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The Economics of Equilibrium with Indivisible Goods

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This paper develops a theory of competitive equilibrium with indivisible goods based entirely on economic conditions on demand. The key idea is to analyze complementarity and substitutability between bundles of goods, rather than merely between goods themselves. This approach allows us to formulate sufficient and essentially necessary conditions for equilibrium existence—which unify settings with complements and settings with substitutes. Our analysis has implications for auction design.

Key words: competitive equilibrium, indivisible goods, auctions

1. INTRODUCTION

Much of economic analysis relies on goods being perfectly divisible. For example, standard convexity conditions—which are critical to ensuring the existence of market-clearing prices (Arrow and Debreu, 1954; McKenzie, 1954)—are incompatible with indivisibilities.

Allowing for indivisibilities is an issue of economic importance due to their relevance for market design. For example, an underlying goal behind many auction designs is to guide markets toward competitive equilibrium outcomes.[1] Moreover, some prominent auction formats directly rely on equilibrium existence as part of their protocols (see, e.g., Milgrom (2009) and Klemperer (2010)). Developing an economic theory of equilibrium with indivisible goods is therefore an issue not only of theoretical interest, but also of practical importance.

One celebrated case for which there is a well-behaved theory of auctions and markets with indivisible goods is the case of gross substitutes. With divisible goods, gross substitutes provides a condition under which competitive equilibria are (dynamically) stable and tâtonnement price dynamics converge to equilibrium (Arrow and Hurwicz,

1. See, e.g., Milgrom (2000, 2009), Ausubel and Milgrom (2002), Ausubel, Cramton, and Milgrom (2006), Klemperer (2010), and Milgrom and Segal (2020).

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1958; Arrow, Block, and Hurwicz, 1959). With indivisible goods, Kelso and Crawford (1982) showed that gross substitutes also provides a sufficient condition for the existence of competitive equilibria. This theoretical result underpins practical auction designs for substitute indivisible goods (Gul and Stacchetti, 2000; Milgrom, 2000, 2009; Ausubel and Milgrom, 2002; Klemperer, 2010). But the gross substitutes condition is also rather restrictive: it entirely rules out complementarities, which are a central feature of many auctions for multiple goods.[2]

If the market contains only two indivisible goods, the case of complements is isomorphic to the case of substitutes. However, with more than two indivisible goods, the case of complements is more delicate. For example, equilibrium does not generally exist when all goods are complements (Bikhchandani and Mamer, 1997), yet certain patterns of complementarities are compatible with the existence of equilibrium (Greenberg and Weber, 1986; Danilov, Koshevoy, and Lang, 2013).[3] As a result, it has been difficult to uncover the economic issues underlying the equilibrium existence problem in its full generality.

The goal of the current paper is to develop a unified economic theory of equilibrium for settings with indivisible goods that allows for complementarity and substitutability. Our main result provides a condition on the structure of complementarity and substitutability that is sufficient, and essentially necessary, for the existence of competitive equilibria in markets for indivisible goods. To formulate this result, we analyze complementarity and substitutability between bundles of goods, rather than merely between goods themselves.

For ease of exposition, we first focus on the case in which each agent demands at most one unit of each good. An essentially necessary condition for guaranteeing equilibrium existence is that no two goods be substitutes for one agent and complements for another—that is, that each pair of goods either be consistently substitutes or consistently complements.[4] For example, if Lex regards apples and bananas as perfect complements and values the basket of fruit at \$1, and Vincent would like to buy either piece of fruit for \$1, then it is impossible to clear the market if Claude has one apple and one banana available for sale and is unwilling to cut the fruit (Henry, 1970; Kelso and Crawford, 1982).

This consistency condition is not sufficient to guarantee the existence of equilibrium (beyond the case of substitutes). For example, it is always satisfied if all goods are complements.

Our key conceptual insight is to analyze the comparative statics of demand for bundles of goods and impose an analogous consistency condition on complementarity and substitutability. More precisely, we introduce a *bundle consistency* condition that requires that each pair of appropriate bundles of goods be either consistently

2. See, e.g., Palacios-Huerta et al. (2024) for a survey. Complementarities are also a central feature of most settings with production (see, e.g., Milgrom and Roberts (1990)).

3. Beyond the auction and exchange economy context, Rostek and Yoder (2020) showed that equilibria exist in matching markets in which all contracts are complementary. However, analogues of this result for exchange economies require including brokers in the economy in a particular way (Rostek and Yoder, 2025a).

4. This observation was made in the quasilinear case to provide a counterexample to equilibrium existence by Kelso and Crawford (1982) and to provide a maximal domain result by Gul and Stacchetti (1999), and extended to the case with income effects by Baldwin, Jagadeesan, Klemperer, and Teytelboym (2023); see also Henry (1970, Section 3.3) and Yang (2017). As we formalize in Section 5.1, this condition is only necessary to guarantee equilibrium existence for domains that are *invariant* in a sense that we introduce.

complements or consistently substitutes. As bundling only affects the presentation of the market in competitive equilibrium analysis, equilibrium existence does not depend on how goods are bundled. Hence, we can show that bundle consistency is an essentially necessary condition for equilibrium existence.

Our main technical contribution is to show that, more surprisingly, bundle consistency is also a sufficient condition for equilibrium existence. Thus, once bundles are also considered, the *only* obstructions to equilibrium existence with indivisible goods are inconsistencies between substitutability and complementarity.

The observation that underpins our results is that the structure of complementarity and substitutability across bundles of goods can expose obstructions to equilibrium existence that are not apparent from the structure of complementarity and substitutability between goods themselves. In particular, our bundle consistency condition has bite even if all goods are (consistently) complements. The following example illustrates these points.

Example ***1 (Bundle Inconsistency with a Three-Cycle of Complements)*** *Suppose that there are three goods—apples, bananas, and coconuts—and that agents see pairs of goods as complements. It turns out that competitive equilibria do not generally exist in this setting (Bikhchandani and Mamer, 1997, page 390). Even though there is no inconsistency between substitutability and complementarity for any pair of goods, an inconsistency emerges when goods are bundled. Indeed, suppose that bananas are bundled with apples, but apples (and coconuts) can also be bought and sold alone. To buy a banana without an apple, an agent would buy the bundle and sell an apple. Due to the complementarity between bananas and coconuts, an increase in the price of coconuts can lower demand for bananas. With bundling, this leads to selling fewer apples, thereby raising excess demand for apples and exposing a substitutability between coconuts and apples.*[5] *To see a bundle inconsistency, note that apples and coconuts also remain complements (for reasons unrelated to bananas).*

Other patterns of complementarities are bundle-consistent; these patterns turn out to be compatible with equilibrium existence. The following variation of Example 1 illustrates.

Example ***2 (Bundle Consistency in Consecutive Games)*** *Suppose that apples and bananas are complements, as are bananas and coconuts, but that unlike in Example 2, coconuts are only complementary to apples in combination with bananas. This pattern of complementarity is compatible with equilibrium existence (Greenberg and Weber, 1986). And indeed, this pattern of complementarity is bundle-consistent. As in Example 1, if bananas are bundled with apples, and apples can be bought and sold alone, then coconuts and apples still become substitutes. However, this substitutability does not create an inconsistency of complementarity and substitutability between apples and coconuts: bundling apples and bananas makes coconuts become complementary only to the apple–banana bundle—rather than to apples.*

The difference between the patterns of complementarity between goods across Examples 1 and 2 is subtle—whether coconuts are complementary directly to apples, or

5. The general principle at play here is that, even with divisible goods, whether a pair of goods (or bundles) is substitutes or complements depends on the choice of definition of other goods or bundles (Weinstein, 2022).

to apples in conjunction with bananas. In particular, all pairs of goods are complements in both examples. Considering the structure of complementarity and substitutability when goods are bundled makes the economic distinction between the two examples transparent.

A conceptual challenge in formalizing the concept of bundle consistency is to identify *which* bundles the agents must see as consistently substitutable or complementary. When goods themselves are complements, the relevant bundles are the ones consisting of sets of goods that some agent sees as (strict) complements. Moreover, when two goods are substitutes, there is an underlying complementarity between either good and the opportunity to sell the other good (Sun and Yang, 2006; Ostrovsky, 2008; Hatfield et al., 2013, 2019).[6] In light of this "hidden" complementarity, we consider bundles that include opportunities to sell. The relevant bundles then consist of goods and opportunities to sell that some agent sees as (strict) complements. With indivisible goods, we show that price effects can capture which bundles consist of strict complements; this logic applies equally well to bundles that include opportunities to sell. Therefore, by treating goods and opportunities to sell symmetrically, our analysis unifies settings with substitutes with those with complements. It shows that considering the comparative statics of demand when complements can be bundled allows us to detect all possible obstructions to equilibrium existence with indivisible goods.

Our results extend to the more general case in which agents can demand multiple units of each good. In that case, in addition to inconsistencies between substitutability and complementarity *across goods or bundles*, it is possible that there be inconsistencies between substitutability and complementarity *across units of the same good*. Intuitively, units of the same good are mechanically substitutes, so complementarities between units of the same good lead to inconsistencies between substitutability and complementarity among units. We therefore require that units of the same good be substitutes for each other—though they may be complementary to (units of) other goods. This *unit consistency* condition is closely related to the condition in standard equilibrium theory that there be no increasing returns to scale: with one indivisible good, increasing returns arise if and only if units of the good are not substitutes. We show that unit and bundle consistency are together sufficient, and essentially necessary, for equilibrium existence when agents can demand multiple units of any good.

Our proofs of equilibrium existence rely on a mathematical characterization of our bundle consistency property. Specifically, we show that bundle consistency is closely related to "total unimodularity"—a mathematical property that is germane to integer programming and discrete convex analysis.[7] This connection lets us develop our sufficiency results via combinatorial- and tropical-geometric results on equilibrium existence that were proven by Danilov, Koshevoy, and Murota (2001) and Baldwin and Klemperer (2019). (We discuss those results in more detail in the literature review and Appendix A.) Note that while the parts of our proofs use combinatorial-geometric machinery, geometric concepts are not at all needed to understand the statements of, or the economic content of, our results.

6. Note that Sun and Yang (2006), Ostrovsky (2008), and Hatfield et al. (2013) extracted an underlying substitutability between two goods from a complementarity between one good and the opportunity to sell the other good; here, we apply the converse, which Hatfield et al. (2019) showed also holds.

7. This connection also shows that starting from the set of all agents' price effects, there is an algorithm to test for bundle consistency that runs in time that is polynomial in the number of goods.

Our analysis has implications for the design of combinatorial auctions and exchanges based on competitive equilibrium pricing. Beyond the case of substitute goods, much existing work has focused on designing the lots that are sold in the auction to be bundles that the bidders view as substitutes—which is possible, for example, in power markets and certain spectrum auctions (Milgrom, 2007, 2009, 2019).[8] However, complementarities, in general, are impervious to being recast as substitutes in this way. Even when no bundling can fully restore substitutability, our analysis highlights how bundling complements can uncover aspects of preferences crucial to clearing markets, and suggests that such bundling can provide new opportunities for eliciting richer preferences and developing practical auction designs.

Related literature. A large literature in economics has studied the equilibrium existence problem beyond the case of substitutes. One strand of this literature (Greenberg and Weber, 1986; Sun and Yang, 2006, 2009; Hatfield et al., 2013; Fleiner et al., 2019) has developed sufficient conditions for equilibrium existence by guaranteeing that there is a family of (strong) substitute bundles; equilibrium existence then follows from Kelso and Crawford (1982). While this approach has led to elegant and economically interpretable sufficient conditions, the ability to bundle goods in a way that restores substitutability is not necessary for equilibrium existence (Danilov, Koshevoy, and Lang, 2013; Baldwin and Klemperer, 2019); bundle consistency is a strictly weaker condition (see Section 7). Thus, while Kelso and Crawford's (1982) substitutability property is not the only fundamental reason for equilibrium existence, our results show that their counterexample is the only fundamental obstruction to equilibrium existence once bundles are considered.

A second strand of literature has focused on developing necessary and sufficient conditions for equilibrium existence for specific valuation profiles (Bikhchandani and Mamer, 1997; Ma, 1998). While these conditions are sharp, they are expressed in terms of properties of integer and linear programs.[9] Our approach does not lead to necessary conditions for specific valuation profiles, but identifies sufficient and essentially necessary conditions in terms of versions of the standard properties of complementarity and substitutability.

Our results have a closer mathematical connection to geometric approaches to the existence of equilibrium with indivisible goods. Danilov, Koshevoy, and Murota (2001) developed a theory of convexity for settings with indivisible goods by using methods from combinatorial geometry. In their framework, each set $\mathcal{D}$ of integer vectors satisfying a condition called "total unimodularity" gives a "class of discrete convexity," defined as the preferences for which the edge directions of the convex hulls of (Hicksian) demand sets are elements of $\mathcal{D}$.[10] They showed each class of discrete convexity gives a domain for equilibrium existence.

For the quasilinear case, Baldwin and Klemperer (2019) took a dual geometric approach to constructing Danilov, Koshevoy, and Murota's (2001) domains for equilibrium existence. Baldwin and Klemperer (2019) represented preferences in terms of the set of price vectors at which demand is nonunique. Their key insight was to observe

8. Example 2 provides a simple illustration of a preference structure that can arise in these markets.

9. These analyses have also been useful for developing sufficient conditions for equilibrium existence under specific functional forms. For example, Candogan et al. (2015) employed Bikhchandani and Mamer's (1997) results to show equilibrium existence for a class of quadratic valuations where the structure of superadditivity and subadditivity of valuations can be described by a tree graph.

10. These demand sets are also required to include all integer points in their convex hulls.

that for quasilinear preferences over indivisible goods and money, this set is a space of the type studied in the mathematical area of tropical geometry. They then classified valuations into "demand types": each demand type is defined by a set $\mathcal{D}$ of integer vectors that places mathematically natural constraints on the tropical-geometric representation. They showed that when $\mathcal{D}$ is totally unimodular, the class of valuations of demand type $\mathcal{D}$ recovers the quasilinear case of the class of discrete convexity corresponding to $\mathcal{D}$; the class of valuations of each (totally) unimodular demand type thus gives a domain for equilibrium existence.[11] Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020) extended part of Baldwin and Klemperer's (2019) work to settings with income effects and recovered classes of discrete convexity beyond the quasilinear case.

Danilov, Koshevoy, and Lang (2003, 2013) showed that there are classes of discrete convexity corresponding to well-understood domains for equilibrium existence with transferable utility, including the case of substitutes.[12] Most classes of discrete convexity, and the corresponding totally unimodular demand types, introduced novel domains. However, the economic content of these new domains for equilibrium existence has remained elusive.[13] Our approach defines mathematically equivalent domains to Danilov, Koshevoy, and Murota (2001), but in terms of economic conditions on the structure of complementarity and substitutability across bundles, rather than relying on mathematical concepts from combinatorial, convex, or tropical geometry; we discuss the relationship in more detail in Appendix A. We also use Danilov, Koshevoy, and Murota's (2001) results (and other aspects of the theory of discrete convexity) as steps in the proofs of our sufficiency results.

Our focus on demand for bundles of goods, rather than goods for themselves, has a conceptual relationship to three recent analyses with divisible goods. Weinstein (2022) introduced a critique of the standard definitions of substitutes and complements based on their dependence on exactly how goods are bundled. This issue arises in our setting, where how all goods are bundled can affect whether two goods are substitutes or complements. We address this issue by focusing on a natural family of relevant bundles. Galeotti et al. (2025) used bundles corresponding to eigenvectors of the Slutsky matrix to investigate optimal tax and subsidy policy. In our setting, there is no Slutsky matrix due to the presence of indivisibilities. Rostek and Yoder (2025b) showed that it is possible to design a core-selecting reallocative auction if there is a way to bundle goods and/or sale opportunities to make all bundles independent for (and efficiently allocated in nonnegative quantities to) all bidders. Our bundle consistency condition only requires consistency between substitutability and complementarity for each pair of bundles, but

11. To recover classes of discrete convexity and obtain equilibrium existence, valuations must also be *pseudoconcave* in that demand sets include all integer points in their convex hulls. See also Footnote 10.

12. Danilov, Koshevoy, and Lang (2003) showed that there is a class of discrete convexity corresponding to the class of (strong) substitutes valuations (Kelso and Crawford, 1982; Gul and Stacchetti, 1999; Milgrom and Strulovici, 2009); see also Shioura and Tamura (2015). Danilov, Koshevoy, and Lang (2013) showed that there are classes of discrete convexity corresponding to the class of "(gross) substitutes and complements" valuations (Sun and Yang, 2006, 2009), and to the class of valuations for "consecutive games" (Greenberg and Weber, 1986); see also Baldwin and Klemperer (2019).

13. Baldwin and Klemperer (2019, page 868) asserted, based on their Proposition 3.3, that "a demand type is defined by a list of vectors that give the possible ways in which [...] demand can change in response to a small generic price change" (see also Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020, page 24) for a similar assertion for settings with income effects). However, this assertion is incorrect even in the quasilinear setting, for reasons that Footnote 35 in Appendix A.2 will explain.

requires that this property hold for all relevant ways of bundling goods and/or sale opportunities.

Subsequent to our original work, Nguyen and Teytelboym (2024) have applied our approach to developing essentially necessary conditions for equilibrium existence in transferable utility economies to relate the equilibrium existence problem for transferable utility economies to the implementability problem for random allocations in pseudomarkets.

2. SETTING

The setup follows Baldwin, Jagadeesan, Klemperer, and Teytelboym's (2023) model of exchange economies with indivisible goods and money. There is a set I of indivisible goods and a set J of agents. Agents have preferences over consumption vectors $\mathbf{x}$ of indivisible goods (or bads) and a continuously divisible numéraire, which we call money. Formally, each agent j has a non-empty domain $X^j \subseteq \{0,1,\ldots,M\}^I$ (where M is a positive integer) of feasible consumption vectors of indivisible goods, and a utility function $U^j = U^j(x_0,\mathbf{x}) : \mathbb{R} \times X^j \to \mathbb{R}$, which we assume is continuous, strictly increasing in the quantity x_0 of money, and satisfies

$$\lim_{x_0 \to -\infty} U^j(x_0,\mathbf{x}) = -\infty \quad \text{and} \quad \lim_{x_0 \to \infty} U^j(x_0,\mathbf{x}) = \infty \qquad \text{for all } \mathbf{x} \in X.$$ [14]

Each agent j is endowed with an amount w_0^j of money and an feasible consumption vector $\mathbf{w}^j$. Given an endowment $(w_0^j,\mathbf{w}^j) \in \mathbb{R} \times X^j$ for each agent j, a *competitive equilibrium* consists of a price vector $\mathbf{p} \in \mathbb{R}^I$ and a profile of consumption vectors $(\mathbf{x}^j)_{j \in J}$ such that

$$\mathbf{x}^j \in \operatorname*{argmax}_{\mathbf{x} \in X^j} \left\{ U^j\left(w_0^j - \mathbf{p}\cdot(\mathbf{x}-\mathbf{w}^j),\mathbf{x}\right)\right\} \text{ for each agent } j \quad \text{and} \quad \sum_{j \in J} \mathbf{x}^j = \sum_{j \in J} \mathbf{w}^j.$$

Here, $\mathbf{p}\cdot(\mathbf{x}-\mathbf{w}^j)$ is the net cost of consuming $\mathbf{x}$ given an endowment $\mathbf{w}^j$ of goods.

To analyze settings both with and without income effects, we focus on (Hicksian) demand. Following Baldwin, Jagadeesan, Klemperer, and Teytelboym (2023), by analogy with the standard divisible good setting, we define (Hicksian) demand as the set of feasible bundles that minimize the expenditure of achieving a target utility level at given prices. Formally, define the (Hicksian) *demand* of an agent j at a price vector $\mathbf{p}$ for a utility level u by

$$D_{\mathrm{H}}^j(\mathbf{p};u) = \operatorname*{argmin}_{\mathbf{x} \in X^j} \left\{ \min_{x_0 \mid U^j(x_0,\mathbf{x}) \geq u} \{x_0 + \mathbf{p}\cdot\mathbf{x}\} \right\}.$$

When an agent does not experience income effects, their utility function has a quasilinear representation $U^j(x_0,\mathbf{x}) = x_0 + V^j(\mathbf{x})$, where $V^j : X^j \to \mathbb{R}$ is a *valuation* for

14. Utility can obviously take range in any open interval instead. Here, agents are allowed to borrow money, but would never choose to borrow and spend an infinite amount of money. All the results continue to hold if agents cannot borrow and find it essential to end up with a positive amount of money (as in Henry (1970) and Baldwin, Jagadeesan, Klemperer, and Teytelboym (2023)).

bundles of indivisible goods. Without income effects, an agent's demand is simply given by

$$D^j(\mathbf{p}) = \underset{\mathbf{x}\in X^j}{\arg\max}\left\{V^j(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}\right\}.$$

In particular, in the case without income effects, an agent's endowment of money does not affect their demand, and we can therefore leave endowments of money unspecified.

Intuitively, a good i is substitutable (resp. complementary) to good k if whenever the price of good i increases, the demand for good k weakly increases (resp. weakly decreases). Formally, we say that a good i is *substitutable* (resp. *complementary*) *to good k for agent j* if for all utility levels u, price vectors $\mathbf{p}$ with $D^j_{\mathrm{H}}(\mathbf{p};u) = \{\mathbf{x}\}$, new prices $p'_i < p_i$, and new demands $\mathbf{x}' \in D^j_{\mathrm{H}}(p'_i, \mathbf{p}_{I\smallsetminus\{i\}};u)$, we have that $x'_k \le x_k$ (resp. $x'_k \ge x_k$).[15] In Appendix B, we show that our definition is equivalent to several alternative ways of defining substitutability and complementarity for pairs of indivisible goods based on conditions from the literature.

If goods are not substitutes (resp. complements) for some agent, then we say that they are strict complements at some prices (resp. strict substitutes at some prices). That is, we say that good i is *strictly complementary* (resp. *strictly substitutable*) *to good k at some prices*, if there exist an agent j, a utility level u, a price vector $\mathbf{p}$ with $D^j_{\mathrm{H}}(\mathbf{p};u) = \{\mathbf{x}\}$, a new price $p'_i < p_i$, and a new demand $\mathbf{x}' \in D^j_{\mathrm{H}}(p'_i, \mathbf{p}_{I\smallsetminus\{i\}};u)$ such that that $x'_k > x_k$ (resp. $x'_k < x_k$).

3. COMPLEMENTARITY AND SUBSTITUTABILITY FOR BUNDLES

Rather than only focusing on whether pairs of goods are substitutes or complements, we will need to consider whether bundles of goods are substitutes or complements.

3.1. *Preferences over bundles*

Formally, a *bundling* is a set $\mathcal{B} \subseteq \{-1,0,1\}^I$ of bundles $\mathbf{b}$ that forms a basis for $\mathbb{R}^I$.

In particular, all consumption vectors that include at most one unit of each good are potential bundles. Moreover, we allow bundles in a bundling to include negative components, which represent sale opportunities included in a bundle. For example, the bundle $\mathbf{b} = (1,-1)$ represents the combination of the first good with an opportunity to sell the second good. Considering bundles that include sale opportunities is important as sale opportunities can be complementary to goods (Sun and Yang, 2006; Ostrovsky, 2008; Hatfield et al., 2013).

Given a (potentially fractional and negative) quantity vector $\mathbf{q} \in \mathbb{R}^{\mathcal{B}}$ of bundles, there is a corresponding (potentially fractional and negative) bundle $\mathbf{x}(\mathbf{q};\mathcal{B}) = \sum_{\mathbf{b}\in\mathcal{B}} q_{\mathbf{b}}\mathbf{b} \in \mathbb{R}^I$. Here, $q_{\mathbf{b}}$ represents the quantity of bundle $\mathbf{b}$ included in the quantity vector $\mathbf{q}$. Since bundlings form bases for $\mathbb{R}^I$, any feasible consumption bundle can be uniquely achieved by consuming only bundles in any bundling, though potentially in fractional and/or negative quantities.

15. The condition that all goods be substitutes corresponds to Baldwin, Jagadeesan, Klemperer, and Teytelboym's (2023) "net substitutes" condition. We omit the modifier "net" as in standard consumer theory.

We next define demand for bundles when bundles, rather than goods, are priced. In the case of an agent j with quasilinear preferences, we can define the *bundled demand*, given a bundling $\mathcal{B}$, at a bundle price vector $\widetilde{\mathbf{p}}\in\mathbb{R}^{\mathcal{B}}$ by

$$\widetilde{D}^j(\widetilde{\mathbf{p}};\mathcal{B})=\underset{\mathbf{q}\in\mathbb{R}^{\mathcal{B}}|\mathbf{x}(\mathbf{q};\mathcal{B})\in X^j}{\operatorname{argmax}}\left\{V^j(\mathbf{x}(\mathbf{q};\mathcal{B}))-\widetilde{\mathbf{p}}\cdot\mathbf{q}\right\}.$$

Here, bundled demand simply expresses demand in terms of bundles from the given bundling. For example, if an agent's demand at certain prices were $(0,1)$ and the bundling were $\{(1,0),(1,1)\}$, then bundled demand at the corresponding bundle prices would be given by $(q_{(1,0)},q_{(1,1)})=(-1,1)$. That is, if bananas are bundled with apples and apples can be bought and sold alone, then to buy an banana alone, an agent would buy the bundle and sell an apple. More generally, allowing for income effects, given an agent j, a utility level u, and a bundling $\mathcal{B}$, we define *bundled demand* at bundle price vector $\widetilde{\mathbf{p}}\in\mathbb{R}^{\mathcal{B}}$ by

$$\widetilde{D}^j_{\mathrm{H}}(\widetilde{\mathbf{p}};u,\mathcal{B})=\underset{\mathbf{q}\in\mathbb{R}^{\mathcal{B}}|\mathbf{x}(\mathbf{q};\mathcal{B})\in X^j}{\operatorname{argmin}}\left\{\min_{x_0|U^j(x_0,\mathbf{x}(\mathbf{q};\mathcal{B}))\geq u}\{x_0+\widetilde{\mathbf{p}}\cdot\mathbf{q}\}\right\}.$$

To define substitutability and complementarity between bundles, analogously to case of goods, we look at the effect of changing the price of one bundle (holding the prices of other bundles fixed) on bundled demand. We also consider whether a pair of bundles is consistent in their substitutability or complementarity.

Definition 1. *Given a bundling $\mathcal{B}$ and bundles $\mathbf{b},\mathbf{c}\in\mathcal{B}$:*

- *Bundle* $\mathbf{b}$ *is* substitutable *(resp.* complementary*) to bundle* $\mathbf{c}$ *if for all agents j, utility levels u, price vectors $\widetilde{\mathbf{p}}$ with $\widetilde{D}^j_{\mathrm{H}}(\widetilde{\mathbf{p}};u,\mathcal{B})=\{\mathbf{q}\}$, new prices $\tilde{p}'_{\mathbf{b}}<\tilde{p}_{\mathbf{b}}$, and new demands $\mathbf{q}'\in\widetilde{D}^j_{\mathrm{H}}(\tilde{p}'_{\mathbf{b}},\widetilde{\mathbf{p}}_{\mathcal{B}\smallsetminus\{\mathbf{b}\}};u,\mathcal{B})$, we have that $q'_{\mathbf{b}}\leq q_{\mathbf{b}}$ (resp. $q'_{\mathbf{b}}\geq q_{\mathbf{b}}$).*[16]
- *Bundle* $\mathbf{b}$ *is* consistent *with bundle* $\mathbf{c}$ *if* $\mathbf{c}$ *is substitutable to, or complementary, to* $\mathbf{b}$.

With two indivisible goods, if each agent can demand at most one unit of each good, consistency between the goods is sufficient and essentially necessary for equilibrium existence (Henry, 1970; Kelso and Crawford, 1982; Gul and Stacchetti, 1999). But with more than two goods, the situation is more complicated.

3.2. *Leading examples revisited*

We now revisit the example from the Introduction that shows that even if all agents view goods as complements, certain bundlings can reveal inconsistencies between substitutability and complementarity that obstruct equilibrium existence.

16. Weinstein (2022) observed that in standard consumer theory with divisible goods, the definitions of substitutes and complements are sensitive to the precise definition (or basis) for goods. As our examples below show, whether agents view goods or bundles as substitutes or complements in the indivisible good setting also depends on the bundling.

Example ***1 (continued)*** *Let the set of goods be* $I=\{\mathsf{apple},\mathsf{banana},\mathsf{coconut}\}$*, and let the set of agents be* $J=\{1,2,3\}$*. Suppose that each agent demands at most one unit of each good (*$M=1$*), and that agents' preferences are quasilinear and are given by the valuations*

$$V^1(\mathbf{x})=3\min\{x_{\mathsf{a}},x_{\mathsf{b}}\},\quad V^2(\mathbf{x})=3\min\{x_{\mathsf{b}},x_{\mathsf{c}}\},\quad V^3(\mathbf{x})=3\min\{x_{\mathsf{a}},x_{\mathsf{c}}\}.$$

All pairs of goods are complements for all agents, so there is no inconsistency between goods. Yet no competitive equilibria exist if the total endowment of indivisible goods in the economy consists of one unit of each good (Bikhchandani and Mamer, 1997, page 390).[17]

Now, let us analyze the comparative statics of bundled demand for all agents. Consider the bundling $\mathcal{B}$ *in which* apple *and* banana *are bundled, but* apple *and* coconut *can be traded separately. Formally, this bundling is given by*

$$\mathcal{B}=\{(1,0,0),(1,1,0),(0,0,1)\},$$

where $(1,1,0)$ *represents the bundle comprised of an* apple *and a* banana*.*

Suppose we start at bundle prices $\widetilde{\mathbf{p}}=(1,2,3)$*. The agents' bundled demands are:*

$$\widetilde{D}^1(\widetilde{\mathbf{p}};\mathcal{B})=\{(0,1,0)\},\quad \widetilde{D}^2(\widetilde{\mathbf{p}};\mathcal{B})=\{(0,0,0)\},\quad \widetilde{D}^3(\widetilde{\mathbf{p}};\mathcal{B})=\{(0,0,0)\}.$$

Indeed, agent 1 wants to consume apple *and* banana*, so agent 1 simply demands the bundle. Agents 2 and 3 do not wish to buy anything.*

Now suppose that the price of coconut *decreases to 1, so the bundle prices change to* $\widetilde{\mathbf{p}}'=(1,2,1)$*. The agents' bundled demands are then:*

$$\widetilde{D}^1(\widetilde{\mathbf{p}}';\mathcal{B})=\{(0,1,0)\},\quad \widetilde{D}^2(\widetilde{\mathbf{p}}';\mathcal{B})=\{(-1,1,1)\},\quad \widetilde{D}^3(\widetilde{\mathbf{p}}';\mathcal{B})=\{(1,0,1)\}.$$

Indeed, agent 1's demand (and hence bundled demand) does not change. Agent 3 now wishes to buy apple *and* coconut*. Agent 2 wishes to end up with* banana *and* coconut*, so under the bundling* $\mathcal{B}$*, buys the bundle of* apple *and* banana *and sells* apple *(in addition to buying* coconut *separately). Thus, following the decrease in the price of* coconut*, agent 2's demand for* apple *falls (since they wish to sell* apple*), while agent 3's demand for* apple *rises.*

So, while apple *and* coconut *remain complements for agent 3, these goods become substitutes for agent 2 due to bundling. Hence, given bundling* $\mathcal{B}$*, the trivial bundle corresponding to an* apple *is not consistent with the trivial bundle corresponding to a* coconut*.* ■

In Example 1, bundling two complements reveals an inconsistency between two goods (which we regard as trivial bundles). However, it is not the presence of complements *per se* that creates this bundle inconsistency, but rather the pattern of preferences over

17. To see this, note that efficiency requires allocating one agent at least two goods that they desire. Without loss of generality, suppose that agent 1 is allocated apple and banana (as well as, possibly, coconut). Then, we must have $p_{\mathsf{coconut}}=0$ and $p_{\mathsf{apple}}+p_{\mathsf{banana}}\leq 3$. But for agent 2 not to demand banana and for agent 3 not to demand apple, we must have that $p_{\mathsf{banana}}\geq 3$ and $p_{\mathsf{apple}}\geq 3$, respectively, which is a contradiction.

complementary goods that is key to a possible bundle inconsistency, which in turn serves as an obstruction to equilibrium existence. The following example highlights that bundle consistency (and equilibrium existence) is compatible with agents viewing all goods as complements.

Example *2* ***(continued)*** *To formalize this example, let us replace agent 3's valuation in Example 1 by the following valuation:*

$$V^3(\mathbf{x}) = 3\min\{x_{\mathsf{a}}, x_{\mathsf{b}}, x_{\mathsf{c}}\}.$$

Given the bundling $\mathcal{B} = \{(1,0,0),(1,1,0),(0,0,1)\}$ *and bundle prices* $\widetilde{\mathbf{p}} = (1,2,3)$*, agents bundled demands are as in Example 1:*

$$\widetilde{D}^1(\widetilde{\mathbf{p}};\mathcal{B}) = \{(0,1,0)\}, \quad \widetilde{D}^2(\widetilde{\mathbf{p}};\mathcal{B}) = \{(0,0,0)\}, \quad \widetilde{D}^3(\widetilde{\mathbf{p}};\mathcal{B}) = \{(0,0,0)\}.$$

But following the decrease in the price of coconut *to 1 (making bundle prices* $\widetilde{\mathbf{p}}' = (1,2,1)$*), agents' bundled demands are:*

$$\widetilde{D}^1(\widetilde{\mathbf{p}}';\mathcal{B}) = \{(0,1,0)\}, \quad \widetilde{D}^2(\widetilde{\mathbf{p}}';\mathcal{B}) = \{(-1,1,1)\}, \quad \widetilde{D}^3(\widetilde{\mathbf{p}}';\mathcal{B}) = \{(0,1,1)\}.$$

Thus, agent 3 now demands the bundle of apple *and* banana *as well as the* coconut*, but not the bundle consisting of* apple *alone.*

So, apple *and* coconut *become substitutes for all agents. In particular, given bundling* $\mathcal{B}$*, the trivial bundles corresponding to an* apple *and a* coconut*, respectively, are consistent. Moreover, competitive equilibria exist for all endowments (Greenberg and Weber, 1986).* ■

While the patterns of complementarities between goods are only subtly different across the two examples, they have dramatically different implications for equilibrium existence, which bundling reveals. One might hope that to reveal all obstructions to equilibrium existence, it would suffice to consider bundlings in which only complementary goods are bundled. However, in Online Appendix E.1, we provide an example showing that to reveal all obstructions to equilibrium existence, we must include “hidden” complementarities between goods and opportunities to sell other goods in the complements that we bundle. Nevertheless, we will show that the challenge of identifying bundlings relevant for obstructions to equilibrium existence can be entirely addressed by considering bundles that correspond to price effects.

4. PRICE EFFECTS AND BUNDLE CONSISTENCY

The goal of this section is to use price effects to identify which bundlings are critical for equilibrium existence: they either contain bundles of strictly complementary goods or bundles of strict “hidden” complements—i.e., goods and strictly complementary opportunities to sell other goods. Price effects will allow us to measure both complementarities between goods and “hidden” complementarities between goods and sale opportunities.

The definition of (compensated) price effects is analogous to settings with divisible goods.[18]

18. When an agent's utility function is quasilinear, the agent does not experience income effects so the compensated price effect is just the overall price effect.

Definition 2. *A* (compensated) price effect for agent j and good i *is a nonzero vector* $\Delta\mathbf{x}$ *for which there exist a utility level* u*, a price vector* $\mathbf{p}$*, and a new price* $p'_i < p_i$ *with* $D^j_{\mathrm{H}}(\mathbf{p};u) = \{\mathbf{x}\}$ *and* $\mathbf{x}+\Delta\mathbf{x} \in D^j_{\mathrm{H}}(p'_i,\mathbf{p}_{I\smallsetminus\{i\}};u)$.

Substitutability and complementarity are conditions on price effects. Consider the change in (compensated) demand $\mathbf{x}'-\mathbf{x}=\Delta\mathbf{x}$ following a fall in the price of good i. By the (compensated) law of demand, the ith entry of the $\Delta\mathbf{x}$ is positive.[19] If the kth entry of $\Delta\mathbf{x}$ is positive, then the price effect $\Delta\mathbf{x}$ exhibits a strict complementarity between goods i and k.

On the other hand, if the kth entry of $\Delta\mathbf{x}$ is negative, then the price effects reveal exhibits a strict substitutability between goods i and k. In this second case, as demand for good k falls, it is equivalent that the agent would like to sell good k. Hence, in the second case, the price effect $\Delta\mathbf{x}$ reveals a strict "hidden" complementarity between good i and the opportunity to sell good k.

These relationships apply with divisible goods as well. However, as we will show in Sections 5.2 and 6.1, price effects are a more powerful reflection of the structure of complementarity and substitutability in the indivisible good case than in the divisible good case.

The following definition summarizes which bundles are critical to identifying obstructions to equilibrium existence with indivisible goods.

Definition 3. *A bundle* $\mathbf{b} \in \{-1,0,1\}^I$ *is* relevant *if it is either a price effect* $\Delta\mathbf{x}$ *for some agent* j *or a bundle* $\mathbf{e}^i$ *consisting only of good* i*.*

For example, if all goods are complements (as in Examples 1 and 2), then each relevant bundle is a bundle of goods (i.e., it does not include opportunities to sell goods). But if some goods are substitutes, then some bundles that include both some goods and opportunities to sell other goods are relevant. Sections 5.2 and 6.1 provide more precise intuition regarding why relevant bundles capture bundles of complements in our setting with indivisible goods.

We can now introduce the condition that turns out to be key to equilibrium existence.

Definition 4. *Preferences are* bundle-consistent *if for each bundling* $\mathcal{B}$ *that is comprised solely of relevant bundles, each pair of bundles* $\mathbf{b},\mathbf{c} \in \mathcal{B}$ *are consistent.*

Definition 4 says that any two bundles must be either substitutes or complements whenever we look at any bundling that is comprised only of bundles that correspond to price effects,[20] and trivial bundles that correspond to goods.

5. WHEN AGENTS DEMAND AT MOST ONE UNIT OF EACH GOOD

To develop the connection between bundle consistency and equilibrium existence, we start by considering the case in which each agent demands at most one unit of each

19. See Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Lemma 1) for a precise statement applicable to settings with indivisible goods.

20. Note that by the definition of bundlings, we restrict here to price effects that contain at most one unit of each good and sale opportunity.

good. Our first result shows that bundle consistency is in fact sufficient for equilibrium existence.

Theorem 1. *Suppose that each agent demands at most one unit of each good. If preferences are bundle-consistent, then competitive equilibria exist.*

Theorem 1 gives a simple, economically interpretable condition for equilibrium existence that is expressed in terms of comparative statics of demand. As we will show in Section 5.1, Theorem 1 encompasses many equilibrium existence results from the literature.

To prove Theorem 1, we prove that each bundle-consistent domain of preferences lies within a "class of discrete convexity" in the sense of Danilov, Koshevoy, and Murota (2001). To show this, we first use a mild extension of Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorem 1) to show that when agents demand at most one unit of each good, price effects carry mathematical information about the tropical-geometric representation of preferences studied by Baldwin and Klemperer (2019). We then develop a technical result inspired by a result of Schrijver (1998), which relates bundle consistency to the "total unimodularity" property from integer programming that lies at the heart of discrete convex analysis. This second step also turns out to provide a computationally simple way to test for bundle consistency, which we develop in Section 7. We can then conclude equilibrium existence using the equilibrium existence results of Danilov, Koshevoy, and Murota (2001). The detailed arguments are in Appendix C.

5.1. *Necessity of bundle consistency*

We now show that bundle consistency is necessary for equilibrium existence, in a much stronger sense than previous necessity results in the literature.

In specific economies, it is possible that some agents view a pair of good as complements, other agents view them as substitutes, and yet equilibrium exists (Bikhchandani and Mamer, 1997; Ma, 1998). Thus, to formulate a necessity result, we require that the preference domain for existence to be sufficiently rich in that it is "invariant" under a simple family of transformations. We focus on the case of transferable utility economies here; Online Appendix F develops an analogous result for economies with income effects.

Definition 5. *A domain $\mathcal{V}$ of valuations is* invariant *if:*

(i) for all valuations $V \in \mathcal{V}$ and all price vectors $\mathbf{p} \in \mathbb{R}^I_{\geq 0}$, writing $\check{V}(\mathbf{x}) = V(\mathbf{x}) + \mathbf{p} \cdot \mathbf{x}$, we have that $\check{V} \in \mathcal{V}$; and

(ii) for each good i, $\mathcal{V}$ includes a valuation $\check{V}^i : \check{X}^i \to \mathbb{R}$ such that $\{\mathbf{0}, \mathbf{e}^i\} \subseteq \check{X}^i \subseteq \{0,1\}^I$.

Part (i) of the definition of invariance requires closure under the addition of positive linear functions. Subsequent work by Nguyen and Teytelboym (2024) offers two sufficient conditions for Part (i). First, any domain defined by conditions on the demand correspondence at all prices satisfies Part (i). Second, any domain defined by conditions on how the addition of a good can affect the marginal value of another good satisfies Part (i). Examples of domains that satisfy Part (i) thus include many conditions for existence in the literature: substitutes (Kelso and Crawford, 1982), consecutive games (Greenberg and Weber, 1986), substitutes and complements (Sun and Yang, 2006), and sign-consistent tree valuations (Candogan et al., 2015), as well as classes of discrete convexity (Danilov, Koshevoy, and Murota, 2001) and (totally) unimodular demand types (Baldwin and Klemperer, 2019). On the other hand, domains based on conditions on specific valuation profiles, as proposed by Bikhchandani and Mamer (1997) and Ma (1998), do not satisfy Part (i) of invariance.

Furthermore, given any bundle-consistent domain of valuations, if the domain does not satisfy Part (i) of invariance, then there is a larger, bundle-consistent domain that satisfies Part (i) of invariance. The larger domain can be constructed by simply adding nonnegative linear functions to valuations in the original domain.

Part (ii) of invariance requires that, for each good, the domain include a valuation of an agent who sometimes demands one unit of that good and sometimes demands nothing. Part (ii) gives meaning to "goods" in our model: if it were not satisfied, then agents would never demand (a unit of) each good separately, and it would not be clear why the primitive goods were defined correctly (or why each "good" should be priced separately). We therefore view Part (ii) of invariance as economically innocuous.[21]

The following result states that for invariant domains of valuations, bundle consistency is necessary for equilibrium existence. This provides a partial converse to Theorem 1.

Theorem 2. *If competitive equilibria exist in all economies in which agents have valuations in an invariant domain $\mathcal{V}$, then the valuations in $\mathcal{V}$ are bundle-consistent.*

Theorem 2 implies that bundle consistency encompasses most previous domains for equilibrium existence.[22] Indeed, bundle consistency subsumes substitutes (Kelso and Crawford, 1982), consecutive games (Greenberg and Weber, 1986), and substitutes and complements (Sun and Yang, 2006), as well as classes of discrete convexity (Danilov, Koshevoy, and Murota, 2001) and totally unimodular demand types (Baldwin and Klemperer, 2019). Indeed, these domains all place price-independent restrictions on demand, and guarantee existence in all exchange economies with preferences in the domain.[23]

Intuitively, consistency between a pair of goods or bundles is necessary to simultaneously clear the markets for those goods or bundles in any invariant domain. As equilibrium existence requires that all markets clear simultaneously, each pair of goods or bundles must be consistent for equilibrium to exist for all valuations in an invariant domain.

To understand the logic behind Theorem 2 in more detail, let us first show why consistency between all pairs of goods is a necessary condition for equilibrium to exist for

21. Note also that Part (ii) of the definition of invariance is automatically satisfied if the economy can contain a seller who seeks to efficiently assign (up to) one unit of each good to agents.

22. Theorem 2 also provides a formal sense in which inconsistencies between *goods* obstructs equilibrium existence beyond the case of substitutes (see Footnote 4).

23. Note, however, bundle consistency does not subsume equilibrium existence results that rely on special conditions on the joint structure of preferences or endowments in the economy. For example, it does not subsume conditions for equilibrium existence developed by Bikhchandani and Mamer (1997) and Ma (1998) that are based on the entire profile of valuations (Part (i) of invariance fails for these domains). Moreover, it does not subsume Candogan et al.'s (2015) existence result for the domain of "sign-consistent tree valuations" because that result requires conditions on the total endowment of indivisible goods (see Section 3.3 of their Supplemental Material), and hence does not apply in all exchange economies.

all sets of valuations in an invariant domain. The logic is based on the counterexample to equilibrium existence introduced by Henry (1970) and Kelso and Crawford (1982). Suppose that apple and banana are strict complements at some prices under valuation $V^{\mathsf{c}} \in \mathcal{V}$, and strict substitutes at some prices under valuation $V^{\mathsf{s}} \in \mathcal{V}$. Using the first part of invariance, we can assume that when all goods are priced at the same price p, agent c is indifferent between obtaining both apple and banana and neither (also in conjunction with potentially different sets of other goods), and agent s is indifferent between obtaining either apple or banana (in conjunction with potentially different sets of other goods).[24] Using the second part of invariance, we can add multiple additional agents for each good in the economy other than apple and banana who are indifferent between getting the good and not at price p. When each such agent is endowed with one unit of their good, the only possible equilibrium price of goods other than apple and banana is p.[25] But then if there is one unit of each of apple and banana, it is not possible to simultaneously clear the markets for apple and banana (for similar reasons to Henry's (1970) and Kelso and Crawford's (1982) examples).[26]

The logic behind why bundle consistency is necessary proceeds along similar lines. If two bundles $\mathbf{b}^1$ and $\mathbf{b}^2$ are inconsistent in a bundling $\mathcal{B}$ that is comprised of relevant bundles, then we can construct an economy in which the markets for $\mathbf{b}^1$ and $\mathbf{b}^2$ cannot simultaneously clear at particular prices for the other bundles, obstructing equilibrium existence. Here, the relevance of the bundles in the bundling ensures that we can add agents indifferent between getting those bundles and not in a way that pins down the equilibrium prices of the bundles other than $\mathbf{b}^1$ and $\mathbf{b}^2$ at prices at which $\mathbf{b}^1$ and $\mathbf{b}^2$ are inconsistent.

5.2. *Interpretation of price effects*

Why are price effects so powerful in identifying all the complementarities and "hidden" complementarities that need to be bundled to identify obstructions to equilibrium existence? As in the divisible goods case, price effects for a good i clearly reveal *pairwise* complementarities and substitutabilities between good i and other goods. (Such substitutabilities are equivalent to pairwise "hidden" complementarities between good i and opportunities to sell other goods.) However, it is not obvious whether or not such price effects capture complementarities between *sets* of goods, or either complementarities or substitutabilities involving goods other than i. Nevertheless, in the setting with indivisible goods, it turns out that price effects contain far more information about the complementarities when we observe a simultaneous change in demand of

24. Thus, Part (i) of the definition of invariance is required for Theorem 2 because bundle inconsistencies can come from conflicts between complementarities and substitutabilities that arise at different prices. However, if preferences are sufficiently homogeneous, this part of the definition of invariance can be dispensed with, as we show by example in Online Appendix E.2.

25. In our proofs, using a more careful analysis of potential equilibrium price vectors, we can work with simpler constructions that contains only one additional agent for each good in the economy.

26. More precisely, for s to demand neither apple nor banana, both would have to be priced greater than p, in which case c would also demand neither of them. For s to demand both apple and banana, both would have to be priced less than p, in which case c would also demand both of them. Thus, the only possible equilibrium allocation would be for each of c and s to get one of apple and banana. For c to demand apple on its own, apple would have to be priced less than p, and for s to demand banana on its own, banana would have to be priced lower than apple, hence also less than p. But then c would demand both apple and banana.

multiple goods; this additional information leads to a precise intuition behind why relevant bundles correspond to bundles of complements.

Proposition 1. *Suppose that agent j demands at most one unit of each good. Let $\Delta\mathbf{x}$ is a (compensated) price effect for agent j and good i, and let k,ℓ be distinct goods.*

- *(a) If $(\Delta\mathbf{x})_k,(\Delta\mathbf{x})_\ell > 0$ or $(\Delta\mathbf{x})_k,(\Delta\mathbf{x})_\ell < 0$, then k and ℓ are strict complements at some prices.*
- *(b) If $(\Delta\mathbf{x})_k > 0 > (\Delta\mathbf{x})_\ell$, then k and ℓ are strict substitutes at some prices.*

Proposition 1 says that if following a price decrease there is a change in demand of two goods in the same (resp. opposite) direction, then there must be a strict complementarity (resp. substitutability) between the goods at some prices. For example, if decreasing the price of apple makes the demands for banana and coconut both rise (or both fall), then banana and coconut must be strict complements at some prices.[27] On the other hand, if decreasing the price of apple makes the demand for banana rise but the demand for coconut fall, then banana and coconut must be strict substitutes at some prices. That is, there must be a strict "hidden" complementarity between banana and an opportunity to sell coconut.

As a result, Proposition 1 provides a simple interpretation of which bundles are relevant. First, bundling goods whose quantities in a price effect have the same sign corresponds to bundling strict complements. Second, bundling goods whose quantities in a price effect are positive with opportunities to sell goods whose quantities in the price effect are negative corresponds to bundling strict "hidden" complements.

Thus, relevant bundles effectively correspond to bundles of complements. From this point of view, our results show that considering the comparative statics of demand when complements can be bundled allows us to detect all possible obstructions to equilibrium existence.

6. WHEN AGENTS DEMAND MULTIPLE UNITS OF ANY GOOD

We now turn to the case in which each agent may demand multiple units of any good. The definitions of price effects, relevant bundles, and bundle consistency carry over verbatim from Section 4. By definition, relevant bundles include at most one unit of each good.

However, stronger conditions are needed for equilibrium existence when each agent can demand multiple units of any good than in the case in which each agent demands at most one unit of each good. The issue is that we need to rule out the possibility of increasing returns scale in units (e.g., minimum viable quantity of units), as increasing returns can obstruct equilibrium existence for reasons familiar from classic general equilibrium theory. We will therefore impose a condition called "unit consistency'' that will ensure that units of the same good are substitutes for each other.

To define unit consistency formally, we regard each unit of each good as a separate item. Formally, an *item* consists of a good i and a serial number $1 \leq m \leq M$. Hence, the set of items is $\overline{I} = I \times \{1,2,\ldots,M\}$. Given a bundle $\overline{\mathbf{x}} \in \{0,1\}^{\overline{I}}$ of items, there is a

27. In classical consumer theory with divisible goods, one would not be able to make such an inference without knowing the effect of changing the price of banana on demand for coconut.

corresponding bundle $\mathbf{x} = \pi(\overline{\mathbf{x}})$ of goods defined by

$$\pi(\overline{\mathbf{x}})_i = \sum_{m=1}^{M} \mathbf{x}_{(i,m)}.$$

Given any utility function U^j for goods and money, there is a corresponding utility function $\overline{U}^j : \mathbb{R} \times \pi^{-1}(X^j) \to \mathbb{R}$ for items and money defined by $\overline{U}^j(x_0, \overline{\mathbf{x}}) = U^j(x_0, \pi(\overline{\mathbf{x}}))$. We can now formally state the definition of unit consistency.

Definition 6. *Utility function U^j is* unit-consistent *if for the utility function $\overline{U}^j$, for all goods i and serial numbers $1 \leq m < m' \leq M$, the items (i,m) and (i,m') are substitutes.*[28]

Imposing unit consistency turns out to be sufficient to deal with the additional difficulties of the setting when agents can demand multiple units of any good.

Theorem 3. *If preferences are unit- and bundle-consistent, then competitive equilibria exist.*

Theorem 3 generalizes Theorem 1 for settings in which agents can demand multiple units of any good. This extension encompasses an existence result for substitutes due to Milgrom and Strulovici (2009), as well as existence results of Danilov, Koshevoy, and Murota (2001) and Baldwin and Klemperer (2019) (see Appendix A).

To prove Theorem 3, we prove that each domain of preferences that is unit- and bundle-consistent lies within a "class of discrete convexity" in the sense of Danilov, Koshevoy, and Murota (2001) (see Proposition 4 in Appendix A for a precise statement). The proof of this connection is fairly involved in the case when agents can demand multiple units of any good, and proceeds in three steps. A first step is to show that under unit consistency, price effects continue to carry mathematical information about the tropical-geometric representation of preferences studied by Baldwin and Klemperer (2019) even when agents can demand multiple units of any good.[29] A second step is to show that under unit consistency, bundle consistency is closely related to the "total unimodularity" property even when agents can demand multiple units of any good. A third step is to show that unit and bundle consistency imply that demand sets cannot miss integer points in their convex hulls; here, we rely on a combinatorial-geometric result underlying Danilov and Koshevoy's (2004) theory of discrete convexity. We can then conclude equilibrium existence using the equilibrium existence results of Danilov, Koshevoy, and Murota (2001). The detailed arguments are in Online Appendix D.

28. As Lemma 3 in Online Appendix D.1 shows, unit consistency implies a property introduced by Milgrom and Strulovici (2009) that at each price vector, the set of demanded quantities of each good consists of consecutive integers. In general, unit consistency is stronger than this "consecutive integer property."

29. In the case in which agents demand at most one unit of each good, this connection was developed by Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorem 1); we use a version of that case of the connection in the proof of Theorem 1. While Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorems 2 and 3) offered versions of their Theorem 1 for the case in which agents can demand multiple units of any good, those versions rely on decompositions of price effects as linear combinations of various other vectors, rather than price effects themselves. As a result, those results are not useful for the current paper's economic approach to equilibrium existence.

Our final main result shows that unit and bundle consistency are essentially necessary to guarantee equilibrium existence under when agents can demand multiple units of any good, thereby generalizing Theorem 2 and showing that unit consistency is also essentially necessary for equilibrium existence.

Theorem 4. *If competitive equilibria exist in all economies in which agents have valuations in an invariant domain $\mathcal{V}$, then the valuations in $\mathcal{V}$ are unit- and bundle-consistent.*

The proof of Theorem 4 shows first the necessity of unit consistency and then the necessity of bundle consistency. The proof of the necessity of unit consistency follows similar logic to Milgrom and Strulovici (2009), but is more involved due to the possibility of complementarities. The proof of the necessity of bundle consistency given unit consistency applies logic similar to the proof of Theorem 2. The detailed arguments are in Online Appendix D.

6.1. *Interpretation of price effects*

We already know that when agents demand at most one unit of each good, price effects pinpoint the structure of complementarities and substitutabilities between sets of goods; this feature allows the comparative statics of demand for relevant bundles to detect all possible obstructions to equilibrium existence. The following analogue of Proposition 1 shows that price effects that lie in $\{-1,0,1\}^I$ identify strict complementarities and substitutabilities across sets of goods even when agents can demand multiple units of some goods.

Proposition 2. *Suppose that agent j's preferences are unit-consistent. Let $\Delta\mathbf{x}\in\{-1,0,1\}^I$ be a (compensated) price effect for agent j and good i, and let k,ℓ be distinct goods.*

- *(a) If $(\Delta\mathbf{x})_k,(\Delta\mathbf{x})_\ell>0$ or $(\Delta\mathbf{x})_k,(\Delta\mathbf{x})_\ell<0$, then k and ℓ are strict complements at some prices.*
- *(b) If $(\Delta\mathbf{x})_k>0>(\Delta\mathbf{x})_\ell$, then k and ℓ are strict substitutes at some prices.*

While Proposition 2 is broadly similar to Proposition 1, note that Proposition 2 restricts to price effects in which demand for each good changes by at most one unit (as in the definition of relevance).[30] This additional restriction is crucial. Indeed, consider an agent who a banana as a strict complement to each of an apple and a coconut (and is interested in up to one unit of each of apple and coconut), but views apple and coconut as independent:

$$V(\mathbf{x})=\max_{0\le y\le x_{\mathsf{b}}}\left\{3\min\{x_{\mathsf{a}},y,1\}+3\min\{x_{\mathsf{b}}-y,x_{\mathsf{c}},1\}\right\}.$$

At price vector $\mathbf{p}=(1,3,1)$, the agent demands nothing, i.e., $D^j(\mathbf{p})=\{(0,0,0)\}$. Now suppose that the price of banana falls, so $\mathbf{p}'=(1,1,1)$. The agent then demands apple,

30. This restriction is automatic when agents can demand at most one unit of each good. Proposition 2 also imposes unit consistency, which also holds vacuously when agents can demand at most one unit of each good.

coconut and two units of banana, so $D^j(\mathbf{p}') = \{(1,2,1)\}$. Thus, decreasing the price of banana makes the demands for apple and coconut increase simultaneously—which would suggest that apple and coconut are strict complements, even though the agent in fact views them as independent goods. The reason for the discrepancy is that the hypothesis of Proposition 2 is not satisfied, as we considered a price effect in which demand for banana changed by two units.

Proposition 2 shows that by restricting to price effects in which demand changes by at most one unit per good, we can still use price effects identify strict complementarities and strict substitutabilities between sets of goods. Since strict substitutabilities correspond to strict "hidden" complementarities involving opportunities to sell, relevant bundles hence effectively correspond to bundles of complements that include at most one unit of each good or sale opportunity. Thus, the intuition behind relevance extends to the multiunit setting.

7. TESTING FOR BUNDLE CONSISTENCY

While bundle consistency is an economically natural condition, one may ask whether there is a computationally straightforward way to test whether a given domain of preferences is bundle-consistent, and whether there is a mathematical recipe to construct bundle-consistent domains of preferences. To answer these questions, we connect bundle consistency to the mathematical property of total unimodularity. Recall that a set S of integer vectors is *totally unimodular* if every square submatrix of the matrix whose columns are the elements of S has determinant 0 or ± 1. It turns out that one can test for bundle consistency by checking whether the set of all price effects is totally unimodular. (When agents can demand multiple units of any good, we need to restrict attention to price effects that lie in $\{-1,0,1\}^I$.)

Proposition 3. *(a) Suppose that each agent demands at most one unit of each good. Preferences are bundle-consistent if and only if the set of all agents' price effects is totally unimodular.*

(b) Suppose that preferences are unit-consistent. Preferences are bundle-consistent if and only if the set of all agents' price effects that lie in $\{-1,0,1\}^I$ is totally unimodular.

Proposition 3 provides a useful test for bundle consistency.[31] For example, it is well-known that the total unimodularity of a set of integer vectors can be tested in time that is polynomial in the dimensionality (see, e.g., Schrijver (1998, Theorem 20.3)). Hence, Proposition 3 implies that starting from the set of all agents' price effects, bundle consistency can be checked in time that is polynomial in the number of goods.

In addition to the value of Proposition 3 as a test for bundle consistency, the "only if" direction of the proposition comprises a critical element of the proof of Theorem 1, and the crux of one of the steps in the proof of Theorem 3, as discussed in Sections 5 and 6.

31. Furthermore, as the total unimodularity of a set of integer vectors is unaffected by adding or removing elementary basis vectors (see, e.g., Equation (43)(v) in Schrijver (1998, page 200)), one can remove elementary basis vectors from the set of price effects before checking total unimodularity. In economic terms, the bundle consistency of a family of preferences is unaffected by adding or removing (unit-consistent) valuations that are additively separable across goods.

The following example illustrates how to apply Proposition 3 to verify bundle consistency.

Example ***3 (Danilov, Koshevoy, and Lang, 2013, Example 4)*** *Let the set of goods be* $I=\{\mathsf{apple},\mathsf{banana},\mathsf{coconut},\mathsf{date}\}$ *and the set of agents be* $J=\{1,2,3,4\}$. *Suppose that each agent demands at most one unit of each good (*$M=1$*), and that agents' preferences are quasilinear and are given by the valuations:*

$$V^1(\mathbf{x})=x_{\mathsf{a}}+x_{\mathsf{b}}+x_{\mathsf{c}}+x_{\mathsf{d}},\quad V^2(\mathbf{x})=3\min\{x_{\mathsf{a}},x_{\mathsf{b}}\},\quad V^3(\mathbf{x})=3\min\{x_{\mathsf{b}},x_{\mathsf{c}}\},$$

$$V^4(\mathbf{x})=3\min\{x_{\mathsf{c}},x_{\mathsf{d}}\},\quad V^5(\mathbf{x})=3\min\{x_{\mathsf{d}},x_{\mathsf{a}}\},\quad V^6(\mathbf{x})=3\min\{x_{\mathsf{a}},x_{\mathsf{b}},x_{\mathsf{c}},x_{\mathsf{d}}\}.$$

Thus, agent 1 sees the goods as independent, agents 2–5 regard pairs of consecutive goods as complements, and agent 6 regards all goods as complements.

The set of all price effects consists of the columns of the following matrix:

$$\begin{pmatrix} 1&0&0&0&1&0&0&1&1\\ 0&1&0&0&1&1&0&0&1\\ 0&0&1&0&0&1&1&0&1\\ 0&0&0&1&0&0&1&1&1 \end{pmatrix} \tag{7.1}$$

The first four columns describe the price effects for agent 1, the next four columns describe the price effects for agents 2–5, and the final column describes the price effects for agent 6. It is easy to verify that this matrix is totally unimodular. Therefore, in this example, agents' preferences are bundle-consistent (by Proposition 3(a)), so equilibria exist for all endowments under these preferences (by Theorem 1). ■

In Example 3, there is *no* bundling for which all agents view the bundles as (strong) substitutes (Danilov, Koshevoy, and Lang, 2013). Example 3 therefore illustrates that our equilibrium existence result cannot be deduced from Kelso and Crawford's (1982) sufficient conditions by identifying a family of substitute bundles, unlike for the equilibrium existence results of Greenberg and Weber (1986) and Sun and Yang (2006).

Furthermore, Proposition 3 gives a recipe to construct bundle-consistent domain of preferences. Indeed, one can obtain a bundle-consistent domain of preferences by constraining price effects that lie in $\{-1,0,1\}^I$ to lie in any given totally unimodular set of vectors.

Corollary 1. Let $\mathcal{D}$ be a totally unimodular set of vectors. The class of unit-consistent preferences all of whose price effects that lie in $\{-1,0,1\}^I$ also lie in $\mathcal{D}$, is bundle-consistent.

For example, the class of unit-consistent preferences whose price effects that lie in $\{-1,0,1\}^I$ also lie among the columns of (7.1) is bundle-consistent. Equilibrium existence for this domain is guaranteed by Theorem 3, but cannot be deduced from Kelso and Crawford (1982).

8. CONCLUSION

We showed that with indivisible goods, the existence of equilibrium is fundamentally determined by whether there is consistency between the substitutability and complementarity for appropriate pairs of bundles of goods. When agents can demand more than one unit of any good, units of the same good must also not be complementary. Our analysis provides a unified, economic framework for understanding equilibrium with indivisible goods that allows for complementarity, substitutability, and income effects.

Our results have implications for the design of sealed-bid multi-item auctions that guarantee market-clearing. Currently, parsimonious bidding languages based on ideas from producer theory have been developed only for the case of quasilinear substitutes preferences, or of preferences that can be made to reflect substitutability by carefully defining the lots that are sold (Milgrom, 2009).[32] However, many auction settings exhibit complex patterns of complementarity and substitutability that cannot be recast as substitutes. Our results suggest that by bundling certain complements and using consumer theory, natural bidding languages can be developed that would allow bidders to express richer preferences that, in addition to substitutes, may exhibit irreducible complementarities and income effects.

A. RELATIONSHIP TO GEOMETRIC APPROACHES

In this section, we explain the mathematical relationship between our equilibrium existence results and two geometric approaches to equilibrium existence, expanding on the Introduction. In Appendix A.1, we discuss the combinatorial-geometric “discrete convexity” approach developed by Danilov, Koshevoy, and Murota (2001). In Appendix A.2, we discuss the tropical-geometric approach developed by Baldwin and Klemperer (2019).

A.1. *Relationship to discrete convexity*

The key combinatorial-geometric objects that Danilov, Koshevoy, and Murota (2001) worked with are classes of “discrete convex sets.” Such classes are closely related to totally unimodular sets of integer vectors (Danilov and Koshevoy, 2004). Throughout this appendix, we let $\mathcal{D}\subseteq\mathbb{Z}^I$ consist of vectors whose components have no nontrivial common factors; we also suppose that $\mathcal{D}$ is closed under negation. Letting $\mathcal{D}$ be totally unimodular, a finite set $S\subseteq\mathbb{Z}^I$ of integer vectors is *$\mathcal{D}$-convex* if it includes all integer vectors in its convex hull, and each edge of the convex hull is parallel to an element of $\mathcal{D}$.[33]

To develop their equilibrium existence result, Danilov, Koshevoy, and Murota (2001, page 261) assumed that $\mathbf{0}\in X^j$ and studied the functions

$$q^j_m(\mathbf{x})=\left(U^j(\cdot,\mathbf{x})\right)^{-1}\left(U^j(m,\mathbf{0})\right)$$

They then introduced a notion of discrete convexity for functions: letting $\mathcal{D}$ be a totally unimodular set of vectors, a function $q^j:X^j\to\mathbb{R}$ is *$\mathcal{D}$-convex* if for all $\mathbf{p}\in\mathbb{R}^I$, the set

$$\operatorname*{argmin}_{\mathbf{x}\in X^j}\left\{q^j(\mathbf{x})+\mathbf{p}\cdot\mathbf{x}\right\}$$

is $\mathcal{D}$-convex as a set of integer vectors. We say that a utility function U^j is *$\mathcal{D}$-quasiconcave* if for each real number m, the function q^j_m is $\mathcal{D}$-convex.

32. Klemperer (2010) took a geometric approach to obtain versions of the bidding languages proposed by Milgrom (2009) for the case of substitute goods, focusing on the two-good case for which a particular representation of bidders’ preferences can in fact be visualized. As discussed by Baldwin and Klemperer (2019, Section 6.7), tropical geometry can be used to design and analyze bidding languages beyond the two-good case (Baldwin, Klemperer, and Lock, 2024), and for multi-item auctions more generally. The results of this paper imply generally that any outcome that can be implemented by a tropical-geometric auction format can instead be achieved using bidding languages based on producer theory. Unlike the tropical-geometric approach, the consumer theory approach also applies in settings with income effects.

33. For much of this appendix, we follow the terminology introduced by Danilov, Koshevoy, and Lang (2013).

The classes of $\mathcal{D}$-quasiconcave utility functions for totally unimodular $\mathcal{D}$ are the *classes of discrete convexity* introduced by Danilov, Koshevoy, and Murota (2001). Under some technical assumptions, Danilov, Koshevoy, and Murota (2001, Theorems 2 and 4) showed that competitive equilibria exist in exchange economies with indivisible goods if all agents' utility functions belong to the same class of discrete convexity.[34]

Mathematically, there is a close connection between our existence results and Danilov, Koshevoy, and Murota's (2001). The following proposition explains the connection: in addition to each class of discrete convexity being unit- and bundle-consistent (by Theorem 4), each class of unit- and bundle-consistent preferences belongs to a class of discrete convexity.

Proposition 4. *A family of utility functions belong to a single class of discrete convexity if and only if the family is unit- and bundle-consistent.*

We prove Proposition 4 in Online Appendix D. In addition to illustrating the mathematical connection to Danilov, Koshevoy, and Murota (2001), Proposition 4 is technically useful in establishing Theorem 3. More precisely, we prove Theorem 3 by combining the "if" direction of Proposition 4 with Danilov, Koshevoy, and Murota's (2001) combinatorial-geometric equilibrium existence results (and Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym's (2020) analogous tropical-geometric equilibrium existence results).

Moreover, it turns out that classes of discrete convexity that can be characterized in terms of unit consistency and the compensated price effects: the elements of $\mathcal{D}$ prescribe the possible compensated price effects that are in $\{-1,0,1\}^I$.

Proposition 5. *Let $\mathcal{D}$ be totally unimodular. A utility function is $\mathcal{D}$-quasiconcave if and only if it is unit-consistent and each compensated price effect that lies in $\{-1,0,1\}^I$ lies in $\mathcal{D}$.*

We prove Proposition 4 in Online Appendix D. We use the "if" direction of Proposition 5 to prove the "if" direction of Proposition 4. The case of Proposition 5 under which agents demand at most one unit of each good and utility is quasilinear is a case of Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorem 1), as we discuss in detail in the sequel. The argument is much more involved beyond that case. For the "if" direction, both the unit consistency of preferences and the total unimodularity of $\mathcal{D}$ play critical roles in ensuring that demand sets include all points in their convex hulls; the formal proof relies on a mathematical result of Danilov and Koshevoy (2004). For the "only if" direction, price effects are not generally elements of $\{-1,0,1\}^I$, and deducing unit consistency relies on the total unimodularity of $\mathcal{D}$ and Danilov, Koshevoy, and Murota's (2001) existence results.

A.2. *Relationship to tropical-geometric approaches*

In the quasilinear case, Baldwin and Klemperer (2019) studied the set of price vectors at which demand is nonunique, which they call the "locus of indifference prices" (LIP) using methods from the mathematics of tropical geometry. The "tropical" algebraic structure mathematically underlying quasilinear utility maximization implies that the LIP can be formally decomposed into linear components called "facets." The geometric aspects of the LIP that Baldwin and Klemperer (2019) focused on were the normal vectors to facets.

Formally, given a valuation V^j, Baldwin and Klemperer (2019, Definition 2.2(2)) defined a *LIP facet* to be a set $F \subseteq \mathbb{R}^I$ of price vectors, that lies within exactly one hyperplane, for which there are bundles $\mathbf{x},\mathbf{x}' \in X$ such that $F = \{\mathbf{p} \in \mathbb{R}^I \mid \mathbf{x},\mathbf{x}' \in D^j(\mathbf{p})\}$. By construction, each LIP facet has a well-defined normal direction, which turns out to contain integer vectors (Baldwin and Klemperer, 2019, page 873). Thus, Baldwin and Klemperer (2019, Definition 3.1) defined a valuation V^j to be of *demand type* $\mathcal{D}$ if each LIP facet for V^j is normal to an element of $\mathcal{D}$.[35] To avoid having to deal with the intricacies of LIPs

34. The key additional technical condition Danilov, Koshevoy, and Murota (2001) imposed was an analogue of the standard free disposal condition. Note that Danilov, Koshevoy, and Murota (2001) also allowed for production and for unbounded sets of feasible consumption vectors, which are beyond the scope of this paper.

35. Baldwin and Klemperer (2019, page 868) asserted, based on their Proposition 3.3, that "[demand type] vectors give the possible ways in which [...] demand can change in response to a small generic price

and to stay closer to Danilov, Koshevoy, and Murota (2001), we work with an equivalent alternative definition.[36]

Definition *7 **(Baldwin and Klemperer, 2019, page 882)** A valuation V^j is* of demand type $\mathcal{D}$ *if for every price vector* $\mathbf{p}$ *for which* $\operatorname{Conv} D^j(\mathbf{p})$ *is one-dimensional,* $\operatorname{Conv} D^j(\mathbf{p})$ *is parallel to an element of* $\mathcal{D}$.

Let us call a valuation *pseudoconcave* if every demand set includes every integer point in its convex hull. Baldwin and Klemperer (2019, Theorem 4.3) showed that equilibria are guaranteed to exist in transferable utility economies if all agents' valuations are pseudoconcave and of the same totally unimodular demand type.[37]

Baldwin and Klemperer's (2019) results also apply slightly more generally when $\mathcal{D}$ is "unimodular." However, the only unimodular demand types that satisfy Part (ii) of the definition of invariance are the totally unimodular demand types. A failure of that property suggests that the natural objects that are bought are sold should be bundles rather than goods. Redefining the natural bundles as goods converts any unimodular type to a totally unimodular one (Baldwin and Klemperer, 2019, page 909). We therefore focus on total unimodularity, rather than the slightly weaker property of unimodularity.

Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020) extended some of Baldwin and Klemperer's (2019) analysis to settings with income effects by reducing to the quasilinear case. Given a utility function U^j and a utility level u, Baldwin, Jagadeesan, Klemperer, and Teytelboym (2023, Definition 1) defined the *Hicksian valuation* $V^j_{\mathrm{H}}(\cdot;u): X^j \to \mathbb{R}$ by

$$V^j_{\mathrm{H}}(\mathbf{x};u) = -\left(U^j(\cdot,\mathbf{x})\right)^{-1}(u);$$

their Lemma 1 (see Fact 1 below) shows that for all price vectors, demand for this Hicksian valuation matches Hicksian demand for the original utility function at utility level u.

Fact *1 **(Baldwin, Jagadeesan, Klemperer, and Teytelboym, 2023, Lemma 1)** For all price vectors* $\mathbf{p}$ *and utility levels* u*, we have that*

$$D^j_{\mathrm{H}}(\mathbf{p};u) = \operatorname*{argmax}_{\mathbf{x}\in X^j} \left\{V^j_{\mathrm{H}}(\mathbf{x};u) - \mathbf{p}\cdot\mathbf{x}\right\}.$$

Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020, Definitions 9 and 10) defined a utility function to be of *demand type* $\mathcal{D}$ if all of its Hicksian valuations are of demand type $\mathcal{D}$, and to be *quasipseudoconcave* if all of its Hicksian valuations are pseudoconcave.[38] Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020, Theorem 3) then showed that equilibria are guaranteed to exist even in the presence of income effects if all agents' utility functions are quasipseudoconcave and of the same totally unimodular demand type.

change." However, small changes in prices can give changes in demand that are not in the directions of normal vectors to LIP facets even generically. (Geometrically, this issue arises because generic small changes in prices can cross multiple LIP facets.) For example, suppose that there are two goods and that the domain is $X^j = \{0,1\}^2$. The additive valuation V^j defined by $V^j(\mathbf{x}) = x_1 + x_2$ is of demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1)\}$. Nevertheless, small changes in prices from $(1-\epsilon_1, 1-\epsilon_2)$ to $(1+\delta_1, 1+\delta_2)$ (for any $\epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0$) change demand from $(1,1)$ to $(0,0)$—i.e., not by an element of $\mathcal{D}$. The same issue applies to an analogous assertion of Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020, page 24) for settings with income effects.

36. As discussed by Baldwin and Klemperer (2019, page 882), the equivalence between the two definitions follows from their Proposition 2.20; in their terminology, one-dimensional sets of the form $\operatorname{Conv} D^j(\mathbf{p})$ are called *edges of the demand complex* for V^j (Baldwin and Klemperer, 2019, Definition 2.15.4). See also Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Fact B.2).

37. Baldwin and Klemperer (2019, Theorem 4.3) also showed that if $\mathcal{D}$ contains the elementary basis vectors but is not totally unimodular, then the class of pseudoconcave valuations of demand type $\mathcal{D}$ does not give a domain for equilibrium existence.

38. Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020) called this condition "quasiconcavity"; we use the term quasipseudoconcavity to parallel the term pseudoconcavity for a condition on valuations.

As Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020, Footnote 35) showed, if $\mathcal{D}$ is totally unimodular, a utility function is quasipseudoconcave and of demand type $\mathcal{D}$ if and only if it is $\mathcal{D}$-quasiconcave. Hence, totally unimodular demand types provide the same domains for equilibrium existence introduced by Danilov, Koshevoy, and Murota (2001). In particular, Propositions 4 and 5 therefore also give the mathematical relationship between unit and bundle consistency and the tropical-geometric approach to equilibrium existence.

Proposition 4′. *A family of valuations (resp. utility functions) are all (quasi)pseudoconcave and belong to a single totally unimodular demand type if and only if the family is unit- and bundle-consistent.*

Proposition 5′. *Let $\mathcal{D}$ be a totally unimodular. A valuation (resp. utility function) is (quasi)pseudoconcave and of demand type $\mathcal{D}$ if and only if it is unit-consistent and each (compensated) price effect that lies in $\{-1,0,1\}^I$ lies in $\mathcal{D}$.*

Beyond the totally unimodular case, it turns out that a version of the characterization of the utility functions of a demand type in terms of (compensated) price effects given by Proposition 5′ persists provided that utility functions are assumed to be unit-consistent.

We first explain the connection for the case in which agents demand at most one unit of each good, where it is easiest to state and follows straightforwardly from Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorem 1).

Proposition 6. *If agent j demands at most one unit of each good, then V^j (resp. U^j) is of demand type $\mathcal{D}$ if and only if each (compensated) price effect for agent j lies in $\mathcal{D}$.*[39]

We use Proposition 6 to prove our results for the case in which each agent demands at most one unit of each good in Appendix C.

More generally, unit consistency on its own provides a sufficient condition for being of a demand type in terms of price effects, and the condition also becomes necessary if each pair of goods is either substitutes or complements for an agent.

Proposition 7. *(a) If V^j (resp. U^j) is unit-consistent and each (compensated) price effect for agent j that lies in $\{-1,0,1\}^I$ also lies in $\mathcal{D}$, then V^j (resp. U^j) is of demand type $\mathcal{D}$.*

(b) If V^j (resp. U^j) is unit-consistent and each pair of goods is either substitutes or complements for agent j, then V^j (resp. U^j) is of demand type $\mathcal{D}$ if and only if each (compensated) price effect for agent j that lies in $\{-1,0,1\}^I$ also lies in $\mathcal{D}$.

We prove Proposition 7 in Online Appendix D. In addition to further clarifying the mathematical relationship between our approach and the tropical-geometric approach of Baldwin and Klemperer (2019) and Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020), Proposition 7 is technically useful in establishing the other propositions, and our main results, when agents can demand multiple units of any good (see Online Appendix D).

The cases of Propositions 5′ and 7(b) under which agents demand at most one unit of each good and utility is quasilinear corresponds to Proposition 6, which does not require the hypothesis that each pair of goods is consistent for either direction. The arguments are much more involved beyond that case, where price effects are not generally elements of $\{-1,0,1\}^I$; and critical roles are played by unit consistency, total unimodularity (for Proposition 5′), and the consistency of each pair of goods (for the "only if" direction of Proposition 7(b)).

39. The "if" direction of Proposition 6 is a special case of the "if" direction of Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorem 1). The "only if" direction of Proposition 6 mildly extends the "only if" direction of Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorem 1) to provide a constraint on price effects for which demand at the final prices may not be unique; the argument is in Appendix C.

B. ON THE DEFINITIONS OF SUBSTITUTES AND COMPLEMENTS

In this appendix, we prove that our definitions of substitutability and complementarity between *pairs* of indivisible goods are equivalent to alternative definitions based on definitions from the literature that require that *all* goods be substitutes.

Our definition of substitutability or complementarity for pairs of goods requires that for any decrease in the price p_i of a good i from prices $\mathbf{p}$ at which demand is unique, every selection $\mathbf{x}'$ from demand at the new prices $(p'_i,\mathbf{p}_{I\smallsetminus\{i\}})$ be compatible with substitutability or complementarity. By contrast, in definitions of substitutability between all goods given in the literature, Kelso and Crawford (1982) (see also Milgrom and Strulovici (2009)) allow demand to be non-unique at $\mathbf{p}$, and require that a selection $\mathbf{x}$ from demand at $\mathbf{p}$ that is compatible with substitutability or complementarity exist for each selection $\mathbf{x}'$ from demand at $(p'_i,\mathbf{p}_{I\smallsetminus\{i\}})$.[40] On the other hand, an analogous definition in Ausubel and Milgrom (2002) only considers new prices at which demand is unique.

In the case of the definition of substitutability between *all* goods in which agents each demand at one unit of each indivisible good, it is known that Kelso and Crawford's (1982) and Ausubel and Milgrom's (2002) definitions coincide with each other (Danilov, Koshevoy, and Lang, 2003, Corollary 5; Hatfield et al., 2019, Theorem A.1) and hence with our definition. However, the case of the definition of substitutability between *all* goods when agents can demand more than one unit of an indivisible good, Kelso and Crawford's (1982) approach yields a strictly stronger condition than Ausubel and Milgrom's (2002) definition (Danilov, Koshevoy, and Lang, 2003, Example 6; Baldwin and Klemperer, 2014, Footnote 47).

Nevertheless, we can show that when in defining substitutability or complementarity between *pairs* of indivisible goods, extensions of Kelso and Crawford's (1982) and Ausubel and Milgrom's (2002) approaches coincide with each other, and with our definition, even when agents can demand more than one unit of any indivisible good. We also connect these definitions to demand types.

Lemma 1. *The following properties are equivalent:*

(1) *Good i is substitutable (resp. complementary) to good k for agent j.*

(2) *For all utility levels u, price vectors $\mathbf{p}$, new prices $p'_i<p_i$, and new demands $\mathbf{x}'\in D^j_{\mathrm{H}}(p'_i,\mathbf{p}_{I\smallsetminus\{i\}};u)$, there exists $\mathbf{x}\in D^j_{\mathrm{H}}(\mathbf{p};u)$ such that $x'_k\le x_k$ (resp. $x'_k\ge x_k$).*[41]

(3) *For all utility levels u, price vectors $\mathbf{p}$, and new prices $p'_i<p_i$ with $D^j_{\mathrm{H}}(\mathbf{p};u)=\{\mathbf{x}\}$ and $D^j_{\mathrm{H}}(p'_i,\mathbf{p}_{I\smallsetminus\{i\}};u)=\{\mathbf{x}'\}$, we have that $x'_k\le x_k$ (resp. $x'_k\ge x_k$).*[42]

(4) *There exists $\mathcal{D}$ such that U^j is of demand type $\mathcal{D}$ and for all $\mathbf{d}\in\mathcal{D}$, the product d_id_k is nonpositive (resp. nonnegative).*[43]

Proof. The implications (2) $\Longrightarrow$ (1) $\Longrightarrow$ (3) are obvious. Hence, it suffices to prove the implications (3) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (4) $\Longrightarrow$ (3). To prove these implications, in light of Fact 1, we can assume that U^j is quasilinear.

To prove the implication (3) $\Longrightarrow$ (2), let $\mathbf{p}$ be a price vector, let $p'_i<p_i$ be a new price, and let $\mathbf{x}'\in D^j(\mathbf{p}')$. Let $\widehat{\mathbf{x}}'$ be an extreme point of $\operatorname{Conv} D^j(\mathbf{p})$ with highest (resp. lowest) k-component. By the upper hemicontinuity of demand, there exists $\varepsilon>0$ such that $D^j(\mathbf{p}+\mathbf{s})\subseteq D^j(\mathbf{p})$ for all $\|\mathbf{s}\|<\varepsilon$. Writing $\mathbf{p}'=(p'_i,\mathbf{p}_{I\smallsetminus\{i\}})$, by Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Claim B.2), there exists $\mathbf{s}\in\mathbb{R}^I$ with $\|\mathbf{s}\|<\varepsilon$ such that $D^j(\mathbf{p}'+\mathbf{s})=\{\widehat{\mathbf{x}}'\}$. Letting $\mathbf{x}$ be an extreme point of $\operatorname{Conv} D^j(\mathbf{p}+\mathbf{s})$, by construction, we have $\mathbf{x}\in D^j(\mathbf{p})$. By the upper hemicontinuity of demand, there exists $\varepsilon'>0$ such

40. Kelso and Crawford (1982) formulated this as a condition on how demand changes when prices increase; our notation here (less naturally) considers decreases in prices for consistency with our other definitions.

41. This property is based on Kelso and Crawford's (1982) definition of "gross substitutes" (see also Milgrom and Strulovici's (2009) definition of "weak substitutes").

42. This property is based on Ausubel and Milgrom's (2002) definition of substitutes.

43. The case of the equivalence between Properties (3) and (4) for which all goods are substitutes or all goods are complements (rather than only a particular pair of goods), and U^j is quasilinear, correspond to Baldwin and Klemperer (2019, Propositions 3.6 and 3.8). See also Danilov, Koshevoy, and Lang (2003, Theorem 1).

that $D^j(\mathbf{p}'+\mathbf{s}+\mathbf{s}')=\{\widehat{\mathbf{x}}'\}$ for all $\|\mathbf{s}'\|<\varepsilon'$. By Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Claim B.2), there exists $\mathbf{s}'\in\mathbb{R}^I$ with $\|\mathbf{s}'\|<\varepsilon'$ such that $D^j(\mathbf{p}+\mathbf{s}+\mathbf{s}')=\{\mathbf{x}\}$. Applying Property (3) to the decrease in the price of good i from $p_i+s_i+s'_i$ to $p'_i+s_i+s'_i$ starting at price vector $\mathbf{p}+\mathbf{s}+\mathbf{s}'$ yields that $\hat{x}'_k\le x_k$ (resp. $\hat{x}'_k\ge x_k$). By the construction of $\widehat{\mathbf{x}}$ and the Krein–Millman Theorem, we have that $x'_k\le\hat{x}'_k$ (resp. $x'_k\ge\hat{x}'_k$), so $x'_k\le x_k$ (resp. $x'_k\ge x_k$)—yielding (2).

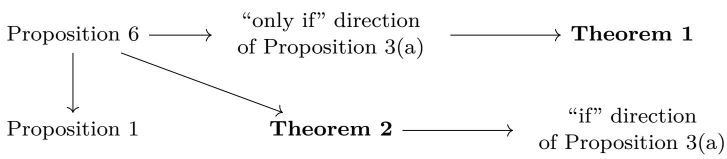
To prove the implication (1)$\Longrightarrow$(4), we prove the contrapositive. Suppose that for all $\mathcal{D}$ such that V^j is of demand type $\mathcal{D}$, there exists $\mathbf{d}\in\mathcal{D}$ such that the product d_id_k is positive (resp. negative). Then, there exists a price vector $\mathbf{p}$ such that $\operatorname{Conv}D^j(\mathbf{p})$ is a line segment parallel to a vector $\mathbf{d}$ such the product d_id_k is positive (resp. negative). Without loss of generality, assume that $d_i>0$, so $d_k>0$ (resp. $d_k<0$). Let the endpoints of this line segment be $\mathbf{x}$ and $\mathbf{x}'$, where $\mathbf{x}'-\mathbf{x}=m\mathbf{d}$ and $m>0$. Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Claim B.1) implies that there exists $\varepsilon>0$ such that $D^j(\mathbf{p}+\varepsilon\mathbf{e}^i)=\{\mathbf{x}\}$. Thus, considering the reduction in the price of i by ε starting from price vector $\mathbf{p}+\varepsilon\mathbf{e}^i$, we see that good i is not substitutable (resp. complementary) to good k for agent j.

The quasilinear case of the implication (4)$\Longrightarrow$(3) follows from Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Proposition D.2). ||

C. PROOFS FOR CASE WHEN AGENTS DEMAND AT MOST ONE UNIT OF EACH GOOD

In this appendix, we prove our results for the case in which each agent demands at most one unit of each good. Specifically, we prove Theorems 1 and 2, and Propositions 1, 3(a), and 6. Except for Proposition 6, these results are special cases of results for the case in each agent can demand multiple units of any good. However, the arguments for the general case are much more involved (see Online Appendix D), so it is instructive to develop our analysis by starting with an important special case for which simpler arguments are available.

We present the proofs of results in their logical order, which differs from the order in which results are stated in the text. The logical dependencies between results are as follows.



Note that Lemma 2, stated in the proof of the "only if" direction of Proposition 3(a), is also used in the proof of Theorem 2.

C.1. *Proof of Proposition 6*

By Fact 1, we can assume that agent j's utility function is quasilinear.

Proof of the "if" direction. The "if" direction follows directly from the "if" direction of Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorem 1). ||

Proof of the "only if" direction. Let $\Delta\mathbf{x}$ be a price effect for agent j. Then, there exists a good i, a price vector $\mathbf{p}$ with $D^j(\mathbf{p})=\{\mathbf{x}\}$, and a new price $p'_i<p_i$ for good i such that $\mathbf{x}+\Delta\mathbf{x}\in D^j(p'_i,\mathbf{p}_{I\smallsetminus\{i\}})$. By the upper hemicontinuity of demand, there exists $\varepsilon>0$ such that $D^j(\mathbf{p}+\mathbf{s})\subseteq D^j(\mathbf{p})$ for all $\|\mathbf{s}\|<\varepsilon$. As agent j demands at most one unit of each good, we have $D^j(p'_i,\mathbf{p}_{I\smallsetminus\{i\}})\subseteq\{0,1\}^I$, and hence $\mathbf{x}+\Delta\mathbf{x}$ is an extreme point of $D^j(p'_i,\mathbf{p}_{I\smallsetminus\{i\}})$. Writing $\mathbf{p}'=(p'_i,\mathbf{p}_{I\smallsetminus\{i\}})$, by Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Claim B.2), there exists $\mathbf{s}\in\mathbb{R}^I$ with $\|\mathbf{s}\|<\varepsilon$ such that $D^j(\mathbf{p}'+\mathbf{s})=\{\mathbf{x}+\Delta\mathbf{x}\}$. By the definition of ε, we have $D^j(\mathbf{p}+\mathbf{s})=\{\mathbf{x}\}$, so lowering the price of good i from $\mathbf{p}+\mathbf{s}$ makes agent j change from uniquely demanding $\mathbf{x}$ to uniquely demanding $\mathbf{x}+\Delta\mathbf{x}$. Hence, the "only if" direction of Baldwin, Jagadeesan, Klemperer, and Teytelboym (2021, Theorm 1) implies that $\Delta\mathbf{x}\in\mathcal{D}$. ||

C.2. *Proof of Proposition 1*

Suppose that U^j is of demand type $\mathcal{D}$. By Proposition 6, we have $\Delta\mathbf{x}\in\mathcal{D}$.

Proof of Part (a). Note that $(\Delta\mathbf{x})_k(\Delta\mathbf{x})_\ell > 0$ and that $\Delta\mathbf{x} \in \mathcal{D}$. Since $\mathcal{D}$ was arbitrary with U^j of demand type $\mathcal{D}$, the contrapositive of the $(1) \Longrightarrow (4)$ implication of Lemma 1 implies that k and ℓ must be strict complements at some prices. ‖

Proof of Part (b). Note that $(\Delta\mathbf{x})_k(\Delta\mathbf{x})_\ell < 0$ and that $\Delta\mathbf{x} \in \mathcal{D}$. Since $\mathcal{D}$ was arbitrary with U^j of demand type $\mathcal{D}$, the contrapositive of the $(1) \Longrightarrow (4)$ implication of Lemma 1 implies that k and ℓ must be strict substitutes at some prices. ‖

C.3. *Proof of the "only if" direction of Proposition 3(a)*

The proof relies on a more technical connection between bundle consistency and total unimodularity, which we also use in the proof of Theorem 2.

Lemma 2. *Suppose that each agent demands at most one unit of each good. Let S be a set of compensated price effects. If S is not totally unimodular, then there exists a bundling $\mathcal{B} \subseteq S \cup \{\mathbf{e}^i | i \in I\}$ that contains a pair of inconsistent bundles and excludes at least two elements of $S \smallsetminus \{\mathbf{e}^i \mid i \in I\}$.*

The proof of the "only if" direction of Proposition 3(a) does not use the conclusion that $\mathcal{B}$ excludes at least two elements of $S \smallsetminus \{\mathbf{e}^i \mid i \in I\}$; we only use this conclusion in the proof of Theorem 2. To prove Lemma 2, we use the following technical linear algebraic claim.[44]

Claim 1. *If $S \subseteq \{-1,0,1\}^I$ is not totally unimodular, then there exist linearly independent vectors $\mathbf{s}^1,\ldots,\mathbf{s}^{|I|} \in S \cup \{\mathbf{e}^i \mid i \in I\}$ and vectors $\mathbf{d},\mathbf{d}' \in S \smallsetminus \{\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}\} \smallsetminus \{\mathbf{e}^i \mid i \in I\}$ such that letting G be the $I \times |I|$ matrix whose columns are $\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}$, there exist $1 \le k,\ell \le |I|$ with $(G^{-1}\mathbf{d})_k(G^{-1}\mathbf{d})_\ell > 0$ and $(G^{-1}\mathbf{d}')_k(G^{-1}\mathbf{d}')_\ell < 0$.*

Proof. By removing elementary basis vectors or their negations from S if necessary, we can assume that S does not contain any such vectors.[45] By further removing vectors from S if necessary, we can assume that there exists a set $I' \subseteq I$ of goods such that a matrix B with columns elements of $\{\mathbf{s}_{I'} \mid \mathbf{s} \in S\}$ is square of determininant not 0 or ± 1.

The remainder of the proof follows Schrijver (1998, pages 270–271). Consider the matrix $C = [B\,\mathrm{Id}_{|I'| \times |I'|}]$; and a matrix $\widetilde{C}$ obtained from C by row addition and subtraction and row multiplication by -1 such that (i) all entries in the first $|I'|$ columns of $\widetilde{C}$ are 0 or ± 1, (ii) the columns of $\widetilde{C}$ include the elementary basis vectors $\mathbf{e}^i$ for $i \in I'$, and (iii) the first $|I'|$ columns of $\widetilde{C}$ include as many elementary basis vectors as possible. As row addition and subtraction and row multiplication by -1 do not change the magnitude of the determinant, letting $\tilde{B}$ be the square matrix consisting of the first $|I'|$ columns of $\widetilde{C}$, we have $\det \tilde{B} = \pm \det B$.

Since $\det B \notin \{-1,0,1\}$, we have $\det \tilde{B} \notin \{-1,0,1\}$. Hence, the columns of $\tilde{B}$ cannot include all of the elementary basis vectors $\mathbf{e}^i$ for $i \in I'$. Suppose that $\mathbf{e}^k$ is not a column of $\tilde{B}$. As $\det \tilde{B} \neq 0$, there exists a column $\mathbf{c}$ of $\tilde{B}$ (say the jth column of $\tilde{B}$) with $c_k \neq 0$.

Consider the matrix $\overline{C}$ adding (resp. subtracting) the kth row of $\widetilde{C}$ to (resp. from) each row ℓ with $c_\ell = -c_k$ (resp. $c_\ell = c_k$). Note that $\overline{C}$ satisfies (ii) by construction, and its first $|I'|$ columns include strictly more elementary basis vectors than $\widetilde{C}$. Hence, by the definition of $\widetilde{C}$, there must be an entry of $\overline{C}$ that is not in $\{-1,0,1\}$. By construction, the kth row of $\overline{C}$ is the kth row of C. Therefore, there must exist (j',ℓ) with $j' \le |I'|$ and $\ell \neq k$ such that $\overline{C}_{\ell,j'} \notin \{-1,0,1\}$. Let $\mathbf{c}'$ be the j'th column of C. As $\overline{C}_{\ell,j'} = c'_\ell - \frac{c_\ell}{c_k} c'_k$ (by construction) and the entries of $\tilde{B}$ are all 0 or ± 1, we must have $c'_\ell = -\frac{c_\ell}{c_k} c'_k \neq 0$. We then have $c_k, c_\ell, c'_k, c'_\ell \in \{-1,1\}$ and hence $c_k c_\ell = -c'_k c_\ell$. Without loss of generality, by exchanging the roles of $\mathbf{c},\mathbf{c}'$ if necessary, we can assume that $c_k c_\ell > 0 > c'_k c'_\ell$.

For each $1 \le n \le |I'|$, suppose that the $\sigma(n)$th column of $\widetilde{C}$ is the nth elementary basis vector, and let $\mathbf{s}^n$ be the $\sigma(n)$th column of C. Let $\mathbf{s}^{|I'|+1},\ldots,\mathbf{s}^{|I|}$ be the elementary basis vectors $\{\mathbf{e}^i \mid i \in$

44. We also use Claim 1 in Online Appendix D in the proof of the "only if" direction of Proposition 3(b).

45. Indeed, the total unimodularity of a set of integer vectors is unaffected by adding or removing elementary basis vectors or their negations from the set (see, e.g., Equation (43)(v) in Schrijver (1998, page 200)).

$I\smallsetminus I'\}$ in any order. Similarly, let $\mathbf{d}$ and $\mathbf{d}'$ be the the jth and j'th columns of $\widetilde{C}$, respectively. By construction, the first $|I'|$ components of $G^{-1}\mathbf{d}$ (resp. $G^{-1}\mathbf{d}'$) are given by $\mathbf{c}$ (resp. $\mathbf{c}'$). In particular, we have $(G^{-1}\mathbf{d})_k(G^{-1}\mathbf{d})_\ell>0$ and $(G^{-1}\mathbf{d}')_k(G^{-1}\mathbf{d}')_\ell<0$. As $G^{-1}\mathbf{s}^i=\mathbf{e}^i$ for $1\leq i\leq|I|$, by construction, we have that $\mathbf{d},\mathbf{d}'\notin\{\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}\}$, and the lemma follows. ‖

Proof of Lemma 2. Let $S'\subseteq S\cup\{\mathbf{e}^i\,|\,i\in I\}$ be a minimal subset that is not totally unimodular. Claim 1 implies that there exist linearly independent vectors $\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}\in S'$, vectors $\mathbf{d},\mathbf{d}'\in S'\smallsetminus\{\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}\}\smallsetminus\{\mathbf{e}^i\,|\,i\in I\}$, and $1\leq k,\ell\leq|I|$ such that letting G be the $I\times|I|$ matrix whose columns are $\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}$, we have that $(G^{-1}\mathbf{d})_k(G^{-1}\mathbf{d})_\ell>0$ and that $(G^{-1}\mathbf{d}')_k(G^{-1}\mathbf{d}')_\ell<0$. The minimality of S' implies that G must have determinant ±1.

For each agent j, define a transformed utility function $\widetilde{U}^j:\mathbb{R}\times(G^{-1}X^j)\to\mathbb{R}$ by $\widetilde{U}^j(x_0,\mathbf{q})=U^j(x_0,G\mathbf{q})$.[46] By Proposition 6, all agents' preferences are of demand type $\mathcal{D}$ only if $\mathcal{D}\supseteq S$. Baldwin and Klemperer (2019, Proposition 3.11) then implies that all the transformed preferences are of demand type $\mathcal{D}$ only if $\mathcal{D}\supseteq G^{-1}S$. Since $(G^{-1}\mathbf{d})_k(G^{-1}\mathbf{d})_\ell>0$, the contrapositive of the $(1)\Longrightarrow(4)$ implication of Lemma 1 implies that under the transformed utility functions, there is an agent for whom the kth and ℓth commodities are not substitutes. Similarly, since $(G^{-1}\mathbf{d}')_k(G^{-1}\mathbf{d}')_\ell<0$, under the transformed utility functions, there is an agent for whom the kth and ℓth commodities are not complements.

Consider the bundling $\mathcal{B}=\{\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}\}$. The definition of the transformed utility functions implies that $\widetilde{D}^j_{\mathrm{H}}(\widetilde{\mathbf{p}};u,\mathcal{B})$ is Hicksian demand for $\widetilde{U}^j$ at price vector $\widetilde{\mathbf{p}}$ and utility level u. Hence, for the bundling $\mathcal{B}$, bundles $\mathbf{s}^k$ and $\mathbf{s}^\ell$ are neither complements nor substitutes. Since $\mathbf{d},\mathbf{d}'\in S'\smallsetminus\{\mathbf{s}^1,\ldots,\mathbf{s}^{|I|}\}\smallsetminus\{\mathbf{e}^i\,|\,i\in I\}$, the bundling $\mathcal{B}$ excludes at least two elements of $S'\smallsetminus\{\mathbf{e}^i\,|\,i\in I\}$ (namely $\mathbf{d}$ and $\mathbf{d}'$), and hence at least two elements of $S\smallsetminus\{\mathbf{e}^i\,|\,i\in I\}$. ‖

To complete the proof of the "only if" direction of Proposition 3(a), let S be the set of all compensated price effects for all agents and goods. As any bundling $\mathcal{B}\subseteq S\cup\{\mathbf{e}^i\,|\,i\in I\}$ is comprised solely of relevant bundles, which must be consistent by hypothesis, the contrapositive of Lemma 2 implies that S is totally unimodular.

C.4. *Proof of Theorem 1*

Let S be the set of all compensated price effects for all agents and goods that lie in $\{-1,0,1\}^I$, and let $\mathcal{D}=S\cup-S$. The "only if" direction of Proposition 3(a) implies that S (and hence $\mathcal{D}$) is totally unimodular. By Proposition 6, all agents' preferences are of demand type $\mathcal{D}$. Since each agent demands at most one unit of each good, all utility functions are quasipseudoconcave, hence $\mathcal{D}$-quasiconcave. Danilov, Koshevoy, and Murota (2001, Theorem 2 and 4) or Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2020, Theorem 3) therefore guarantees that competitive equilibria exists.

C.5. *Proof of Theorem 2*

We prove the contrapositive of the theorem. Suppose that $\mathcal{V}$ is invariant but not bundle-consistent; we show that there exist valuations in $\mathcal{V}$ and endowments for which competitive equilibria do not exist.

The proof uses the following fact regarding equilibrium prices in transferable utility economies.[47] The fact is that if competitive equilibrium exists, then every price vector at which $\mathbf{0}$ lies in the convex hull of aggregate excess demand is the price vector of a competitive equilibrium. Its contrapositive lets us prove equilibrium does not exist by constructing a non-equilibrium price vector at which $\mathbf{0}$ lies in the convex hull of aggregate excess demand.

Fact *2 **(Milgrom and Strulovici, 2009, Proposition 2 and Theorem 18; Baldwin and Klemperer, 2019, Lemma 2.11)** In transferable utility economies, if competitive equilibria exist and a price vector* $\mathbf{p}$ *satisfies* $\sum_{j\in J}\mathbf{e}^j\in\mathrm{Conv}\sum_{j\in J}D^j(\mathbf{p})$, *then* $\mathbf{p}$ *is a price vector of a competitive equilibrium.*

46. This extends a construction of Baldwin and Klemperer (2019, page 885) to settings with income effects.

47. We also Fact 2 in Online Appendix D in the proofs of the "if" directions of Propositions 5 and $5'$, and of Theorem 4.

The strategy of the proof is to consider a bundling $\mathcal{B}$ comprised of relevant bundles and two valuations in $\mathcal{V}$ that exhibit an inconsistency between two relevant bundles $\mathbf{b}^1,\mathbf{b}^2$. Using Part (i) of the definition of invariance, we can make the inconsistency occur at a single price vector $\mathbf{p}$. We then introduce other agents in the economy with valuations in $\mathcal{V}$ to make it so that aggregate excess demand for $\mathbf{b}^1$ at $\mathbf{p}$ is always -1 or 1, but that $\mathbf{0}$ lies in the convex hull of aggregate excess demand. These additional agents have bundles in $\mathcal{B}$ as price effects. Due to the relevance of bundles in $\mathcal{B}$, every bundle in $\mathcal{B}$ that is not a price effect is an elementary basis vector, in the construction, we use Part (ii) of the definition of invariance to construct such valuations in $\mathcal{V}$ whose price effects are elementary basis vectors.

As a preliminary step, we use the definition of invariance to construct valuations in $\mathcal{V}$ under which demand is $\{\mathbf{0},\mathbf{e}^i\}$ when all goods are priced at a particular price $K>0$.[48]

Claim 2. *Let $\mathcal{V}$ be an invariant domain. There exists a constant $K>0$ and, for each good i, a valuation $\check{V}^i:\check{X}^i\to\mathbb{R}$ in the domain such that $\{\mathbf{0},\mathbf{e}^i\}\subseteq\check{X}^i\subseteq\{0,1\}^I$, and $\check{V}^i(\mathbf{x})-\check{V}^i(\mathbf{0})\le K\sum_{k\in I}x_k$ for all $\mathbf{x}\in\check{X}^i$, with equality if and only if $\mathbf{x}\in\{\mathbf{0},\mathbf{e}^i\}$.*

Proof. For each good i, let $\check{V}^i\in\mathcal{V}$ be a valuation whose set of feasible consumption vectors $\check{X}^i$ satisfies $\{\mathbf{0},\mathbf{e}^i\}\subseteq\check{X}^i\subseteq\{0,1\}^I$. We can normalize $\check{V}^i(\mathbf{0})=0$ for all i without loss of generality. Let $K>0$ be a constant such that for all goods i, we have $K\ge\check{V}^i(\mathbf{e}^i)$ and

$$\check{V}^i(\mathbf{x})-x_i\check{V}^i(\mathbf{e}^i)<K\sum_{k\in I\smallsetminus\{i\}}x_k \quad \text{for all } \mathbf{x}\in\check{X}^i\smallsetminus\{\mathbf{0},\mathbf{e}^i\};$$

such a constant K exists as the hypothesis that $\check{X}^i\subseteq\{0,1\}^I$ implies that $\sum_{k\neq i}x_k>0$ for all $\mathbf{x}\in\check{X}^i\smallsetminus\{\mathbf{0},\mathbf{e}^i\}$. By invariance, by adding a linear valuation $(K-\check{V}^i(\mathbf{e}^i))x_i$ to each valuation V^i, we can ensure that $\check{V}^i(\mathbf{e}^i)=K$ for all i and preserve the property that $\check{V}^i\in\mathcal{V}$. It follows that $\check{V}^i(\mathbf{x})<K\sum_{k\in I}x_k$ for all i and $\mathbf{x}\in X^i\smallsetminus\{\mathbf{0},\mathbf{e}^i\}$, as desired. ‖

For the remainder of the proof, let $K>0$ and $(\check{V}^i:\check{X}^i\to\mathbb{R})_{i\in I}$ be as in Claim 2.

Let $\mathcal{B}$ be a bundling consisting of relevant bundles that contains two bundles that are inconsistent. Among all such bundlings, consider one that includes as many elementary basis vectors as possible. Let $\mathcal{B}=\{\mathbf{b}^1,\dots,\mathbf{b}^{|I|}\}$, where $\mathbf{b}^1$ and $\mathbf{b}^2$ are inconsistent given $\mathcal{B}$.

We claim that $\mathcal{B}$ is totally unimodular. Suppose for the sake of deriving a contradiction that $\mathcal{B}$ is not totally unimodular. Applying Lemma 2 to $S=\mathcal{B}$, there exists a bundling $\mathcal{B}'\subseteq\mathcal{B}\cup\{\mathbf{e}^i\mid i\in I\}$ that excludes at least two elements of $\mathcal{B}\smallsetminus\{\mathbf{e}^i\mid i\in I\}$ but includes a pair of inconsistent bundles. Then, $\mathcal{B}'$ contains at least two more elementary basis vectors than $\mathcal{B}$—contradicting the choice of $\mathcal{B}$. Hence, $\mathcal{B}$ must be totally unimodular.

In particular, letting G be the matrix whose columns are $\mathbf{b}^1,\dots,\mathbf{b}^{|I|}$ in that order, the determinant of G is ±1. We use G to transform valuations and demand type vectors.

Let $V^{\mathsf{c}}\in\mathcal{V}$ (resp. $V^{\mathsf{s}}\in\mathcal{V}$) be a valuation under which $\mathbf{b}^1$ and $\mathbf{b}^2$ are not substitutes (resp. complements) given bundling $\mathcal{B}$. Given a valuation V^j, define a transformed $G^*V^j:G^{-1}X^j\to\mathbb{R}$ by $(G^*V^j)(\mathbf{q})=V^j(G\mathbf{q})$ (Baldwin and Klemperer, 2019, page 885). By construction, demand for G^*V^j at price vector $\widetilde{\mathbf{p}}$ is $\widetilde{D}^j(\widetilde{\mathbf{p}};\mathcal{B})$. Hence, the first and second goods are not substitutes (resp. complements) under G^*V^{c} (resp. G^*V^{s}). By the contrapositive of the $(4)\Longrightarrow(1)$ implication of Lemma 1, if G^*V^{c} (resp. G^*V^{s}) is of demand type $\mathcal{D}$, there must exist $\mathbf{d}\in\mathcal{D}$ with $d_1d_2>0$ (resp. $d_1d_2<0$). Baldwin and Klemperer (2019, Proposition 3.11) implies that V^j is of demand type $\mathcal{D}$ if and only if that G^*V^j is of demand type $G^{-1}\mathcal{D}$. Hence, if V^{c} (resp. V^{s}) is of demand type $\mathcal{D}$, then there must exist $\mathbf{d}\in\mathcal{D}$ with $(G^{-1}\mathbf{d})_1(G^{-1}\mathbf{d})_2>0$ (resp. $(G^{-1}\mathbf{d})_1(G^{-1}\mathbf{d})_2<0$). Thus, there exists a price vector $\mathbf{p}^{\mathsf{c}}$ (resp. $\mathbf{p}^{\mathsf{s}}$) such that $\operatorname{Conv}D^{\mathsf{c}}(\mathbf{p}^{\mathsf{c}})$ (resp. $\operatorname{Conv}D^{\mathsf{s}}(\mathbf{p}^{\mathsf{s}})$) is one-dimensional and parallel to an integer vector $\mathbf{d}^{\mathsf{c}}$ (resp. $\mathbf{d}^{\mathsf{s}}$) whose components have no nontrivial common factors such that $(G^{-1}\mathbf{d}^{\mathsf{c}})_1(G^{-1}\mathbf{d}^{\mathsf{c}})_2>0$ (resp. $(G^{-1}\mathbf{d}^{\mathsf{s}})_1(G^{-1}\mathbf{d}^{\mathsf{s}})_2<0$).

We next show that $\mathcal{B}\cup\{\mathbf{d}^{\mathsf{c}}\}$ and $\mathcal{B}\cup\{\mathbf{d}^{\mathsf{s}}\}$ are totally unimodular. To show this, let $\overline{\mathcal{D}}$ be the set of all integer vectors whose components have no nontrivial common factors, and let $\mathcal{D}^{\mathsf{c}}=\overline{\mathcal{D}}\smallsetminus\{\pm\mathbf{d}^{\mathsf{c}}\}$. Since V^{c} is not of demand type $\mathcal{D}^{\mathsf{c}}$, by Proposition 6, $\pm\mathbf{d}^{\mathsf{c}}$ must be a price effect for V^{c}. Now, suppose for the sake of deriving a contradiction that $\mathcal{B}\cup\{\mathbf{d}^{\mathsf{c}}\}$ is not totally unimodular. Applying Lemma 2 to $S=\mathcal{B}\cup\{\mathbf{d}^{\mathsf{c}}\}$, there exists a bundling $\mathcal{B}'\subseteq\mathcal{B}\cup\{\mathbf{d}^{\mathsf{c}}\}\cup\{\mathbf{e}^i\mid i\in I\}$ that excludes at least two elements of $(\mathcal{B}\cup\{\mathbf{d}^{\mathsf{c}}\})\smallsetminus\{\mathbf{e}^i\mid i\in I\}$ but includes a pair of inconsistent bundles. Then, $\mathcal{B}'$ contains at least one more elementary basis vectors than $\mathcal{B}$—contradicting the choice of $\mathcal{B}$. Hence, $\mathcal{B}\cup\{\mathbf{d}^{\mathsf{c}}\}$ must be totally unimodular. Similarly, $\mathcal{B}\cup\{\mathbf{d}^{\mathsf{s}}\}$ must be totally unimodular.

48. We also use Claim 2 in Online Appendix D in the proof of Theorem 4.

In particular, for each $1 \le i \le |I|$, the matrix obtained by replacing the ith column of G by $\mathbf{d}^{\mathsf{c}}$ must have determinant 0 or ± 1. Multiplying by G^{-1} on the left, we obtain that the matrix whose columns are $\mathbf{e}^1,\ldots,\mathbf{e}^{i-1},G^{-1}\mathbf{d}^{\mathsf{c}},\mathbf{e}^{i+1},\ldots,\mathbf{e}^{|I|}$ in that order must also have determinant 0 or ± 1. But that matrix has determinant $(G^{-1}\mathbf{d}^{\mathsf{c}})_i$. Hence, the components of $G^{-1}\mathbf{d}^{\mathsf{c}}$ must each be 0 or ± 1. Similarly, the components of $G^{-1}\mathbf{d}^{\mathsf{s}}$ must each be 0 or ± 1.

Without loss of generality, we can assume that $(G^{-1}\mathbf{d}^{\mathsf{c}})_1 = (G^{-1}\mathbf{d}^{\mathsf{s}})_1 = 1$. We must then have that $(G^{-1}\mathbf{d}^{\mathsf{c}})_2 = 1$ and $(G^{-1}\mathbf{d}^{\mathsf{s}})_2 = -1$. It follows that

$$\frac{1}{2}G^{-1}\mathbf{d}^{\mathsf{c}} + \frac{1}{2}G^{-1}\mathbf{d}^{\mathsf{s}} + \sum_{j=3}^{|I|} \frac{-\left[(G^{-1}\mathbf{d}^{\mathsf{c}})_j + (G^{-1}\mathbf{d}^{\mathsf{s}})_j\right]}{2}\mathbf{e}^j = \mathbf{e}^1.$$

Multiplying both sides by G on the left yields that

$$\frac{1}{2}\mathbf{d}^{\mathsf{c}} + \frac{1}{2}\mathbf{d}^{\mathsf{s}} + \sum_{j=3}^{|I|} \frac{-\left[(G^{-1}\mathbf{d}^{\mathsf{c}})_j + (G^{-1}\mathbf{d}^{\mathsf{s}})_j\right]}{2}\mathbf{b}^j = \mathbf{b}^1, \tag{C.1}$$

where we have $\left|\frac{(G^{-1}\mathbf{d}^{\mathsf{c}})_j + (G^{-1}\mathbf{d}^{\mathsf{s}})_j}{2}\right| \le 1$ for $3 \le j \le |I|$ since $|(G^{-1}\mathbf{d}^{\mathsf{c}})_j|,|(G^{-1}\mathbf{d}^{\mathsf{s}})_j| \le 1$.

We next show that for each $1 \le j \le |I|$, there exists a valuation $V^j \in \mathcal{V}$ for which $\mathbf{b}^j$ is a price effect. As $\mathcal{B}$ is comprised solely of relevant bundles, each bundle $\mathbf{b}^j$ is a price effect for some valuation in $V^j \in \mathcal{V}$ or an elementary basis vector. If $\mathbf{b}^j$ is an elementary basis vector $\mathbf{e}^i$, then considering the valuation $V^j = \check{V}^i \in \mathcal{V}$, Claim 2 implies that demand when all goods are priced above K is $\{\mathbf{0}\}$, and lowering the price of good i to K changes demand to $\{\mathbf{0},\mathbf{e}^i\}$. In particular, $\mathbf{e}^i$ is a price effect for V^j.

For each $3 \le j \le |I|$, Proposition 6 tells us that V^j is of demand type $\mathcal{D}$ only if $\mathbf{b}^j \in \mathcal{D}$. Thus, there exists a price vector $\mathbf{p}^j$ such that $\operatorname{Conv} D^j(\mathbf{p}^j)$ is one-dimensional and parallel to $\mathbf{b}^j$. For $j = 1,2$, as $\mathbf{b}^j$ is a price effect for V^j, there exists a price vector $\mathbf{p}^j$ and a bundle $\mathbf{x}^j$ such that $D^j(\mathbf{p}^j) = \{\mathbf{x}^j\}$ and $\mathbf{x}^j + \mathbf{b}^j \in X^j$. By invariance, adding linear functions to each valuation V^j and $V^{\mathsf{c}},V^{\mathsf{s}}$ if necessary, we can ensure that $\mathbf{p}^j = \mathbf{p}^{\mathsf{c}} = \mathbf{p}^{\mathsf{s}} = \mathbf{p}$ for $1 \le j \le |I|$.

For $j \in \{\mathsf{c},\mathsf{s}\}$, let one endpoint of $\operatorname{Conv} D^j(\mathbf{p})$ be $\mathbf{w}^j$ such that $\operatorname{Conv} D^j(\mathbf{p})$ is a subset of the ray with endpoint $\mathbf{w}^j$ and direction $\mathbf{d}^j$. Let $\mathbf{w}^1 = \mathbf{x}^1 + \mathbf{b}^1$ and $\mathbf{w}^2 = \mathbf{x}^2$. For $3 \le j \le |I|$, let one endpoint of $\operatorname{Conv} D^j(\mathbf{p})$ be $\mathbf{w}^j$ such that $\operatorname{Conv} D^j(\mathbf{p})$ is a subset of the ray with endpoint $\mathbf{w}^j$ and direction $-\mathbf{b}^j$ (resp. $\mathbf{b}^j$) if $(G^{-1}\mathbf{d}^1)_j + (G^{-1}\mathbf{d}^2)_j \ge 0$ (resp. $(G^{-1}\mathbf{d}^1)_j + (G^{-1}\mathbf{d}^2)_j < 0$).

Consider the economy with agents $1,2,\ldots,|I|,\mathsf{c},\mathsf{s}$, where agent j has valuation V^j and endowment $\mathbf{w}^j$ of indivisible goods. By construction, at the price vector $\mathbf{p}$:

- excess demand for c (resp. s) is a subset of $\mathbb{R}_{\ge 0}\mathbf{d}^{\mathsf{s}}$ (resp. $\mathbb{R}_{\ge 0}\mathbf{d}^{\mathsf{s}}$) that includes at least one nonzero integer vector,
- excess demand for agent 1 is $-\mathbf{b}^1$, and excess demand for agent 2 is $\mathbf{0}$;
- if $(G^{-1}\mathbf{d}^1)_j + (G^{-1}\mathbf{d}^2)_j \ge 0$, excess demand for agent $3 \le j \le |I|$ is a subset of $\mathbb{R}_{\le 0}\mathbf{b}^j$ that includes at least nonzero integer vector; and
- if $(G^{-1}\mathbf{d}^1)_j + (G^{-1}\mathbf{d}^2)_j < 0$, excess demand for agent $3 \le j \le |I|$ is a subset of $\mathbb{R}_{\ge 0}\mathbf{b}^j$ that includes at least nonzero integer vector.

Hence, it follows from (C.1) that $\mathbf{0}$ lies in the convex hull of aggregate excess demand at price vector $\mathbf{p}$. However, as the vectors $\mathbf{d}^1,\mathbf{d}^2,\mathbf{b}^3,\mathbf{b}^4,\ldots,\mathbf{b}^{|I|}$ are linearly independent and the summands $\frac{1}{2}\mathbf{d}^1$ and $\frac{1}{2}\mathbf{d}^2$ are not integer, $\mathbf{0}$ does not lie in aggregate excess demand at price vector $\mathbf{p}$. Hence, by the contrapositive of Fact 2, no competitive equilibria can exist.

C.6. *Proof of the "if" direction of Proposition 3(a)*

By Fact 1, we can assume that agent j's utility function is quasilinear.

Let S be the set of all compensated price effects for all agents and goods, and let $\mathcal{D} = S \cup -S$, which is totally unimodular by hypothesis. Let $\mathcal{D}' = \mathcal{D} \cup \{\pm\mathbf{e}^i \mid i \in I\}$, which is also totally unimodular because the total unimodularity of a set of integer vectors is unaffected by adding elementary basis vectors and their negations (see, e.g., Equation (43)(v) in Schrijver (1998, page 200)). Proposition 6 implies that preferences are all of demand type $\mathcal{D}'$.

Consider the class $\mathcal{V}$ of all valuations of demand type $\mathcal{D}'$ under which at most one unit of each good is demanded. Part (i) of the definition of invariance is automatically satisfied. As $\mathcal{D}'$ contains the elementary basis vectors, the zero valuation $V_0:\{0,1\}^I \to \mathbb{R}$ defined by $V_0(\mathbf{x})=0$ is $\mathcal{D}'$-quasiconcave; hence, taking $\check{V}^i = V_0$ for all goods i, we can see that Part (ii) of the definition of invariance is satisfied too. Since each agent demands at most one unit of each good, all valuations in $\mathcal{V}$ are pseudoconcave. Because the pseudoconcave valuations of demand type $\mathcal{D}'$ form a domain for equilibrium existence (Danilov, Koshevoy, and Murota, 2001, Theorems 2 and 4; Baldwin and Klemperer, 2019, Theorem 4.3), Theorem 2 implies that $\mathcal{V}$ is bundle-consistent. Hence, preferences are bundle-consistent.

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