2)}{(k_2+2)(n-4)} - \frac{1}{n-2} - \frac{(k_2-1)(k_2-2)}{k_2(n-4)} - \frac{2}{k_2(k_2+2)}\right)z + \frac{2k_2+2}{k_2(k_2+2)}y\right)}_{:=w} \\ &\quad = \underbrace{\left(\frac{1}{k_2+2} + \frac{k_2(k_2-2)}{(k_2+2)(n-4)} - \frac{1}{k_2} - \frac{(k_2-1)(k_2-2)}{k_2(n-4)}\right)z + \frac{1}{k_2+2}y}_{:=v}. \end{aligned}$$

Hence we can write the fairness condition as $pw = v$ and $p = \frac{v}{w}$. To guarantee that $\frac{v}{w} \in [0,1]$ it is enough to verify that $v \leq w$ and $v > 0$. We can bound v by

$$\begin{aligned} v &= -\left(\frac{2}{k_2(k_2+2)} + \frac{(k_2-2)^2}{k_2(k_2+2)(n-4)}\right)z + \frac{1}{k_2+2}y \\ &= \frac{1}{k_2+2}\left(y - \left(\frac{2}{k_2} + \frac{(k_2-2)^2}{k_2(n-4)}\right)z\right) \\ &\geq \frac{1}{k_2+2}(y-z) > 0, \end{aligned}$$

where we used in the first inequality that $k_2 \leq n-3$ and in the second the assumption $y > z$. Moreover

$$w \geq v \quad \Longleftrightarrow \quad \frac{y}{k_2} \geq \frac{z}{n-2}.$$

Thus there exists a $p \in [0,1]$, namely $p = \frac{v}{w}$, such that the fairness condition is satisfied and we have found a valid amendment.

Case *(iv)***:** $(k_1 = k_2 = 1)$ Suppose that $k_1 = k_2 = 1$. As before, let $p \in [0,1]$ be fixed later. With probability p, we assign to the third agent the value z, i.e. $A_3 = z$, and distribute y randomly among the three remaining agents $\{1,2,4\}$. In the other case with probability $1-p$, assign to the third agent the value y, and distribute z randomly among the three remaining agents. Since the third agent has originally obtained the value z and obtains in the amendment either the payoff y or z, the majority condition is fulfilled. Next, we compute the expected payoff. For $i = 3$ we have

$$\mathbb{E}[A_3] = pz + (1-p)y,$$

and for $i \in \{1,2,4\}$

$$\mathbb{E}[A_i] = \frac{1}{3}\left(py + (1-p)z\right).$$

Thus, the fairness condition is equivalent to

$$pz + (1-p)y = \frac{1}{3}\left(py + (1-p)z\right) \quad \Longleftrightarrow \quad p = \frac{3y - z}{4(y-z)}.$$

Since $y \geq 3z$ by assumption, this fraction is positive and in $[0,1]$. Thus, we can always find a $p \in [0,1]$ such that the fairness condition holds, and therefore the conditions $(i) - (iii)$ are fulfilled.

□

Proof of Theorem 2. Since for each fixed k the payoff $V_k(\delta)$ is continuous in δ at $\delta = 1$ and the set $\{1, \ldots, n-1\}$ is finite, any strict ranking of the values V_k obtained at $\delta = 1$ remains valid for all $\delta < 1$ sufficiently close to 1. We solve Equations (19)–(21) for

$\overline{V}_k$:

$$\overline{V}_k = \begin{cases} \frac{2k^2(n-1)+2(n-1)n+k(n^2-2n-1)}{2n(k^2(n-1)+n(n-1)-k)}, & \text{if } k \leq (n-3)/2, \\ \frac{n+k(n-2)-1}{k^2(n-1)+n(n-1)-k}, & \text{if } k \geq (n-1)/2. \end{cases} \tag{47}$$

We have already argued that the generalized $\widehat{k}$-SSR can only be an SSPE if $\overline{V}_{\widehat{k}} \geq \overline{V}_k$ for every $k = 1, \ldots, n-1$.

As an auxiliary, it is useful to define the continuous function

$$\nu(\kappa) := \frac{2\kappa^2(n-1) + 2(n-1)n + \kappa(n^2 - 2n - 1)}{2n(\kappa^2(n-1) + n(n-1) - \kappa)}$$

for the real-valued variable κ. We observe that $\nu(\kappa)$ exists for all $\kappa \in [1, (n-3)/2]$. Its derivative can be written as

$$\nu'(\kappa) = -\frac{(n-1)^3(\kappa^2 - n)}{(2n(\kappa^2(n-1) - \kappa + n(n-1)))^2},$$

and we can easily verify that

$$\begin{aligned} \nu'(\kappa) &> 0 \text{ if } \kappa \in [1, \sqrt{n}), \\ \nu'(\kappa) &= 0 \text{ if } \kappa = \sqrt{n}, \\ \nu'(\kappa) &< 0 \text{ if } \kappa \in (\sqrt{n}, (n-3)/2]. \end{aligned}$$

We see that $\nu(\kappa)$ attains its unique maximum at the point $\kappa = \sqrt{n}$. So far, we have shown that the generalized $\widehat{k}$-SSR can only be an SSPE if either $\sqrt{n} - 1 < \widehat{k} < \sqrt{n} + 1$ or $\widehat{k} \geq \frac{n-1}{2}$. In order to complete the proof of the theorem, we want to exclude the latter possibility.

To this end, we first note that $\overline{V}_{\widehat{k}} \leq \nu(\widehat{k})$ for any $\widehat{k} \geq \frac{n-1}{2}$. In order to see this, consider the following sequence of implications:

$$\begin{aligned}
\widehat{k} &\geq \frac{n-1}{2}, \\
0 &\leq 2\widehat{k} - (n-1), \\
0 &\leq \widehat{k}^2 \left(\frac{n-1}{n} \right) - \left(\frac{\widehat{k}}{2} \right) \left(\frac{(n-1)^2}{n} \right), \\
0 &\leq \widehat{k}^2 \left(\frac{n-1}{n} \right) + \widehat{k} \left(\frac{n^2 - 2n - 1}{2n} \right) - \widehat{k}(n-2), \\
n + \widehat{k}(n-2) - 1 &\leq \widehat{k}^2 \left(\frac{n-1}{n} \right) + (n-1) + \frac{\widehat{k}}{2n} \left(n^2 - 2n - 1 \right), \\
\overline{V}_{\widehat{k}} &\leq \nu(\widehat{k}).
\end{aligned}$$

Now suppose by way of contradiction that for some $\widehat{k} \geq \frac{n-1}{2}$, the generalized $\widehat{k}$-SSR is an SSPE. Then, for any $k \in \{1, \dots n-1\} \setminus \{\widehat{k}\}$ it holds that

$$\overline{V}_k \leq \overline{V}_{\widehat{k}} \leq \nu(\widehat{k}).$$

Due to the premise that $n \geq 15$[24], we have that $\sqrt{n} + 1 < \frac{n-1}{2}$. Thus there is an integer $\widetilde{k} \in (\sqrt{n} - 1, \ \sqrt{n} + 1)$ such that

$$\nu(\widetilde{k}) = \overline{V}_{\widetilde{k}} > \nu(\widehat{k}) \geq \overline{V}_{\widehat{k}}.$$

Since $\overline{V}_{\widetilde{k}} > \overline{V}_{\widehat{k}}$, we have obtained the desired contradiction and the proof of the theorem is complete.

$\square$

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[24] The bound $n \geq 15$ is sufficient but not tight. The only inequality needed is $\sqrt{n} + 1 < \frac{n-1}{2}$, which is equivalent to $n > 9$. Under the assumption that n is odd, this reduces to $n \geq 11$. Thus the argument remains valid with the weaker threshold $n \geq 11$.

Data Availability Statement (DAS)

The code underlying Figure 1 and Tables 1 & 2 is available online:

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