Auctions with Frictions: Recruitment, Entry, and Limited Commitment*

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Abstract

Auction models are convenient abstractions of informal price-formation processes that arise in markets for assets or services. These processes involve frictions like bidder recruitment costs for sellers, participation costs for bidders, and limitations on sellers' commitment abilities. This paper develops an auction model that captures such frictions. We derive novel insights, notably that outcomes are often inefficient, that markets sometimes unravel, and that the observability of competition may have a large effect.

Recruiting and motivating bidders are crucial in auctions, possibly impacting revenue more than the details of the bidding mechanism.[1] Recruitment is often challenging due to the substantial costs bidders incur to evaluate items, secure financing, and prepare bids. Sellers' recruitment costs and bidders' participation costs are particularly likely to be significant in the sale of idiosyncratic assets. Another salient feature of such auctions is the seller's limited ability to commit to the extent of participation, to its disclosure to bidders, and sometimes even to the rules.

This paper investigates how sellers' recruitment efforts and bidders' entry decisions jointly determine auction participation and outcomes when the seller's commitment ability is limited.

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[1]See Bulow and Klemperer (1996) for the revenue effects of attracting an additional bidder.

The main novelty of the model lies in its integration of these elements: costly recruitment, costly participation, and limited commitment. It captures the fundamental conflict between the seller's desire to recruit more bidders—to intensify competition and draw high-value bidders—and the bidders' apprehension about costly entry into an overly competitive auction. Without the seller's ability to commit to participation levels, this tension can result in excessive recruitment efforts and, occasionally, a total cessation of trade. These intrinsic inefficiencies are the subject of our first set of insights.

A related set of insights pertains to the impact of bidders' ability to observe the level of participation (or the seller's ability to disclose it credibly). We identify conditions on the combination of recruitment and entry costs under which observability can promote or suppress trade. As we explain later, these insights can also be interpreted as a comparison of the first-price auction (FPA) and the second-price auction (SPA) in the presence of recruitment and entry costs.

The aforementioned tensions and insights have not been studied before, since the existing literature has studied entry and recruitment separately. They may help explain the viability of costly intermediary services that recruit bidders, facilitate seller commitment, and reduce bidders' costs. Sellers' willingness to pay 20 to 30 percent of their revenues to auction houses like Christie's and Sotheby's may partially reflect the reduction of the inefficiencies exposed here.[2]

The model features a seller who offers a single item for sale. In the recruitment stage, the seller makes a costly effort to attract bidders. The random number of bidders contacted follows a Poisson distribution whose mean γ is determined by the seller's effort. A contacted bidder decides whether to incur a cost to discover his own private value and participate in the auction. The bidding stage is a first-price auction (FPA). The seller cannot commit to the level of the recruitment effort (which is unobservable to the bidders) or to a reserve price. We consider two variants of the auction stage: the *PO scenario* ("participation-observable") and the *PU scenario* ("participation-unobservable").

In the PO scenario, bidders observe the number of auction participants before bidding (e.g., when the bidding is in person at the auction site). This observability reinforces the recruitment incentives, since greater participation induces more aggressive bidding. This in turn contributes to the inefficiencies mentioned above, namely, excessive recruitment and potential market shutdown. In the PU scenario, bidders do not observe the number

[2] Auction houses are also rewarded for other useful services, such as quality certification. Ashenfelter and Graddy (2003) describe the fees and other institutional details for such auction houses.

of participants (and the seller cannot credibly disclose it).[3] This unobservability generates an incentive for the seller to secretly reduce recruitment, since that does not depress the bidding. This may give rise to multiple equilibria sustained by different levels of fulfilled expectations. In particular, an equilibrium with no trade always exists.

The comparison of these two scenarios exposes the important role of (un)observability when the seller cannot credibly commit to a target recruitment level. The profit and trade volume when the competition is observable (in the unique PO equilibrium) may be higher or lower than when it is unobservable (in the most profitable PU equilibrium). Generally, when the marginal recruitment cost is relatively high and the bidders' entry cost relatively low, bidders are willing to enter, and the main challenge for the seller is to induce aggressive bidding. The PO scenario is more effective in this regard, as increased recruitment leads to greater observed competition, which in turn induces higher bids. This is not the case, however, in the PU scenario, where competition is not observable. When the marginal recruitment cost is even higher within this range, only the PO scenario supports trade.

In contrast, when the bidders' entry cost is relatively large and the marginal recruitment cost is relatively small, the seller's main challenge is to convince bidders to enter. The weaker recruitment incentive in the PU scenario ensures that bidders are less concerned about strong competition and more willing to enter than in the PO scenario. Consequently, the PU scenario is more profitable, and when the bidders' entry cost is even higher within this range, only the PU scenario supports trade.

This comparison also indicates in which situations the seller would prefer to either conceal or disclose participation information, if this could be done credibly, which is not possible in our model.

The same results can be viewed from another angle: they apply directly to a comparison of the first-price auction (FPA) and second-price auction (SPA), both with unobservable participation. The PU scenario is by definition an FPA with unobservable participation. By standard revenue-equivalence arguments explained later in the paper, the PO equilibrium outcome is equivalent in terms of payoffs and participation to the outcome of the dominant-strategy equilibrium of the SPA in the same environment with observable participation. But since the dominant strategy of the SPA is independent of whether participation is observable, the same equilibrium outcome would prevail when participation is unobservable. As is well known, in the absence of the frictions considered here—costly recruitment, costly participation, and lack of commitment—the FPA and

[3] For example, even with in-person bidding, other bidders may be "shills" or may be bidding via agents.

SPA formats yield equal profit and surplus. The interesting insight here is that with such frictions, the two formats are not equivalent. Since they affect the seller's recruitment incentives differently, which in turn affect the bidders's entry decisions, their ranking depends on the recruitment and entry costs.

The core of our model is the interaction between costly recruitment and costly participation when the seller cannot commit. Other features, like Poisson arrivals or whether bidders learn their values before or after entry, are not essential for the main insights. Several extensions illustrate the robustness of our qualitative findings. In particular, in the online appendix we consider the case in which bidders know their values before entry, as well as the cases in which the seller can set an entry fee/subsidy or a reserve price. In an earlier version of the paper, we established the same main insights with deterministic (rather than Poissonian) recruitment.

Anecdotal evidence. Ample anecdotal evidence demonstrates the relevance of the key elements in our auction model: limited seller commitment, recruitment costs, and entry costs.

Limited commitment. Boone and Mulherin (2009, 2007) and Subramanian (2011) study merger and takeover proceedings, which often involve auctions of some form. Despite the high stakes involved, such auctions are often conducted in a way that suggests limited seller commitment. First, many of these auctions (the majority, according to Boone and Mulherin (2009).) are "informal," in the sense that they are a mixture of auctions and negotiations rather than "a structured process where the rules are laid out in advance." Second, sellers seem unable to credibly commit ex-ante to a level of participation or its disclosure.[4] Sellers' commitment ability is sometimes further limited by confidentiality agreements with certain bidders (see also Gentry and Stroup, 2019), or other legal considerations, such as the reluctance of courts to enforce certain contract clauses.

Recruitment costs. Subramanian (2011) describes the critical role of bidder recruitment in merger and acquisition auctions. Milgrom (2004) states that, based on his consulting experience, the marketing of an auction is often more critical for its success than clever design. Fees paid by sellers to intermediaries go partly towards recruitment efforts. Recruitment costs may also reflect implicit costs, such as the costly disclosure of sensitive information to motivate potential buyers.[5]

[4] Subramanian (2011) provides examples of sellers trying to increase competitive pressure by using fictitious bidders.

[5] Bulow and Klemperer (1996, p. 190) mention the implicit costs of revealing information as an addi-

Entry costs. An extensive empirical literature documents the importance of bidders' entry and participation costs; see, for example, Gentry and Stroup (2019) and the work discussed there.

The process of obtaining bids for home repair provides an example of an informal auction that will be familiar to many readers, in which both recruitment and entry costs play a major role. A homeowner may wish to suggest to prospective contractors that they have some competition, but not so much as to scare them away.

Related literature. Our model's main novelty is the combination of costly recruitment and costly entry with limited commitment. Although strands of the literature discuss each of these frictions in isolation, we are not aware of any references that discuss all three jointly or that derive insights similar to ours.

An extensive literature on auctions with costly entry has found that when bidders enter before learning their values, their entry decisions are efficient, and the seller's incentives align with the social planner's as she obtains the full surplus (McAfee and McMillan, 1987; Levin and Smith, 1994; Crémer et al., 2007).[6] These models correspond to versions of our model with positive entry costs and an exogenously given expected number of potential bidders.

Szech (2011) examines costly recruitment in an FPA where all contacted bidders enter. Her model corresponds to our PO scenario with costless entry. She shows that the seller's profit-maximizing choice of recruitment effort generally exceeds the efficient one. Lauermann and Wolinsky (2017, 2022) also feature costly recruitment and costless entry, but in a common-value setting. They focus on different questions related to information aggregation with a privately informed seller.

Milgrom (1987); McAfee and Vincent (1997) nd Liu et al. (2019) study limited commitment to a reserve price in auctions with a fixed set of bidders.

Our model can be viewed as a simultaneous search model in which the seller is the searcher. Renaming the actors turns our model into a stochastic version of the simultaneous search model of Burdett and Judd (1983), with the added features of heterogeneous production costs and price-quoting costs.

tional (unmodeled) reason for restricting bidder numbers. To give an idea of recruitment in practice, the first case discussed in Boone and Mulherin (2009) is the sale of a firm, Blount Inc., where 65 potential buyers were contacted, of which 28 signed confidentiality agreements, and 2 submitted a bid (Lehman Brothers won).

[6] However, this is not the case when bidders have private information at entry (Samuelson, 1985; Ye, 2007). Moreover, with affiliated types, revenue equivalence fails, and so the auction format and the disclosure of the number of participants affect the bidders' entry (Murto and Välimäki, 2025).

1 The PO auction: Observable participation

1.1 The model

A seller organizes an auction to sell an indivisible item that has value 0 to her. She makes recruitment effort $\gamma \geq 0$, resulting in a Poisson-distributed number of prospective bidders with mean γ; that is, the probability of t bidders being contacted is $\frac{\gamma^t}{t!}e^{-\gamma}$. The cost of effort γ is γs, for some $s > 0$.[7]

A prospective bidder i who decides to participate incurs a cost $c > 0$. He then observes his own value v_i for the item and the total number n of bidders who chose to enter the auction (including i himself). The v_i are private values, independently and identically distributed with a cumulative distribution function (CDF) G, with support $[0, 1]$, a continuous density g, and increasing virtual values, $v - \frac{1-G(v)}{g(v)}$. The bidders do not observe γ. Finally, the participating bidders submit bids. The highest bidder wins and pays his bid.

When an auction ends with winning bid p, the payoff is $p-\gamma s$ for the seller, v_i-p-c for the winning bidder i, $-c$ for each participating bidder who lost, and 0 for each contacted bidder who declined entry.

1.2 Interaction: Strategies and equilibrium

The seller's strategy is the recruitment effort $\gamma \geq 0$. Bidder i's strategy is (q_i, β_i), where $q_i \in [0, 1]$ is the entry probability and $\beta_i : [0, 1] \times \{1, 2, ...\} \to [0, 1]$ describes i's bid as a function of his information (v_i, n)—that is, his private value and the number of participating bidders. Bidder i's belief concerning the seller's effort, conditional on being contacted—but before observing (v_i, n)—is a probability measure μ_i on $[0, \infty)$.

We study symmetric behavior in which all bidders employ the same strategy (q, β) and hold the same belief μ. An **equilibrium** consists of γ^*, q^*, and β^* such that:

(E1) The effort γ^* maximizes the seller's expected payoff given q^* and β^*.

(E2) There exists a belief μ such that

- (i) q^* and β^* maximize each bidder's payoff, given μ and the other bidders' strategy (q^*, β^*);
- (ii) if $\gamma^* > 0$, then $\mu(\gamma^*) = 1$, i.e., the belief is confirmed on the path;
- (iii) if $\gamma^* = 0$, then the seller's payoff is nonnegative for any $\hat{\gamma}$ in the support of μ, given (q^*, β^*).

[7]The assumption of constant marginal cost of effort is made for simplicity and is not essential, as explained in Section A.2 of the online appendix.

Thus, the equilibrium allows only pure recruitment and bidding strategies; mixing is allowed only in the bidders' entry decisions, $q \in [0,1]$.[8]

Off-path beliefs arise only when $\gamma^* = 0$, but their role is not negligible since this is an important case of extreme market failure. Equilibrium condition E2(iii) imposes a refinement on the off-path beliefs, which allows us to rule out no-trade equilibria that rely on unfounded beliefs. This will be discussed in Section 5.2, where we present alternative ways to obtain the needed refinement.

The random number of actual participants in the auction is Poisson distributed with mean

$$\lambda := q\gamma.$$

As pointed out by Myerson (1998), λ is both the expected number of participants from an outsider's perspective, and the expected number of competitors of a participating bidder from his own perspective (that accounts for having been sampled).

For convenience, we will mostly use λ (instead of γ). Thus, the bidders' belief μ will be over λ, and the equilibrium will be expressed in terms of $\lambda^* := q^*\gamma^*$.

2 Equilibrium analysis for the PO scenario

2.1 Solving backward

The interaction in the PO scenario unfolds in three stages: recruitment, entry, and bidding. We can solve for the equilibrium backward.

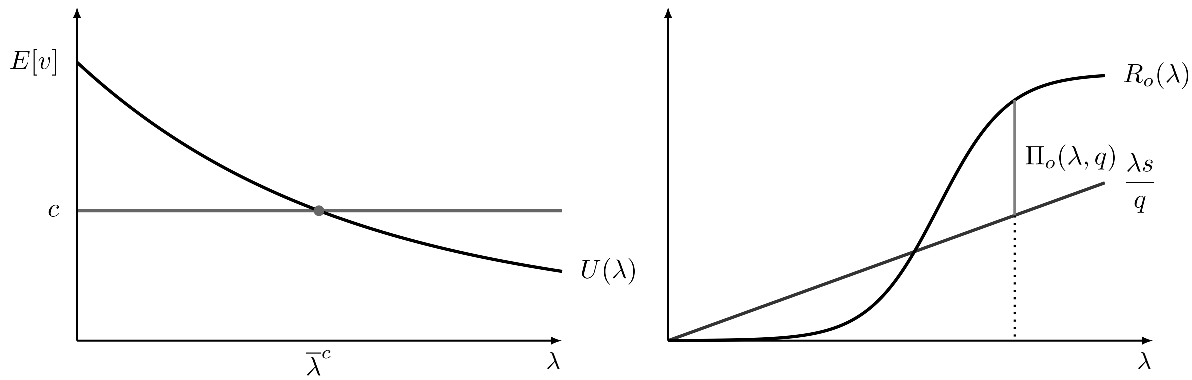
Stage 3: Bidding. Once the number of participants n is realized, the ensuing auction is a standard symmetric FPA with independent private values drawn from the CDF G. Such an auction has a unique symmetric equilibrium (see, e.g., Krishna, 2009),

$$\beta_{FPA}(v,n) = v - \int_0^v \left[\frac{G(y)}{G(v)}\right]^{n-1} dy, \tag{1}$$

and so $\beta^* = \beta_{FPA}$ is the bidding strategy in every equilibrium. The main properties used below are that $\beta_{FPA}(v,n)$ is increasing in v and n, with $\beta_{FPA}(v,n) = 0$ if $n = 1$; and $\beta_{FPA}(v,n) \to v$ as n becomes large.

Stage 2: Entry. Let $U(\lambda)$ be the bidders' ex-ante expected payoff (gross of the cost of entry), given a Poisson-distributed number of participating bidders with mean λ who use

[8] The online appendix shows that for generic s, there is no equilibrium with a mixed recruitment strategy.



(a) Bidders' payoff $U(\lambda)$.

(b) Revenue, cost, and profit.

Figure 1: Illustration of payoffs and profits.

β_{FPA}. The essential features of U are shown by Figure 1a: it is continuous and decreasing in λ, with $U(0) = E[v]$ and $\lim_{\lambda\to\infty} U(\lambda) = 0$. Intuitively, the properties of β_{FPA} imply that, for any v, both the probability of winning and the payoff conditional on winning are decreasing in n and a larger λ means, on average, higher n. The explicit functional form of U is presented by Claim 2 in the appendix.

Given the bidders' belief μ concerning λ, their optimal entry decision q satisfies:

$$\begin{aligned} E_\mu[U(\lambda)] > c \quad &\Rightarrow \quad q = 1; \\ E_\mu[U(\lambda)] < c \quad &\Rightarrow \quad q = 0. \end{aligned} \tag{2}$$

Since $c \geq U(0)$ means that no bidder enters, we consider only $c < U(0)$. Since U is continuous and strictly decreasing to 0, then for every $c \in (0, U(0))$, there is a unique $\bar{\lambda}^c > 0$ such that

$$U(\overline{\lambda}^c) = c. \tag{3}$$

This is the bidders' break-even participation level: given λ, a bidder's expected payoff from entering is nonnegative if and only if $\lambda \leq \overline{\lambda}^c$. The upper bar in $\overline{\lambda}^c$ will serve as a reminder that this is the maximal scale acceptable to bidders.

Thus, in any equilibrium,

$$\lambda^* \leq \overline{\lambda}^c, \tag{4}$$

and if $\lambda^* \in (0, \overline{\lambda}^c)$, then q^* must be 1.

Stage 1: Recruitment. Given q^* and β^*, the seller's problem is to choose recruitment effort γ to maximize profit. The choice of effort γ at cost s is equivalent to the choice of $\lambda = q\gamma$ at cost s/q. Let $R_o(\lambda)$ be the seller's expected revenue given the participation level λ and β^*. (The subscript o here and later indicates that participation is observable.) The profit as a function of λ and $q > 0$ is

$$\Pi_o(\lambda, q) = R_o(\lambda) - \lambda \frac{s}{q},$$

with $\Pi_o(0,0) = 0$, and $\Pi_o(\lambda, 0) = -\infty$ for $\lambda > 0$.

In any equilibrium, $\lambda^* \in \arg\max \Pi_o(\lambda, q^*)$. We now turn to the solution to this maximization problem. Figure 1b shows the essential properties of R_o: it is increasing, twice continuously differentiable, $R_o(0) = 0$, and $\lim_{\lambda \to \infty} R_o(\lambda) = 1$. Figure 2 depicts the marginal revenue R_o' and the average revenue $\frac{R_o}{\lambda}$: both curves are single-peaked, are 0 at $\lambda = 0$, asymptote to 0 as $\lambda \to \infty$, and intersect once at the maximum point of $\frac{R_o(\lambda)}{\lambda}$. (All the observations concerning R_o' and $\frac{R_o}{\lambda}$ are verified by Claim 3 in the appendix.)

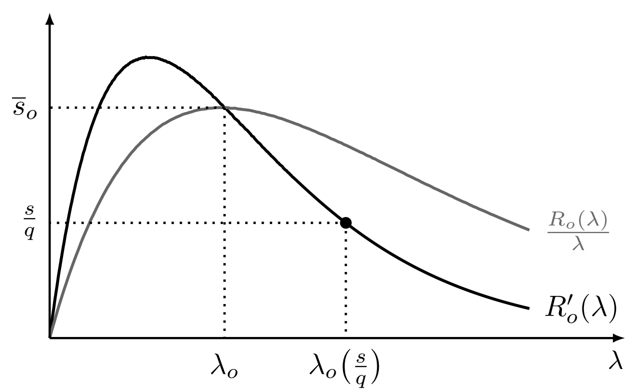


Figure 2: Marginal revenue, average revenue, and marginal recruitment cost.

Let

$$\bar{s}_o := \max_{\lambda} \frac{R_o(\lambda)}{\lambda},$$

and, for $s \leq \max_\lambda R_o'(\lambda)$, let $\lambda_o(\frac{s}{q})$ denote the larger of the two solutions of $R_o'(\lambda) = \frac{s}{q}$. That is, $\lambda_o(\frac{s}{q})$ is the unique solution for the following necessary conditions[9] for $\lambda > 0$ to maximize $\Pi_o(\lambda, q)$:

$$R_o'(\lambda) = \frac{s}{q} \text{ and } R_o''(\lambda) \leq 0. \tag{5}$$

Thus, $\Pi_o(\lambda, q)$ is maximized either at 0 or at $\lambda_o(\frac{s}{q})$, depending on whether $\Pi_o(\lambda_o(\frac{s}{q}), q)$

[9]These conditions are not sufficient since R_0 and hence Π_o are not concave in λ.

is smaller or greater than 0; specifically,

$$\arg\max \Pi_o(\lambda, q) = \begin{cases} \lambda_o(\frac{s}{q}) & \text{if } \frac{s}{q} < \bar{s}_o \\ \{0, \lambda_o(\frac{s}{q})\} & \text{if } \frac{s}{q} = \bar{s}_o \\ 0 & \text{if } \frac{s}{q} > \bar{s}_o. \end{cases} \tag{6}$$

Letting $\underline{\lambda}_o := \lambda_o(\bar{s}_o)$, it follows that in any equilibrium,

$$\lambda^* = \begin{cases} \lambda_o(\frac{s}{q^*}) & \text{if } \frac{s}{q} < \bar{s}_o \\ 0 \text{ or } \underline{\lambda}_o & \text{if } \frac{s}{q} = \bar{s}_o \\ 0 & \text{if } \frac{s}{q} > \bar{s}_o. \end{cases} \tag{7}$$

The conclusion from solving backwards through the three stages above is that equilibrium is characterized by a pair (λ^*, q^*) such that λ^* satisfies (7) and there exists belief μ such that q^* satisfies (2) and either $\lambda^* > 0$ and $\mu(\lambda^*) = 1$, or $\lambda^* = 0$ and $\Pi_o(\hat{\lambda}, q^*) = 0$ for all $\hat{\lambda}$ in the support of μ.

Observe that $\underline{\lambda}_o$, which also satisfies

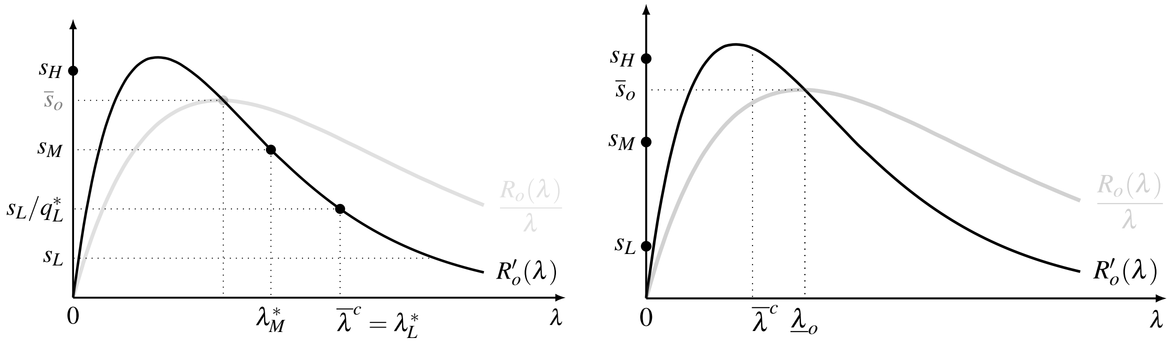
$$\underline{\lambda}_o = \arg\max_{\lambda} \frac{R_o(\lambda)}{\lambda},$$

is the minimal scale for equilibrium—λ^* is either 0 or at least $\underline{\lambda}_o$ (the lower bar in $\underline{\lambda}_o$ serves as a reminder of that). The positive minimal scale is a consequence of the initial λ range of increasing average revenue, which is due to the sharp reduction over that initial range in the probability of there being fewer than two participants and the associated zero revenue.

2.2 The equilibrium outcome

Figure 3 describes all three types of equilibria that may arise for different (c, s) combinations. The magnitude of c is reflected in the figure by the position of $\overline{\lambda}^c$—the maximal participation acceptable for bidders— which is decreasing in c. The unique equilibrium for each of the depicted configurations is marked by a large dot.

No-trade equilibrium. Equilibria with no trade ($\lambda^* = 0$) arise in two cases. First, when $s > \bar{s}_o$ (like s_H in Figure 3), there are no gains from trade—recruitment is simply too costly to be profitable. Recall that, by definition, $\bar{s}_o = \max_\lambda \frac{R_o(\lambda)}{\lambda}$. Therefore, $s > \bar{s}_o$ implies that $s\lambda > R_o(\lambda)$, for any $\lambda > 0$, which means that profit is nonnegative only at $\lambda = 0$.



(a) $\overline{\lambda}^c > \underline{\lambda}_o$ (b) $\overline{\lambda}^c < \underline{\lambda}_o$

Figure 3: PO equilibria.

Second, when $\overline{\lambda}^c < \underline{\lambda}_o$ (Figure 3-B), the problem is not the absence of gains from trade but rather the seller's inability to commit to a participation level that is acceptable to bidders. Bidder optimality (4) implies that any equilibrium λ^* must not exceed $\overline{\lambda}^c$. At the same time, the seller cannot credibly offer a positive $\lambda \leq \underline{\lambda}_o$ since if such a λ is profitable for some s and q, then by (6) profit would be larger at some $\lambda \geq \underline{\lambda}_o$. Therefore, no $\lambda^* > 0$ is compatible with equilibrium—it cannot be both profit-maximizing, which requires $\lambda^* \geq \underline{\lambda}_o$, and acceptable to bidders, which requires $\lambda^* \leq \overline{\lambda}^c < \underline{\lambda}_o$. This is so even if a small s implies substantial potential gains from trade.

Equilibria with trade ($\lambda^* > 0$) arise when $\overline{\lambda}^c \geq \underline{\lambda}_o$ and $s \leq \bar{s}_o$**.** In all such equilibria λ^* is the interior profit maximizer $\lambda_o(\frac{s}{q^*})$, given the marginal recruitment cost $\frac{s}{q^*}$ (recall that $\lambda_o(\frac{s}{q})$ is the larger solution to $R_o'(\lambda) = \frac{s}{q}$). Figure 3a depicts the two types of such equilibria for s_M and s_L, respectively. The equilibrium configurations are denoted in the figure by (λ_k^*, q_k^*), $k = L, M$, and the corresponding $(\lambda_k^*, \frac{s}{q_k^*})$ are marked in the figure by large dots.

Since $s_M > R_o'(\overline{\lambda}^c)$, the unconstrained profit maximizer $\lambda_o\left(s_M\right)$ is below $\overline{\lambda}^c$ and hence does not deter bidder entry. Therefore, $\lambda_M^* = \lambda_o\left(s_M\right) < \overline{\lambda}^c$ and $q_M^* = 1$.

Since $s_L < R_o'(\overline{\lambda}^c)$, the unconstrained profit maximizer $\lambda_o\left(s_L\right)$ exceeds $\overline{\lambda}^c$ and hence cannot be sustained by equilibrium since bidders would not enter. Therefore, $\lambda_L^* = \overline{\lambda}^c$ and q_L^* adjusts to satisfy $\lambda_o(\frac{s_L}{q_L^*}) = \overline{\lambda}^c$.

The upshot is that to be willing to bear the cost of entry, bidders must believe that λ^* is not too large. In equilibria like (λ_M^*, q_M^*) above, this is achieved via a sufficiently large s, like s_M, assuring bidders that the expected participation at the unconstrained profit maximum, $\lambda_o\left(s\right)$, will be small enough. In equilibria like (λ_L^*, q_L^*) above, where s is small, like s_L, this is achieved through bidders' reluctance to enter (i.e., q^* is sufficiently small), which raises the effective marginal recruitment cost s/q^* to a level that stops the seller from recruiting beyond $\overline{\lambda}^c$.

Figure 4 and Proposition 1 provide the complete characterization of the equilibria for all (c, s) configurations. In particular, only the (c, s) configurations in the shaded Regions I and II give rise to equilibria with trade.

Notice that Figure 4 depicts the (c, s) space rather than the (λ, s) space depicted in Figures 2 and 3. As c increases along the c-axis of Figure 4, $\overline{\lambda}^c$ is decreasing. So the $R'_o(\overline{\lambda}^c)$ and $\frac{R_o(\overline{\lambda}^c)}{\overline{\lambda}^c}$ curves are sort of mirror images of the curves $R'_o(\lambda)$ and $\frac{R_o(\lambda)}{\lambda}$ in the previous figures. In particular, since by definition $c \equiv U(\bar{\lambda}^c)$, the conditions $\overline{\lambda}^c \gtreqless \underline{\lambda}_o$ considered above are equivalent to $c \lesseqgtr U(\underline{\lambda}_o)$ in the (c, s) space of Figure 4.

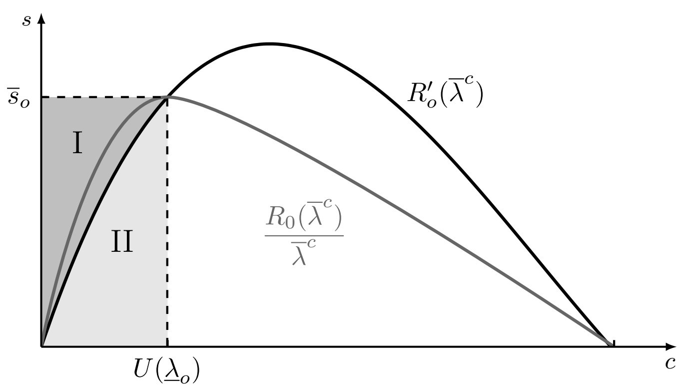


Figure 4: Unique PO equilibrium with trade iff (c, s) in shaded Regions I&II.

Recall that $\lambda_o(s)$ is the larger solution to $R'_o(\lambda) = s$ (alternatively, the solution to the necessary condition for an interior maximum) .

Proposition 1. *There exists a unique equilibrium for all* (c, s) *for which* $c \neq U(\underline{\lambda}_o)$ *and* $s \neq \bar{s}_o$*. The form of the equilibrium varies across the regions of Figure 4 as follows:*

1. *If* (c, s) *is in the unshaded region, then* $\lambda^* = 0$ *(no trade).*

2. *If* (c, s) *is in shaded Region I, then* $\lambda^* = \lambda_o(s)$ *and* $q^* = 1$*.*

3. *If* (c, s) *is in shaded Region II, then* $\lambda^* = \overline{\lambda}^c$ *and* $q^* = s/R'_o(\overline{\lambda}^c)$*.*

If either $s = \bar{s}_o$ or $c = U(\underline{\lambda}_o)$, then $\lambda^* = 0$ and $\lambda^* = \underline{\lambda}_o$ are both equilibrium outcomes (yielding zero profit). The seller's profit maximization condition(7) and the conclusion that follows it imply immediately that the configurations presented by Parts 2 and 3 of Proposition 1 are equilibria (with the belief $\mu(\lambda^*) = 1$). This can also be directly inferred from inspecting Figure 3a. Part 2 is illustrated by the case of $s = s_M$, and Part 3 is illustrated by the case of $s = s_L$. (Both cases were considered above in the discussion of equilibria with trade.)

The two less immediate steps (presented by the proof in the appendix) are, first, the construction of the no-trade equilibrium of Part 1 and, second, the argument ruling out no-trade equilibria when $s < \bar{s}_o$. Both of these steps deal with bidders' off-path beliefs about the expected competition upon being contacted when they expect $\lambda = 0$, by using the equilibrium refinement in condition E2(iii).

Comparative statics. Equilibria with higher λ^* are always more profitable and both λ^* and $\Pi_o(\lambda^*, q^*)$ are non-increasing in c and s. This is because, when $\lambda^* > 0$, we have $R_o'(\lambda^*) = s/q^*$. Therefore, the profit is

$$R_o(\lambda^*) - \lambda^* R_o'(\lambda^*). \tag{8}$$

Since $R_o''(\lambda^*) < 0$, the profit (8) is strictly increasing in λ^*. Thus, equilibria with higher participation are necessarily associated with a higher profit, independently of what the underlying recruitment and entry costs are.

Equilibrium inefficiencies. Subsection 5.1 discusses equilibrium welfare in some detail. Here we point out two types of inefficiency that are immediately obvious from the equilibrium characterization. First, the no-trade equilibrium outcome with $c > U(\underline{\lambda}_o)$ is inefficient when s is sufficiently small. A λ just below $\overline{\lambda}^c$ would be beneficial for the bidders (i.e., $U(\lambda) > c$) and profitable for the seller (because the revenue of nearly $R_o\left(\bar{\lambda}^c\right)$ would exceed the cost with a small s).

Second, a small s necessarily implies an inefficient, wasteful recruitment effort in equilibria with trade. At such equilibria with $s < R_o'(\overline{\lambda}^c)$, participation is fixed at $\overline{\lambda}^c$ and the total recruitment cost is independent of s:

$$\overline{\lambda}^c \frac{s}{q^*} = \overline{\lambda}^c R_o'(\overline{\lambda}^c) = \text{const} > 0 \tag{9}$$

The difference between the actual equilibrium cost and the minimal cost $s\overline{\lambda}^c$ required for recruiting $\overline{\lambda}^c$ is a deadweight loss of the magnitude

$$\text{DWL} = \overline{\lambda}^c \frac{s}{q^*} - s\overline{\lambda}^c = \overline{\lambda}^c (R_o'(\overline{\lambda}^c) - s), \tag{10}$$

where the second equality follows from (9). Thus, for small s near 0, almost all of the recruitment effort is wasteful.

Both of these inefficiencies—no trade and wasteful recruitment efforts—result from the combination of the bidders' cost of entry and the seller's inability to commit (to $\gamma = \overline{\lambda}^c$ or just below it). Indeed, when either one of these conditions is eliminated, these inefficiencies go away. First, the no-trade equilibrium disappears for small c such that $c < U(\underline{\lambda}_o)$, and the deadweight loss in (10) decreases in c and disappears when $R'_o(\overline{\lambda}^c)$, which is decreasing in c, falls below s. Second, as explained in more detail in Subsection 2.3 below, if the seller could commit to γ just below $\overline{\lambda}^c$ (or, equivalently, if the seller's choice of γ were observable to bidders prior to entry), then in both of these situations, bidders will have strict incentives to participate, yielding revenue close to $R_o(\overline{\lambda}^c)$.

2.3 Commitment to recruitment effort

In an alternative scenario in which the seller could commit to the recruitment effort, she first selects γ. Then bidders observe γ and decide whether to enter. Finally, the bidding takes place as before. Recall that γ is just the mean of the Poisson distribution—the actual number of contacted bidders remains uncertain.

The commitment equilibrium must have $\gamma^* \leq \bar{\lambda}^c$ and $q^* = 1$, since any $\gamma > \bar{\lambda}^c$ would trigger a probability of bidders' entry $q = \gamma/\bar{\lambda}^c < 1$, resulting in $\lambda = \bar{\lambda}^c$, and a wasted recruitment cost of $(\gamma - \bar{\lambda}^c)s$. Thus, the seller's problem is

$$\max_{\lambda} \Pi_o(\lambda, 1) \quad \text{s.t.} \quad \lambda \leq \overline{\lambda}^c.$$

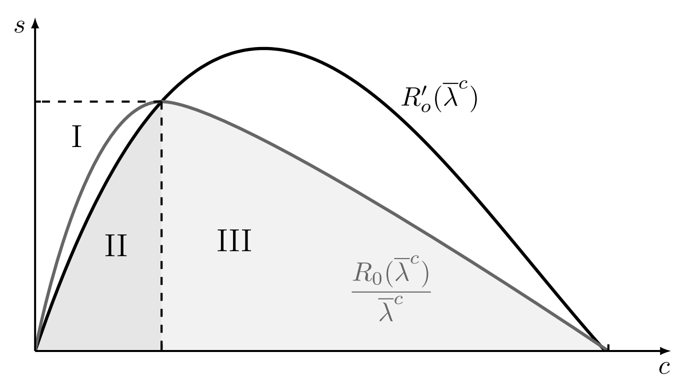


Figure 5: Commitment is strictly profitable only in Regions II and III.

The seller's profit is, of course, weakly higher with commitment. The shaded regions in Figure 5 identify the (c, s) combinations for which the commitment profit is strictly higher.[10]

In Region II, the no-commitment equilibrium has $\lambda^* = \overline{\lambda}^c$ and $q^* < 1$ (see Part 3 of Proposition 1). Therefore, the participation is $\overline{\lambda}^c$ in both regimes, but in the commitment case $q = 1$, which implies that total recruitment cost is lower and hence profit is higher by $(\frac{s}{q^*} - s)\overline{\lambda}^c > 0$ than in the no-commitment case.

In Region III, the no-commitment equilibrium has $\lambda^* = 0$ (see Part 1 of Proposition 1). However, $s < R_o(\bar{\lambda}^c)/\bar{\lambda}^c$ makes commitment to $\gamma = \bar{\lambda}^c$ both profitable for the seller and beneficial for the bidders.

In the unshaded regions of Figure 5, the committment does not increase the profit. In Region I, the no-commitment equilibrium has positive $\lambda^* < \overline{\lambda}^c$ (Part 2 of Proposition 1). Commitment does not help since all contacted bidders enter and the seller does not want to recruit more. In the other unshaded regions, $s > R_o(\lambda)/\lambda$, for all $\lambda \leq \overline{\lambda}^c$. This means that committing to any γ acceptable to bidders will be unprofitable for the seller.

3 The PU auction: Unobservable participation

The PU scenario captures the effects of the bidders' inability to observe the participation and the seller's inability to credibly disclose it. After bidders have sunk their entry costs, it is in the seller's interest to convince them that they have many competitors.[11] This incentive, in conjunction with the seller's inability to credibly disclose participation, implies that the seller cannot convince bidders that the participation exceeds their expectations even when this is indeed the case.

The model is the same as in the PO scenario, except that the bidders *cannot* observe the number of other participants at any stage—neither before nor after entry. The equilibrium definition from Section 1.2 remains the same, except that the bidding strategy β no longer conditions on n. To simplify the exposition, we further restrict attention to equilibria in which the bidders' beliefs have point support, that is, $\mu^*(\hat{\lambda}) = 1$ for some $\hat{\lambda}$.[12] We index the magnitudes for this scenario with the subscript u (for "unobservable").

[10] Figure 5 (like Figure 4 above) depicts the (c, s) space rather than the (λ, s) space depicted in Figures 2 and 3; see the explanation of Figure 4.

[11] Subramanian (2011) gives numerous examples of sellers attempts to inflate the perceived competition: realtors pretend to get calls from other interested parties; companies suggest the existence of additional bidders during takeover negotiations; at Sotheby's and other auction houses, auctioneers make up "chandelier bids" to stimulate bidding.

[12] Since we assume a pure recruitment strategy on the part of the seller, this is a restriction only in the case of $\lambda^* = 0$ (no trade). However, it will become clear that this assumption does not restrict the set of

3.1 Solving backward

As before, the interaction unfolds in three stages—recruitment, entry, and bidding—and the equilibrium can be solved by proceeding backwards from the last stage.

Stage 3: Bidding. Since bidders do not observe the actual participation, this stage is an FPA with an uncertain, Poisson-distributed number of bidders with mean λ. A result in Krishna (2009) implies that the unique symmetric equilibrium of this auction is

$$\beta_\lambda(v) = \sum_{n=0}^{\infty} \frac{P_\lambda(n)\, G(v)^n}{\sum_{n=0}^{\infty} P_\lambda(n)\, G(v)^n} \beta_{FPA}(v, n+1), \tag{11}$$

where $P_\lambda(n) = \frac{\lambda^n}{n!} e^{-\lambda}$ is the probability that the bidder has n competitors in the auction, $G(v)^n$ is the probability that the n others have lower values, and β_{FPA} is the equilibrium bidding strategy given by (1) for the PO scenario[13]. As expected, $\beta_\lambda(v)$ is strictly increasing in v and λ, and $\beta_\lambda(v) \equiv 0$ when $\lambda = 0$.

In equilibrium, $\beta^* = \beta_{\hat{\lambda}}$, where $\hat{\lambda}$ is the bidders' point belief.

Stage 2: Entry and payoff equivalence. Let $u(v, \lambda)$ denote the expected equilibrium payoff of a bidder with value v who bids $\beta_\lambda(v)$,

$$u(v, \lambda) = \sum_{n=0}^{\infty} P_\lambda(n)\, G(v)^n (v - \beta_\lambda(v)) = \sum_{n=0}^{\infty} P_\lambda(n)\, G(v)^n (v - \beta_{FPA}(v, n+1)), \tag{12}$$

where the last expression is obtained by substituting the explicit forms of $\beta_\lambda(v)$ from (11) into the previous term.

The expected ex-ante utility of a prospective bidder (gross of the entry cost) is

$$U(\lambda) = \mathbb{E}_v [u(v, \lambda)]. \tag{13}$$

Payoff equivalence between PO and PU scenarios. The last expression on the right side of (12) is also the expected payoff of a bidder with value v in the PO scenario, when evaluated before the bidder has learned the realized number of participants. Consequently, the bidders' ex-ante expected payoff $U(\lambda)$ is the same for the PU and the PO scenarios. (This is why we use the same notation $U(\lambda)$.)

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conditions under which the no-trade equilibria exist; see also the online appendix.

[13] Krishna (2009), Section 3.2.2 presents such a result for a general probability distribution over n with finite support. Here, we have a Poisson distribution and hence infinite support. However, this does not affect the argument.

This payoff-equivalence result is not surprising. In both scenarios, a bidder with value $v = 0$ gets payoff 0. So, the standard envelope formula implies that, in each scenario, the expected equilibrium payoff of a bidder with value v must be $\int_0^v \Pr(\text{win with value } x)dx$. By the monotonicity of the bidding strategies, Pr(win with value x), and thus, the expected equilibrium payoffs coincide in the two scenarios.[14]

Since the bidders' expected payoffs are the same, the bidders' optimality condition (2) for the entry probability q^* remains unchanged and so does $\overline{\lambda}^c$ (the maximal participation level at which bidders are willing to enter). Hence, $\lambda^* \leq \overline{\lambda}^c$ also holds in every equilibrium of the PU scenario, and, for all $\lambda^* \in (0, \overline{\lambda}^c)$, we have $q^* = 1$.

Stage 1: Recruitment. Let $R_u(\lambda, \beta_{\hat{\lambda}})$ be the expected revenue given λ and $\beta_{\hat{\lambda}}$. Thus, R_u depends directly on the actual participation λ and, through $\beta_{\hat{\lambda}}$, also on bidders' expectation $\hat{\lambda}$. Since the PU and PO scenarios share the same gross total surplus (the expectation of the first-order statistic of v given λ) and, by the payoff equivalence just noted above, also the same total expected bidders' ex-ante expected payoff $\lambda U(\lambda)$, it follows that

$$R_u(\lambda, \beta_\lambda) = \text{Total Surplus}(\lambda) \; - \lambda U(\lambda) = R_o(\lambda). \tag{14}$$

That is, there is revenue equivalence between R_u and the PO revenue R_o when $\hat{\lambda} = \lambda$ (the expected participation coincides with the actual one).

When $q > 0$, the expected profit is

$$\Pi_u(\lambda, \beta, q) = R_u(\lambda, \beta) - \lambda \frac{s}{q};$$

it is 0 when $\lambda = 0$ and $q = 0$, and it is $-\infty$ for $q = 0$ and $\lambda > 0$.

The functions R_u and hence Π_u are concave in λ and strictly so for any β that is not constant at 0 (see Claim 4 in Appendix 7.2). Therefore, for any $\hat{\lambda}$ and q, Π_u has a unique maximum over λ. The maximizing λ satisfies

$$\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}}) \leq \frac{s}{q}, \tag{15}$$

with equality holding for $\lambda > 0$.

[14] Conversely, note that the formula for $u(v, \lambda)$ in (12) could also be derived directly from the envelope formula. Indeed, Krishna (2009) uses the envelope theorem to derive the bidding strategy β_{FPA}, and β_λ is then derived from payoff equivalence. Here, we reversed the order to simplify the presentation.

In sum, (λ^*, q^*) is an equilibrium if and only if

$$\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})_{\hat{\lambda}=\lambda=\lambda^*} \leq \frac{s}{q^*}, \tag{16}$$

with equality holding for $\lambda^* > 0$ and where $q^* = 1$ if $\lambda^* < \overline{\lambda}^c$ and $q^* \in (0,1]$ if $\lambda^* = \overline{\lambda}^c$.

For expositional purposes, let

$$\xi(\lambda) := \frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})|_{\hat{\lambda}=\lambda}. \tag{17}$$

This is the marginal revenue with respect to λ for a fixed bidders' expectation $\hat{\lambda}$ that happens to coincide with the actual λ. Figure 6 depicts the function ξ and its relationship to the marginal revenue curves $\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})$ for two levels of expectations, $\hat{\lambda} = \lambda_1^*$ and $\hat{\lambda} = \lambda_2^*$, respectively. In other words, ξ is the locus of such points, picking up the values of the marginal revenue where the bidders' expectation is correct.

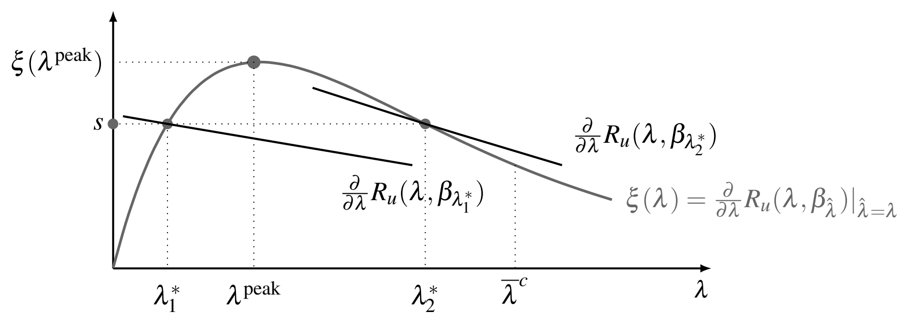


Figure 6: The function $\xi(\lambda)$ and the marginal revenue $\frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})$.

The main properties of ξ shown in the diagram—single-peakedness, continuity, $\xi(0) = 0$, $\xi(\lambda) > 0$ elsewhere, and $\xi(\lambda) \to 0$ as $\lambda \to \infty$—are established by Claim 4 in Appendix 7.2. Of course, λ^{peak} marks the argument of the single peak.

Restating the necessary and sufficient condition (16) in terms of ξ, the pair (λ^*, q^*) is an equilibrium if and only if

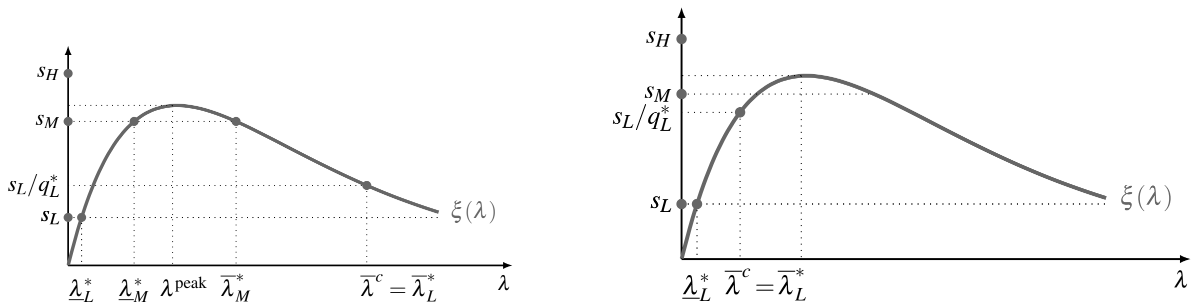
$$\xi(\lambda^*) \leq \frac{s}{q^*} \tag{18}$$

with equality holding for $\lambda^* > 0$ and where $q^* = 1$ if $\lambda^* < \overline{\lambda}^c$ and $q^* \in (0,1]$ if $\lambda^* = \overline{\lambda}^c$.

Figure 6 shows three points—$(0, s)$, (λ_1^*, s) and (λ_2^*, s) (marked by heavy dots)—satisfying condition (18) for $q^* = 1$, which are therefore equilibria for the particular values of s and c shown in the figure (with the value of c reflected by the size of $\overline{\lambda}^c$).

3.2 The equilibrium outcomes

Figure 7 depicts the different types of possible equilibria. The magnitude of c is reflected in the position of $\bar{\lambda}^c$—the maximal participation acceptable to bidders—which is decreasing in c. Panels 7a and 7b depict the cases of $\bar{\lambda}^c > \lambda^{peak}$ and $\bar{\lambda}^c < \lambda^{peak}$, respectively. Large dots mark the equilibria for each level of s. For each of the displayed (c, s) configurations, there are either three equilibria (as is the case for s_M in Panel 7a and for s_L in both panels) or one equilibrium (as is the case for s_M in Panel 7b and for s_H in both panels).



(a) $\overline{\lambda}^c > \lambda^{\text{peak}}$

(b) $\overline{\lambda}^c < \lambda^{\text{peak}}$

Figure 7: PU equilibria.

For $s < \xi(\lambda^{peak})$, let

$$\underline{\lambda}_u(s) \triangleq \min\{\lambda|\xi(\lambda) = s\} \text{ and } \overline{\lambda}_u(s) \triangleq \max\{\lambda|\xi(\lambda) = s\}.$$

There exists a no-trade (λ^*= 0) equilibrium for all $(\mathbf{c}, \mathbf{s})$ (as seen for all the (c, s) combinations in Figure 7). In any such equilibrium, $q^* = 1$, $\beta^*(v) \equiv 0$, and $\mu(0) = 1$. If a bidder is contacted off-path, he still believes that $\lambda = 0$ and bids 0 so that recruitment is indeed unprofitable.

The no-trade equilibrium is the unique equilibrium if $s > \xi(\lambda^{peak})$ (as is the case for s_H in Figure 7a) or when $\bar{\lambda}^c < \underline{\lambda}_u(s)$ (as is the case for s_M in Figure 7b).

Being unable to commit to the recruitment effort or disclose the level of participation, the seller cannot break out of the no-trade equilibrium (even if s and c are small).

Equilibria with trade (λ^*> **0**) exist when $s < \xi(\lambda^{peak})$ and $\bar{\lambda}^c > \underline{\lambda}_u(s)$ (as is the case for s_M and s_L in Figure 7a and for s_L in Figure 7b). For almost all (c, s) combinations that support an equilibrium with trade, there are two such equilibria.

In Figure 7 the equilibrium magnitudes corresponding to s_M and s_L are marked with subscripts M and L, respectively. In particular, the small and large equilibrium λ's associated with s_k are denoted respectively by $\underline{\lambda}_k^*$ and $\overline{\lambda}_k^*$, where $k = L, M$.

The smaller of the two equilibria with trade that correspond to some s is never constrained by $\bar{\lambda}^c$; it always obtains at the smaller intersection of s with ξ, which means that $\lambda^* = \underline{\lambda}_u(s)$ and $q^* = 1$. (Indeed, in both panels of Figure 7, $\underline{\lambda}_L^* = \underline{\lambda}_u(s_L)$, and in Panel 7a it is also the case that $\underline{\lambda}_M^* = \underline{\lambda}_u(s_M)$.)

The larger of the two equilibria with trade may be constrained or unconstrained by $\bar{\lambda}^c$, depending on the relative magnitudes of c and s. If $\bar{\lambda}^c > \overline{\lambda}_u(s)$, then the larger equilibrium too is unconstrained by $\bar{\lambda}^c$; it obtains at the larger intersection of s with ξ, which means that $\lambda^* = \overline{\lambda}_u(s)$ and $q^* = 1$ (which is the case of $\overline{\lambda}_M^* = \overline{\lambda}_u(s_M) < \bar{\lambda}^c$ in Figure 7a). If $\bar{\lambda}^c < \overline{\lambda}_u(s)$, then the larger equilibrium is constrained by $\bar{\lambda}^c$, which means that $\lambda^* = \bar{\lambda}^c$ and q^* adjusts to satisfy $\xi(\bar{\lambda}^c) = \frac{s}{q^*}$ (which is the case of $\overline{\lambda}_L^* = \bar{\lambda}^c$ in both panels of Figure 7). In short, in the larger equilibrium with trade, $\lambda^* = \min[\overline{\lambda}_u(s), \bar{\lambda}^c]$ and q^* adjusts accordingly.

The multiplicity of equilibria is a consequence of the bidders' inability to observe the participation (or, equivalently, the seller's inability to credibly disclose it). When the bidders expect either the no-trade or the low-trade equilibrium, they bid low. This in turn depresses the seller's incentive to recruit and the bidders' low expectations are indeed fulfilled. If the actual participation could be credibly disclosed to the bidders, these equilibria would be broken by more aggressive recruiting that would be rewarded by more aggressive bidding and higher profit.

Proposition 2 provides the complete characterization of the equilibria for all (c, s) configurations, which is also summarized by Figure 8 further below. Recall that $\underline{\lambda}_u(s)$ and $\overline{\lambda}_u(s)$ are the smaller and larger solutions to $\xi(\lambda) = s$.

Proposition 2.

1. *For every (c, s) there exists an equilibrium with $\lambda^* = 0$.*

2. *The $\lambda^* = 0$ equilibrium is unique if $s > \xi(\lambda^{peak})$ or $\bar{\lambda}^c < \underline{\lambda}_u(s)$.*

3. *If $s < \xi(\lambda^{peak})$ and $\bar{\lambda}^c > \underline{\lambda}_u(s)$, there are also two equilibria with $\lambda^* > 0$:*

$$\begin{aligned} \lambda_1^* &= \underline{\lambda}_u(s);\ q_1^* = 1; \\ \lambda_2^* &= \min[\overline{\lambda}_u(s), \bar{\lambda}^c];\ q_2^* = 1 \text{ if } \overline{\lambda}_u(s) \leq \bar{\lambda}^c \text{ and } q_2^* < 1 \text{ o/w.} \end{aligned}$$

4. *The equilibrium with the larger λ^* is strictly more profitable.*

On the boundary between the conditions of Parts 2 and 3, when $s = \xi(\lambda^{peak})$ or $\lambda^c = \underline{\lambda}_u(s)$, there is only one equilibrium with $\lambda^* > 0$. Parts 1-3 follow immediately from the necessary and sufficient condition (18) and the shape of the function ξ (namely, continuity, single peakedness, $\xi(0) = 0$, and $\xi \to 0$ as $\lambda \to \infty$). Part 4 is a special case of the discussion below.

For the case of $\overline{\lambda}_u(s) \leq \bar{\lambda}^c$, the result of Part 4 follows fairly immediately from the following revealed preference argument. The profit $\Pi_u(\underline{\lambda}_u(s), \beta_{\lambda_2^*}, 1)$ of deviating to $\lambda = \underline{\lambda}_u(s)$ from the equilibrium with $\lambda_2^* = \overline{\lambda}_u(s)$ is larger than the profit $\Pi_u(\underline{\lambda}_u(s), \beta_{\lambda_1^*}, 1)$ of the equilibrium with $\lambda_1^* = \underline{\lambda}_u(s)$, since these two situations differ from each other only in the bidding strategy, and $\beta_{\lambda_1^*}$ is pointwise lower than $\beta_{\lambda_2^*}$.

Comparative statics. As in the PO scenario, the equilibrium profit can be expressed in terms of λ^* alone since $\xi(\lambda^*) = s/q^*$ in any equilibrium with trade:

$$\Pi_u(\lambda^*, \beta_{\lambda^*}, q^*) = R_u(\lambda^*, \beta_{\lambda^*}) - \lambda^* \xi(\lambda^*). \tag{19}$$

Claim 1. *(i) If (λ^*, q^*) and (λ^{**}, q^{**}) are equilibria (for possibly different (c, s) configurations), then*

$$\lambda^{**} > \lambda^* \Longrightarrow \Pi_u(\lambda^{**}, \beta_{\lambda^{**}}, q^{**}) > \Pi_u(\lambda^*, \beta_{\lambda^*}, q^*).$$

(ii) The maximal PU equilibrium profit is weakly decreasing in c and s.

The proof in the appendix shows that (19) happens to equal the expectation of the third-order statistic of v, which is strictly increasing in $\lambda^* > 0$. Therefore, equilibria with higher λ^* are always more profitable. The claim follows because λ^* is weakly decreasing in c and s in the maximal profit PU equilibrium.

Inefficiencies. As in the PO case, when the equilibrium is constrained by $\overline{\lambda}^c$, it involves a deadweight loss in the form of excessive recruitment effort that is negated by $q^* < 1$. Here, the magnitude of the deadweight loss is $(\xi(\overline{\lambda}^c) - s)\overline{\lambda}^c$.

Figure 8 complements Proposition 2 by showing all the (c, s) combinations that sustain an equilibrium with trade. Notice that this figure (like Figure 4 above) depicts the (c, s) space rather than the (λ, s) space of Figure 7.[15] Only the (c, s) configurations in the shaded area under the curve sustain an equilibrium with trade. In Region I, the equilibria are unconstrained by $\bar{\lambda}^c$; in Region II the largest equilibrium is constrained by $\bar{\lambda}^c$.

[15]Since a smaller c is associated with a larger $\bar{\lambda}^c$, the curve $\xi\left(\bar{\lambda}^c\right)$ is a sort of mirror image of the curve $\xi(\lambda)$ in Figure 7. Since by definition $c = U(\bar{\lambda}^c)$, the cost c for which $\bar{\lambda}^c = \lambda^{peak}$ (where ξ peaks) is equal to $U(\lambda^{peak})$.

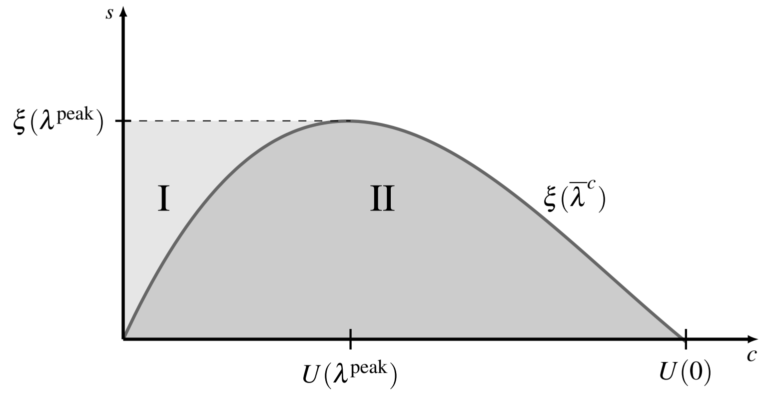


Figure 8: Equilibria with trade only in the shaded regions.

4 Comparison of the PO and PU scenarios

The key difference between the PO and PU scenarios is in the seller's recruitment incentives. In the PO scenario, the seller has two incentives to recruit: (i) to increase the likelihood of high-value bidders and (ii) to encourage more aggressive bidding. In the PU scenario, only the first incentive is present. Therefore, at any λ the marginal incentive to recruit is stronger in the PO scenario.

The differential strength of the recruitment incentives explains why the PU auction can be trapped in a robust no-trade equilibrium even when c and s are small, but this is not so in the PO scenario. If prospective bidders in the PU scenario expect $\lambda = 0$, they will bid 0 if recruited, so the seller has no incentive to recruit. In contrast, since bidders in the PO scenario observe the competition and bid accordingly, regardless of their pre-entry expectation, the marginal incentive to recruit is positive even when bidders expect $\lambda = 0$.

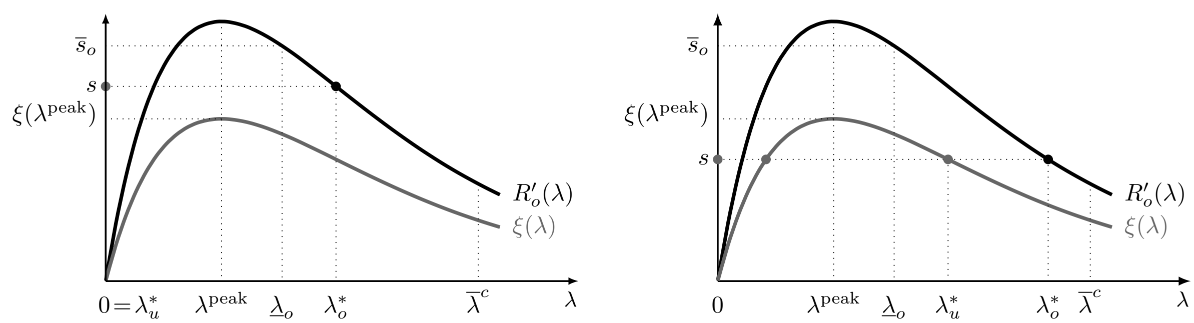
As shown below, the same difference in incentives also explains the difference in trade volumes and profits across these scenarios.

4.1 The effect of entry and recruitment costs

Whether the seller's profit is higher in the PO or the PU scenario depends on how the recruitment incentives interact with the costs s and c. Given s, for sufficiently small c, bidders strictly prefer to enter when they expect the unconstrained profit-maximizing λ. Hence, the seller's main concern is not getting bidders to enter but rather getting them to bid aggressively. In this case, the stronger recruitment incentives of the PO scenario result in greater competition, more aggressive bidding, and higher profit than in the equilibria of the PU scenario. In contrast, given c, for sufficiently small s, bidders are wary of excessive competition and the seller's main concern is bidders' reluctance to enter. In this case, the weaker recruitment incentives in the PU scenario serve as a partial commitment device, making its equilibrium with maximal profit more effective in inducing bidders to enter and thus more profitable than the PO scenario. The upshot is:

- For any $s < \bar{s}_o$, there is a threshold level for c below which the PO equilibrium profit strictly exceeds the maximal PU equilibrium profit.
- For any $c < U(0)$, there is a threshold level for s below which the maximal PU equilibrium profit strictly exceeds the PO equilibrium profit.

We first explain these points intuitively by looking at the respective equilibria for a few representative (c, s) combinations; we then present Proposition 3 and Figure 11 that describe the profit ranking of the two scenarios for all (c, s).



(a) $\xi(\lambda^{\text{peak}}) < s < \overline{s}_o$ (b) $R'(\overline{\lambda}^c) < s < \xi(\lambda^{\text{peak}})$

Figure 9: Comparison of the PO and PU scenarios when c is relatively small.

Figures 9 and 10 combine the marginal-revenue curves of the PO scenario, R_o' and the PU scenario, ξ.[16] The new information delivered by the figures is the relationship displayed between the curves R_o' and ξ. First, Part (ii) in Claim 4 shows that the curves are proportional to each other,[17]

$$R_0'(\lambda) = 2\,\xi\,(\lambda)\,.$$

[16]Recall that (i) $\bar{s}_o$ is the maximal value of the marginal recruitment cost s for which profitable trade in the PO scenario is possible; (ii) $\underline{\lambda}_o$ is the minimal scale of any PO equilibrium with trade; and (iii) $\bar{\lambda}^c$ is the maximal participation acceptable to bidders.

[17]The ordering $R_o'(\lambda) > \xi(\lambda)$ for all $\lambda > 0$ can be derived directly by taking the total derivative on both sides of the revenue equivalence statement $R_o(\lambda) = R_u(\lambda, \beta_{\hat{\lambda}})$. This gives $R_o'(\lambda) = \frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}})\big|_{\hat{\lambda}=\lambda} + \frac{d}{d\hat{\lambda}} R_u(\lambda, \beta_{\hat{\lambda}})\big|_{\hat{\lambda}=\lambda}$. The first term equals $\xi\,(\lambda)$ and the second term satisfies $\frac{d}{d\hat{\lambda}} R_u(\lambda, \beta_{\hat{\lambda}}) > 0$, which holds because $\beta_{\hat{\lambda}}$ is strictly increasing in $\hat{\lambda}$. Thus, as claimed, there is a stronger recruitment incentive in the PO scenario because greater participation makes bidders behave more aggressively.

Second, it follows from (19) and the subsequent discussion that $R_u(\lambda, \beta_\lambda) - \lambda\xi(\lambda) > 0$ for all $\lambda > 0$. Hence, given revenue equivalence $R_u(\lambda, \beta_\lambda) = R_o(\lambda)$,

$$\bar{s}_o \equiv \max_\lambda \frac{R_o(\lambda)}{\lambda} > \xi(\lambda^{peak}).$$

We now present the two main qualitative findings:

PO profits are higher when c is small enough. The two panels of Figure 9 depict the PO and PU equilibria (marked by heavy dots) for (c, s) combinations in which c is small enough to satisfy $R_o'\left(\bar{\lambda}^c\right) \le s$, which means that $\bar{\lambda}^c$ does not constrain the PO profit maximization (it is sufficiently far to the right on both panels). The equilibrium λ's are denoted by λ_o^* (for the unique PO equilibrium) and λ_u^* (for the maximal PU equilibrium).

In Panel 9a, where $\xi(\lambda^{peak}) < s < \bar{s}_o$, the unique PU equilibrium has $\lambda_u^* = 0$ and profit 0, while at the PO equilibrium $\lambda_o^* > 0$ and profit is positive. In Panel 9b, where $s \le \xi(\lambda^{peak})$, the PU equilibrium has $\lambda_u^* = \overline{\lambda}_u(s) > 0$, but the PO equilibrium features even higher participation $\lambda_o^* = \lambda_o(s)$ and higher profit.

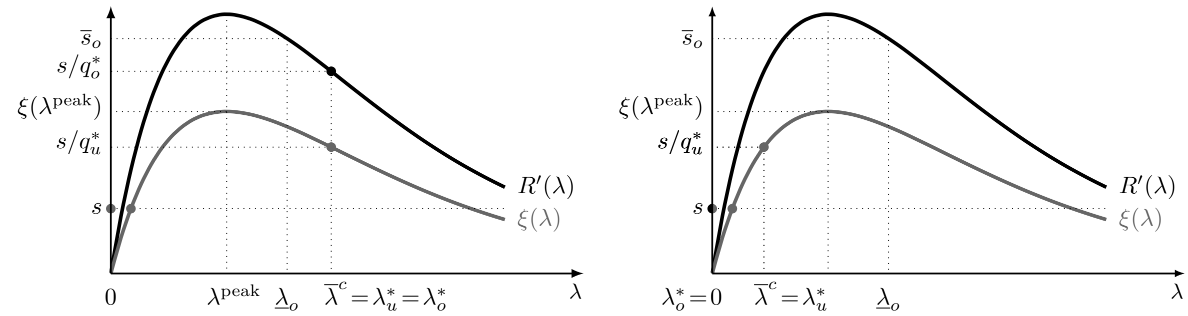
To verify that the PO profit is higher in the case of Panel 9b, recall that by the revenue equivalence (14), for any λ, $R_u(\lambda, \beta_\lambda) = R_o(\lambda) \equiv \int_0^\lambda R_o'(x)dx$. Therefore, the incremental revenue in the PO scenario, $\int_{\lambda_u^*}^{\lambda} R_o'(x)dx$ (i.e., the area under R_o' between λ_u^* and λ_o^* in Panel 9b), clearly exceeds the incremental cost, which is given by the rectangular area $(\lambda_o^* - \lambda_u^*)s$.[18]

Thus, in the cases described in Figure 9, the stronger recruitment incentives of the PO scenario result in larger participation and larger profit, while the smallness of c ensures that the bidders' entry decisions do not counteract these incentives.

In both cases, if committing to λ_o^* were possible, the PU seller would attain the PO equilibrium profit due to the revenue equivalence (14). However, this would not be a PU equilibrium, since the seller would have an incentive to take advantage of the unobservability of the participation and secretly choose $\lambda < \lambda_o^*$. The bidders, anticipating this, would plan to bid less aggressively than if they had expected λ_o^*, which would incentivize the seller to reduce λ further. With $s > \xi(\lambda^{peak})$ this "unraveling" does not stop at any positive λ, whereas with $s \le \xi(\lambda^{peak})$ it settles at $\lambda_u^* > 0$.

PU profits are higher when s is small enough. The two panels of Figure 10 present two types of circumstances under which the maximal PU profit is higher than the PO profit. In both of them $s < \xi\left(\bar{\lambda}^c\right)$ but they differ in the level of c as reflected in the

[18]Alternatively, the seller's revealed preference in the PO scenario for λ_o^* over $\bar{\lambda}_u^*$ also implies the profit ranking.



(a) $\overline{\lambda}^c > \underline{\lambda}_o$ (b) $\overline{\lambda}^c < \underline{\lambda}_o$

Figure 10: Comparison of the PO and PU scenarios when c is relatively large.

positions of $\bar{\lambda}^c$.

Panel 10a shows the case in which $\bar{\lambda}^c > \underline{\lambda}_o$ (i.e., the bound imposed by the bidders' entry decisions is above the minimum scale of any PO equilibrium with trade). Since $s < \xi\left(\bar{\lambda}^c\right) < R'_o\left(\bar{\lambda}^c\right)$, the unconstrained profit maximizers in the two scenarios, $\overline{\lambda}_u(s)$ and $\lambda_o(s)$, exceed $\bar{\lambda}^c$. Therefore, $\bar{\lambda}^c$ constrains the equilibrium participation in both scenarios, $\lambda_o^* = \lambda_u^* = \bar{\lambda}^c$, which implies that the revenues are also the same. However, the cost of inducing expected participation $\bar{\lambda}^c$ in the PO scenario, $\bar{\lambda}^c R'_o\left(\bar{\lambda}^c\right)$, is larger than that cost in the PU scenario, $\bar{\lambda}^c \xi\left(\bar{\lambda}^c\right)$. Therefore, the maximal PU profit exceeds the PO profit.

The stronger marginal recruitment incentive at $\bar{\lambda}^c$ in the PO scenario, as captured by $R'_o\left(\bar{\lambda}^c\right) > \xi\left(\bar{\lambda}^c\right)$, implies that to prevent the seller from over-recruiting, bidders must be more reluctant to enter in the PO scenario, $q_o^* < q_u^*$. Therefore, the seller must make a greater recruitment effort to achieve the same $\bar{\lambda}^c$.

Panel 10b shows the case of $\bar{\lambda}^c < \underline{\lambda}_o$. This precludes trade in the PO scenario, since a positive λ_o^* must be both above $\underline{\lambda}_o$ (for seller optimality) and below $\bar{\lambda}^c$ (for bidder optimality), which is impossible. In contrast, since $s < \xi\left(\bar{\lambda}^c\right)$, the maximal PU equilibrium has $\lambda_u^* > 0$ and positive profit. This is because the weaker recruitment incentives in the PU scenario make it possible to sustain expected participation levels below $\bar{\lambda}^c < \underline{\lambda}_o$ and thus not shut off bidders' entry.

Thus, in the case of small s, the profit ranking of the PO and PU scenarios is again explained by the difference in recruitment incentives. But here these incentives have the opposite effect, since what limits the profit is the bidders' entry cost, and the seller's main challenge is to overcome the bidders' reluctance to enter. Naturally, the weaker recruitment incentives in the PU scenario make it easier to attract reluctant bidders.

In both cases depicted in Figure 10, if the PO seller could commit to the recruitment effort of the PU equilibrium, then her profit would be the same as the PU equilibrium profit (by revenue equivalence). But this would not be an equilibrium in the PO scenario, since the stronger recruitment incentives there would induce the PO seller to recruit more aggressively. The bidders, anticipating this, would be more reluctant to enter in the case of Panel 10a (thus increasing the marginal recruitment cost) and would choose not to enter at all in the case of Panel 10b. By contrast, in the PU scenario, the weaker recruitment incentives create an effective commitment to less aggressive recruiting and hence low participation.

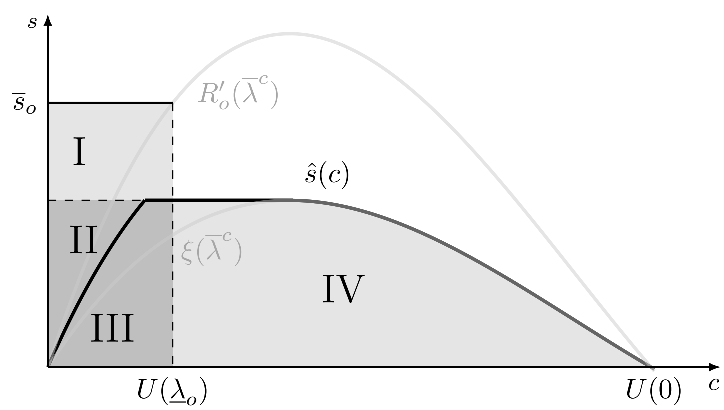


Figure 11: Profit ranking of the PO and PU scenarios.

General Comparison. Figure 11 shows the profit rankings of the PO and PU scenarios for all (c, s) combinations. It compares the profits in the unique PO equilibrium to the maximum profit achievable in a PU equilibrium.[19]

- Region I: Only PO has trade (and is strictly more profitable).
- Region II: Both scenarios have trade; PO is strictly more profitable.
- Region III: Both scenarios have trade; PU is strictly more profitable.
- Region IV: Only PU has trade (and is strictly more profitable).
- Remaining (unshaded) space: No trade in either scenario.

The boundary curve $\hat{s}(c)$ summarizes the profit rankings: under the curve, PU is strictly more profitable; above the curve, PO is strictly more profitable in the shaded region (and elsewhere, profit is 0 in both scenarios). Proposition 3 confirms the shape and location of $\hat{s}(c)$.

[19]To read the figure, recall that higher values of c are associated with lower values of $\bar{\lambda}^c$.

Proposition 3. *The boundary $\hat{s}(c)$ has the following characteristics:*

1. $\xi\left(\bar{\lambda}^c\right) \leq \hat{s}(c) \leq R_o'\left(\bar{\lambda}^c\right)$;

2. $\hat{s}(c)$ *is continuous and strictly increasing on* $[0, c^{\#}]$, *for some* $c^{\#} \in (0, U(\underline{\lambda}_o))$, *with* $\hat{s}(0) = 0$; $\hat{s}\left(c^{\#}\right) = \xi(\lambda^{peak})$;

3. $\hat{s}(c) = \xi(\lambda^{peak})$ *for* $c \in [c^{\#}, U\left(\lambda^{peak}\right)]$;

4. $\hat{s}(c) = \xi(\lambda^c)$ *for* $c \in (U\left(\lambda^{peak}\right), U(0))$.

Disclosure. The comparison of the PO and PU scenarios is also relevant for understanding the seller's disclosure preferences (when disclosure is possible). Suppose that the seller could credibly commit in advance either to always disclose the number of participants prior to the bidding or to never disclose it. This is equivalent to the seller choosing between the PO and PU scenarios. Therefore, the analysis of this section also characterizes the optimal disclosure decision. For example, it implies that the seller may be more likely to prefer to commit to nondisclosure when c is large or s is small.

4.2 A broader perspective on (un)observability

Given the revenue equivalence $R_u(\lambda, \beta_{\hat{\lambda}})|_{\hat{\lambda}=\lambda} = R_o(\lambda)$ established by (14) above, the significant difference between the PO and PU equilibrium outcomes may seem somewhat surprising. Subsection 4.1 explained this difference directly by showing how the (un)observability of the actual participation level affects recruitment incentives. We now provide a broader perspective by looking at the effect of observability beyond the FPA format.

When participation is observable,[20] after the number of participants n is realized, the auction is an ordinary one with n bidders and the bidding is unaffected by the bidders' beliefs $\hat{\lambda}$. As is well known from basic auction theory, all standard auction formats (i.e., those in which the highest-value bidder wins and a bidder with value 0 gets payoff 0) generate the same equilibrium payoffs given n. Therefore, the ex-ante expected payoffs and revenues given $\lambda = \mathbb{E}[n]$ are the same across these formats, which implies identical participation and recruitment incentives and hence identical equilibrium outcomes. In

[20]Since this subsection discusses other auction formats, we spell out "observable participation" and "unobservable participation" rather than use the abbreviations PU and PO, which are reserved for the (un)observability with the FPA format assumed in the main model this paper.

other words, the equilibrium participation and payoffs also do not depend on the details of the auction format (as long as it is standard).

In contrast, when participation is unobservable, the bidding depends on the bidders' belief $\hat{\lambda}$, not the actual λ. This means that the marginal revenue of recruitment is different from its counterpart in the observable participation scenario, and consequently the equilibrium outcome also differs. The effect of $\hat{\lambda}$ on the bidding and hence on recruitment incentives varies across auction formats. In the FPA, where bids are increasing in $\hat{\lambda}$, a low $\hat{\lambda}$ implies low bids and hence weak marginal recruitment incentives. In contrast, the dominant-strategy bids in the second-price auction (SPA) are independent of $\hat{\lambda}$. Therefore, a low $\hat{\lambda}$ need not depress the marginal recruitment incentive. In fact, since the dominant bidding strategies and resulting equilibrium of the SPA are unaffected by observability, the SPA revenue is the same whether participation is observable or not; in both cases, by the argument of the previous paragraph, it is $R_o(\lambda)$.

The SPA format is an extreme case: the bidding is completely belief-independent and so observability is completely irrelevant. In other standard auction formats, such as the all-pay auction or a mechanism that mixes between the FPA and SPA with some predetermined probabilities, the dependence of the bidding on $\hat{\lambda}$ would differ from the dependence in the FPA or SPA format and hence would lead to different equilibrium outcomes (in the unobservable-participation scenario).

This variation of equilibrium outcomes across auction formats (with unobservable participation) should not obscure the fact that when the bidders' beliefs are correct ($\hat{\lambda} = \lambda$), all of these formats yield the same revenue at any given λ (i.e., revenue equivalence holds). The point is that these different formats cannot have the same equilibrium λ, since, as explained above, the marginal revenues and hence the recruitment incentives differ across them.

4.3 Comparison of first- and second-price auctions

The outcome equivalence that we have just pointed out, between the SPA with unobservable participation and the FPA with observable participation, suggests an alternative interpretation of the results of Subsection 4.1 above. All the insights obtained in the comparison of the equilibrium outcomes of the PO and PU scenarios apply verbatim to the comparison of the SPA and FPA formats when participation is unobservable in both.

As noted above, the dominant-strategy equilibrium outcome of the SPA with unobservable participation is also the equilibrium of the SPA with observable participation, and is therefore equivalent (by the revenue equivalence established by (14)) to the equilibrium of the FPA with observable participation.

Thus, the comparison of the PO and PU equilibria in Section 4.1 can be viewed as a comparison between the equilibria of the SPA and FPA formats when participation is unobservable in both. This means that when there are recruitment and participation costs and participation is unobservable, the FPA and SPA are not equivalent in terms of payoffs and participation, because they generate different recruitment incentives.[21]

This latter observation provides an immediate answer to the question of which of these two formats is more profitable for the seller. When participation is observable, the two formats yield the same profit, of course. When participation is unobservable, the answer depends on c and s. The SPA format is more profitable than the (most profitable equilibrium of the) FPA format if and only if the PO profit is higher than the maximal PU profit given (c, s).[22]

However, such "design" questions are outside the scope of this paper. This is because our main interest is in informal auction situations in which the seller has limited power to design the interaction. The FPA without a reserve price seems a more natural model of such a situation. A reserve price and the use of the SPA format require greater commitment power than we would like to assume.

5 Discussion and extensions

5.1 Welfare

Welfare $W(\lambda, q)$ is identified with the total surplus,

$$W(\lambda, q) := T(\lambda) - \lambda \frac{s}{q} - \lambda c,$$

where $T(\lambda) = \int_0^1 v\lambda e^{-\lambda(1-G(v))} g(v)dv = \int_0^1 [1 - e^{-\lambda(1-G(v))}]dv$ is the expected value of the first-order statistic given Poisson(λ)-distributed participation. Let λ^w and q^w denote the

[21] The stronger incentive to recruit in the PO scenario was explained earlier by the more aggressive bidding. Now we note that the PO equilibrium is equivalent (in terms of profit) to the dominant-strategy equilibrium of the SPA, where greater participation does not induce more aggressive bidding (since bidders bid their own values independently of the participation). These observations are not inconsistent: what affects the incentive to recruit is not the aggressive bidding in itself, but rather the higher expected price that it implies. In the SPA, greater participation does not induce more aggressive bidding, but it does translate into a higher expected price.

[22] Even when the most profitable equilibrium of the FPA is more profitable, a seller may still prefer the SPA if she is concerned about being trapped in the less profitable equilibria of the FPA.

welfare-maximizing magnitudes.

Proposition 4. *(i)* $q^w = 1$. *(ii) If* $U(0) > s + c$, *then* λ^w *is the unique value of* λ *satisfying*

$$U(\lambda) = c + s. \tag{20}$$

If $U(0) < s + c$, *then* $\lambda^w = 0$.

Proof. Part (i) is obvious. For Part (ii), note that

$$T'(\lambda) = \int_0^1 (1 - G(v))\, e^{-\lambda(1-G(v))} dv = U(\lambda),$$

where the second equality uses the explicit form of U from (22) in the appendix. Since U is strictly decreasing, T is strictly concave. It follows that (20) is the first-order condition for welfare maximization, and the condition is sufficient, thus proving the claim. ∎

The critical equality is

$$T'(\lambda) = U(\lambda). \tag{21}$$

This equality is not surprising, given the familiar result that in an equilibrium of a standard auction,[23] each bidder's payoff equals his marginal contribution to the total surplus.

There are two types of inefficiency in equilibrium in the PO scenario. First, as we already know, we can have $q^* < 1$ in equilibrium, which immediately means wasted recruitment effort. Second, as we show below, for almost all pairs (s, c) in the PO scenario, $\lambda^* \neq \lambda^w$, and both excessive participation, $\lambda^* > \lambda^w$, and deficient participation, $\lambda^* < \lambda^w$, may arise in equilibrium.

For the equilibrium of the PO scenario to coincide with the welfare maximum, we must have $R_o'(\lambda^*) = s$ and $U(\lambda^*) = s + c$. Since both U and R_o' are independent of s and c, these equalities cannot be expected to hold simultaneously; indeed, they fail for almost all c and s. Thus, in general, the equilibrium does not maximize welfare.

Figure 12 depicts a possible relationship between $U(\lambda)$ and $R_o'(\lambda)$. Its relevant features are consistent with a uniform value distribution, that is, $G(v) = v$.

In this case, since $U(\lambda) < R_o'(\lambda)$ for any $\lambda \geq \underline{\lambda}_o$, it follows that $\lambda^* > \lambda^w$ in any equilibrium with trade. If $\lambda^* < \overline{\lambda}^c$, then $s + c > s = R_o'(\lambda^*) > U(\lambda^*)$; if $\lambda^* = \overline{\lambda}^c$, then $s + c > c = U(\lambda^*)$. In the case of $\lambda^* = \overline{\lambda}^c$, there is also the inefficiency of $q^* < 1$, except when $s = R_o'(\overline{\lambda}^c)$. On the other hand, there is a range of (s, c) combinations such that

[23] An auction in which the highest value bidder wins and the lowest value bidder gets payoff 0.

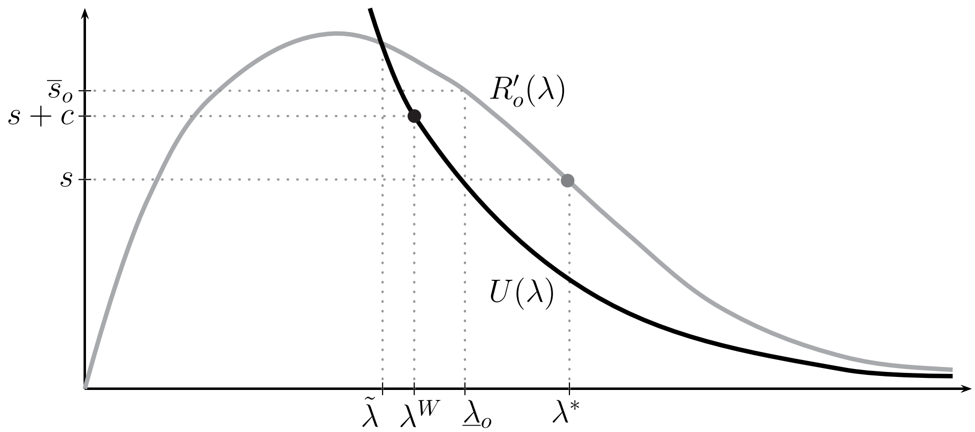


Figure 12: Welfare.

$s + c < U(0)$ requires trade, $\lambda^w > 0$, but either $s > \overline{s}_o$ or $\overline{\lambda}^c < \underline{\lambda}_o$ precludes trade in equilibrium, meaning $\lambda^w > \lambda^* = 0$.

Although this paper does not examine in detail the relationship between equilibrium and welfare in the PU scenario, one can reasonably expect the equilibria to be generally inefficient in that scenario as well. Since the maximal equilibrium in the PU scenario involves lower participation than that of the PO scenario, there will be less inefficiency due to excessive recruiting.[24]

For a general CDF G satisfying our regularity assumptions, we have already established that U is decreasing and R'_o is single-peaked, as shown in Figure 12. The fact that U intersects R'_o for the first time at some point $\widetilde{\lambda} > \lambda^{peak}$ (the maximizer of $R'_o(\lambda)$) also holds for general CDF G (see Claim 5 in the appendix). We have not established analytically all of the details in the figure for general CDF G,[25] but these details will not affect the overall conclusion that the equilibria are suboptimal.

The excessive recruitment noted above for the uniform distribution recalls the result in Szech (2011) that when G exhibits an increasing hazard rate, the equilibrium participation in an FPA with linear recruitment cost will exceed the welfare-maximizing level. Our model is somewhat different because of the stochastic arrival and entry costs, but the insight is similar.

Condition (21) implies that, for a given λ, the individual bidders' entry decisions are efficient. This is the counterpart in our model of a central finding in the literature on

[24]When the value distribution G is uniform, then numerically $\xi < U$, meaning that the seller may often recruit too few bidders.

[25]We have shown that, if G is uniform, $U_o(\lambda)$ and $R'_o(\lambda)$ intersect only once and that $\widetilde{\lambda} < \underline{\lambda}_o$. We have not established these properties for general CDF G. However, loosely speaking, we expect $U_o(\lambda)$ to be mostly below $R'_o(\lambda)$ since $R_o(\lambda)$ is below $T(\lambda)$ and converging to it.

costly entry (see Levin and Smith, 1994).

Note that λ^w maximizes welfare only within the constraints of the original Poisson contacting process. For example, welfare would be higher if the planner could coordinate entry among the contacted bidders so as to avoid excessive participation when the number of contacted bidders realized is too high.[26]

5.2 Uniqueness of equilibrium in the PO scenario

The equilibrium outcome of the PO scenario is unique for almost all values of s and c (except when $s = \bar{s}_o$ or $\overline{\lambda}^c = \underline{\lambda}_o$), given the refinement imposed by the last condition of the equilibrium definition in Section 1.2.[27] Without the refinement, the no-trade outcome is always an equilibrium; more precisely,

- if $s > \bar{s}_o$ or $\overline{\lambda}^c < \underline{\lambda}_o$, then no-trade is still the unique equilibrium outcome;
- if $s < \bar{s}_o$ and $\overline{\lambda}^c > \underline{\lambda}_o$, there are now two equilibrium outcomes: one with $\lambda^* > 0$ and one with $\lambda^* = 0$.

In the case of the second bullet point, the additional no-trade equilibrium $\lambda^* = 0$ is supported by the off-path belief $\mu(\overline{\lambda}^c) = 1$ and $q^* \in (0, \frac{s}{\bar{s}_o})$. That is, bidders contacted off-path conjecture that $\lambda = \overline{\lambda}^c$, which makes them indifferent among all choices of q, including q^*. Such an equilibrium violates the refinement, since $\frac{s}{q^*} > \bar{s}_o$ implies strictly negative profits at any $\lambda > 0$ including $\overline{\lambda}^c$.

Observe that this no-trade equilibrium is unconvincing on other grounds as well. First, when $s < \bar{s}_o$ and $\overline{\lambda}^c > \underline{\lambda}_o$, it is Pareto-dominated by the equilibrium with trade. Second, it is not robust to perturbations. Consider a perturbation in which the seller is required to choose at least an effort $\gamma \geq \varepsilon > 0$, for some small $\varepsilon > 0$. As $\varepsilon \to 0$, this perturbed game has a unique limit outcome that corresponds to the equilibrium with trade. This is because for any $q \in (0,1)$ such that $\frac{s}{q} \geq \bar{s}_o$, the seller's best response is either $\lambda = \varepsilon$ or $\underline{\lambda}_o$ (or mixing between them). However, in all these cases, $\overline{\lambda}^c > \underline{\lambda}_o$ implies that the bidders have a strict incentive to enter, so that $q = 1$.

Formally, since this game is not finite (because it has a continuum of actions and an unbounded number of players), we cannot directly apply the concept of stability in the

[26]This is similar to the observation of Levin and Smith (1994) that the randomness over participation numbers in symmetric mixed equilibria reduces welfare relative to the deterministic participation numbers in asymmetric pure equilibria.

[27]If $\gamma^* = 0$, then no $\hat{\gamma}$ in the support of μ yields negative profits. A slightly more general formulation would require every $\hat{\gamma}$ in the support of μ to be a best response by the seller to q^* and β^*.

sense of Kohlberg and Mertens (1986). However, for a discretized version in which the seller chooses γ from a finite grid (that contains 0, $\overline{\lambda}^c$, and $\underline{\lambda}_o$), we can define a refinement in the spirit of stability, requiring that the equilibrium be immune to all vanishing fully mixed perturbations. It is fairly immediate that the no-trade equilibrium will fail such refinement, while the unique equilibrium with trade will survive it.[28]

We can also indirectly confirm the instability of the no-trade equilibrium by observing that it fails the invariance property of stable equilibrium. To see this, consider the equivalent extensive form in which the seller first chooses between $\gamma = 0$, which terminates the game, and another action, "$\gamma > 0$", which stands for all positive recruitment efforts. After taking the action "$\gamma > 0$", the seller chooses the specific γ and the bidders make their entry decisions. The unique subgame-perfect equilibrium here is the equilibrium with trade, by the same argument as used above for the variation with $\gamma \geq \varepsilon$.

5.3 Pseudo-stability considerations in the PU scenario

The smaller of the two equilibria with trade in the PU scenario is pseudo-unstable in the sense that, following a small displacement of the equilibrium recruitment effort and the corresponding adjustment of bidders' expectations, there is no corrective force that returns the recruitment effort to its equilibrium level. Recall from Section 3.2 (and Figure 7) that in those equilibria it must be that the ξ curve crosses the horizontal s line from below and that $\lambda^* = \underline{\lambda}_u(s) < \overline{\lambda}^c$. Hence, $q^* = 1$ and $\lambda^* = \gamma^*$. Suppose that γ^* is displaced by $\Delta > 0$ such that $\gamma^* + \Delta < \overline{\lambda}^c$. After bidders adjust their expectations and entry decisions, $\lambda = \gamma^* + \Delta$ and $q = 1$. Since $\xi(\lambda) > s$, the seller's profit at λ is higher than in the equilibrium at λ^*. Therefore, the seller has no incentive to cut back the recruitment effort to γ^*.

In contrast, the larger equilibrium with trade is pseudo-stable: the above argument would fail since ξ crosses the horizontal $\frac{s}{q^*}$ from above at the equilibrium at λ^*.

6 Concluding remarks

This paper contributes to the market approach in auction theory, treating auctions as abstractions of less formal price formation.[29] It examines the roles and interactions of

[28] Note, however, that the no-trade equilibrium will survive an analogously defined refinement in the spirit of perfect equilibrium, since we can focus on a sequence of perturbations for which the expectation conditional on $\lambda > 0$ is $\overline{\lambda}^c$.

[29] This perspective was adopted by, for example, Wilson (1977) and Milgrom (1979)

three ubiquitous frictions in such scenarios: costly recruitment, costly bidder entry, and the seller's inability to commit.

A number of natural extensions suggest themselves. Some of them we discuss in the online appendix: nonlinear recruitment cost, entry fees (subsidies), reserve prices, bidder heterogeneity (known values at the time of entry), some further results on welfare, and a numerical analysis for a uniform value distribution.

Many open questions remain. For instance, it might be interesting to study the comparative statics of the outcome with respect to the value distribution: how does the latter affect the seller's recruitment effort and the inefficiency? Relatedly, one could explicitly model a setting in which the bidders acquire information at some cost or in which the seller provides information to bidders at some cost, implicitly subsidizing entry. As we highlighted in the introduction, the fundamental inefficiencies of informal auctions may induce demand for intermediaries; it may therefore be worthwhile to study the role of such intermediaries in our framework.

7 Appendix

7.1 Proofs for the PO scenario

7.1.1 Bidders' ex-ante expected payoff

Claim 2.

$$U(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G(v)]dv. \tag{22}$$

Proof of Claim 2. By (1), β_{FPA} is strictly increasing. Therefore, in an auction with a total of n bidders, a bidder with value v wins with probability $G(v)^{n-1}$. Let $U(n, v)$ denote the equilibrium payoff to a bidder with value v in an auction with a total of n bidders, and note that $U(n, 0) = 0$. The usual envelope argument implies that $U(n, v) = \int_0^v G(x)^{n-1} dx$; see Krishna (2009).

Hence, when n is drawn from a Poisson distribution with mean λ, the expected payoff of type v is

$$\sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^{n-1}}{(n-1)!} U(n, v) = \int_0^v \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n G(x)^n}{n!} dx = \int_0^v e^{-\lambda(1-G(x))} dx.$$

We used that $\sum_{n=0}^{\infty} e^{-\lambda G(x)} \frac{\lambda^n G(x)^n}{n!} = 1$. Therefore, the ex-ante expected payoff is

$$U(\lambda) = \int_0^1 \left(\int_0^v e^{-\lambda(1-G(x))} dx \right) g(v)\, dv.$$

Changing the order of integration yields (22). ■

7.1.2 The seller's revenue

Claim 3. *(i) $R_o(\lambda)$ is strictly increasing, $R_o(0) = 0$, and $\lim_{\lambda\to\infty} R_o(\lambda) = 1$.*

(ii) $R_o(\lambda)$ is continuously differentiable, $R_o'(0) = 0$, $R_o'(\lambda) \to 0$ as $\lambda \to \infty$, and R_o' is single-peaked.

(iii) $\frac{R_o(\lambda)}{\lambda}$ is single-peaked; at its peak, $\frac{R_o(\lambda)}{\lambda} = R_o'(\lambda)$.

Proof of Claim 3. The total surplus (gross of the recruitment costs) is the expectation of the first-order statistic of v given Poisson(λ) distributed participation:

$$\text{Total Surplus}(\lambda) = \int_0^1 \left[1 - e^{-(1-G(v))\lambda}\right] dv.$$

The total surplus is equal to the sum of the revenue, $R_o(\lambda)$, and the bidders' total expected payoff, $\lambda U(\lambda)$. Revenue is the difference of Total Surplus(λ) and $\lambda U(\lambda)$, that is,[30]

$$R_o(\lambda) = \int_0^1 \left[1 - e^{-(1-G(v))\lambda} - e^{-(1-G(v))\lambda} (1 - G(v))\, \lambda\right] dv. \tag{23}$$

Differentiating and rearranging yields

$$\frac{d}{d\lambda} R_o(\lambda) = \int_0^1 \lambda (1 - G(v))^2 e^{-(1-G(v))\lambda} dv. \tag{24}$$

Parts (i) and (ii). Positivity, continuity, and values at $\lambda = 0$ and $\lambda \to \infty$ are obvious from (23) and (24). To establish that R_o' is single-peaked, consider the second derivative:

$$\begin{aligned} \frac{d^2}{d\lambda^2} R_o(\lambda) &= \int_0^1 (1 - G(v))^2 e^{-(1-G(v))\lambda} dv - \int_0^1 \lambda (1 - G(v))^3 e^{-(1-G(v))\lambda} dv \qquad (25) \\ &= e^{-\lambda} \left(\frac{1}{g(0)} - \int_0^1 (1 - G(v))^2 e^{G(v)\lambda} \left[v - \frac{1 - G(v)}{g(v)} \right]_v' dv \right), \end{aligned}$$

[30]Note that the integrand on the right side is 1 minus the CDF of the second-order statistic of v, as expected given the revenue equivalence in the text.

using integration by parts.

Recall that, by assumption, $\left[v - \frac{1-G(v)}{g(v)}\right]'_v > 0$. Thus, the integral on the last line of (25) is positive and increasing in λ, while the first term is positive and independent of λ. Therefore, $\frac{d^2}{d\lambda^2} R_o(\lambda) < 0$ for large λ, and once it turns negative, it stays negative. Inspection of the first line of (25) reveals that $\frac{d^2}{d\lambda^2} R_o(\lambda) > 0$ for $\lambda \in [0, \varepsilon]$ for some $\varepsilon > 0$. The two observations imply that $\frac{d}{d\lambda} R_o(\lambda)$ is single-peaked.

Part (iii). This is an immediate corollary of Parts (i)–(ii) and the fact that $d(R_o(\lambda)/\lambda)d\lambda = \left[R'_o(\lambda) - \frac{R_o(\lambda)}{\lambda}\right]/\lambda$. ■

Proposition 1. There exists a unique equilibrium for all (c, s) such that $c \neq U(\underline{\lambda}_o)$ and $s \neq \bar{s}_o$. The form of the equilibrium varies across the regions of Figure 4 as follows:

1. If (c, s) is in the unshaded region, then $\lambda^* = 0$ (no trade).
2. If (c, s) is in shaded Region I, then $\lambda^* = \lambda_o(s)$ and $q^* = 1$.
3. If (c, s) is in shaded Region II, then $\lambda^* = \overline{\lambda}^c$ and $q^* = s/R'_o(\overline{\lambda}^c)$.

Proof of Proposition 1.

Part 1. The (c, s) in the unshaded region are such that $s > \bar{s}_o$ or $c > U(\underline{\lambda}_o)$. If $s > \bar{s}_o$, then by (7), $\lambda = 0$ is the seller's unique optimal choice for any q^*. Therefore, $\lambda^* = 0$ with $q^* = 1$ and $\mu^*(0) = 1$ is the unique equilibrium in this case.

If $c > U(\underline{\lambda}_o)$ and $s < \bar{s}_o$, then the following is an equilibrium: $\lambda^* = 0$, q^* satisfies $\bar{s}_o = \frac{s}{q^*}$, and μ^* has support on $\{0, \underline{\lambda}_o\}$ with $\mu^*(0)U(0) + \mu^*(\underline{\lambda}_o)U(\underline{\lambda}_o) = c$. Such a μ^* exists because $c \in (U(0), U(\underline{\lambda}_o))$. The choice of μ^* implies $E_{\mu^*}(U(\lambda)) = c$, so q^* is bidder-optimal. The choice of q^* also implies $\max_\lambda \Pi_o(\lambda, q^*) = 0$ and $\arg\max_\lambda \Pi_o(\lambda, q^*) = \{0, \underline{\lambda}_o\}$. Hence, $\lambda^* = 0$ is seller-optimal, and μ^* satisfies equilibrium condition E2(iii) (the refinement).

The uniqueness of this equilibrium is established by the following two arguments. First, there is no equilibrium with $\lambda^* > 0$, since it would have to satisfy $\underline{\lambda}_o \leq \lambda^* \leq \overline{\lambda}^c$. But this contradicts $c > U(\underline{\lambda}_o)$, which implies $\overline{\lambda}^c < \underline{\lambda}_o$.

Second, to verify that there is no other equilibrium with $\lambda^* = 0$, recall that in such an equilibrium seller optimality would imply $\max_\lambda \Pi_o(\lambda, q^*) = 0$. Hence, equilibrium condition E2(iii) implies that $\Pi_o(\lambda, q^*) = 0$ for any λ in the support of μ^*. Observe that $\mu^*(0) < 1$, since $\mu^*(0) = 1$ implies $q^* = 1$, which together with $s < \bar{s}_o$ imply $\Pi_o(\lambda_o(s), 1) > 0$ contradicting $\max_\lambda \Pi_o(\lambda, 1) = 0$. Now, since any $\lambda > 0$ in the support of

μ^* must be profit maximizing and yield profit 0, it follows that such λ and q^* must satisfy $\lambda = \lambda_o(\frac{s}{q^*}) = \underline{\lambda}_o$. Therefore, $\frac{s}{q^*} = \bar{s}_o$, which establishes the uniqueness.

Parts 2&3. For $c < U(\underline{\lambda}_o)$ and $s < \bar{s}_o$, the profiles described in Parts 2 and 3 of Proposition 1 satisfy the bidders' and the seller's optimality conditions. Since $\lambda^* > 0$, the point beliefs are confirmed, and so these are equilibria.

Uniqueness is established by the following four steps.

Step I: There is no equilibrium with $\lambda^* = 0$.

When $\lambda^* = 0$, seller optimality requires that $\max_\lambda \Pi_o(\lambda, q^*) = 0$. So, by the equilibrium refinement, $\Pi_o(\lambda, q^*) = 0$ for any λ in the support of μ^*. Thus, by (7), the support of μ^* is contained in $\{0, \underline{\lambda}_o\}$. Since $c < U(\underline{\lambda}_o)$ implies $\underline{\lambda}_o < \overline{\lambda}^c$, and since U is decreasing, it follows that $E_{\mu^*}[U(\lambda)] > c$, and so $q^* = 1$. However, when $s < \bar{s}_o$, we have $\Pi_o(\underline{\lambda}_o, 1) > 0$, which contradicts the requirement that $\max_\lambda \Pi_o(\cdot, q^*) = 0$; thus, there is no equilibrium with $\lambda^* = 0$. □

Step II: When $\lambda^* > 0$,

$$\lambda^* = \min\{\overline{\lambda}^c, \lambda_o(s)\}. \tag{26}$$

From seller optimality (7), it follows that if $\lambda^* > 0$, then $\lambda^* \geq \underline{\lambda}_o$ and $R'_o(\lambda^*) = \frac{s}{q^*}$. From bidder optimality (2), it follows that q^* may differ from 1 only if $\lambda^* = \overline{\lambda}^c$. Therefore, the only possibilities are $\lambda^* = \overline{\lambda}^c$ or $\lambda^* = \lambda_o(s)$. If $\overline{\lambda}^c > \lambda_o(s)$, then for any q, the fact that R'_0 is decreasing means that $R'_o\left(\overline{\lambda}^c\right) < \frac{s}{q}$, so $\overline{\lambda}^c$ cannot be an equilibrium outcome. If $\overline{\lambda}^c < \lambda_o(s)$, then $U\left(\lambda_o(s)\right) < c$, so $\lambda_o(s)$ cannot be an equilibrium outcome. □

Step III: If $\bar{s}_o > s > R'_o(\overline{\lambda}^c)$ and $\lambda^* > 0$, it must be that $(\lambda^*, q^*) = (\lambda_o(s), 1)$.

Since R'_o is decreasing, $\lambda_o(s) < \overline{\lambda}^c$. Therefore, if $\lambda^* > 0$, it follows from (26) that $\lambda^* = \lambda_o(s)$. And $\lambda_o(s) < \overline{\lambda}^c$ implies $q^* = 1$. □

Step IV: If $s < R'_o(\overline{\lambda}^c)$ and $\lambda^* > 0$, it must be that $(\lambda^*, q^*) = (\overline{\lambda}^c, \frac{s}{R'_o(\overline{\lambda}^c)})$.

Since R'_o is decreasing, $\lambda_o(s) > \overline{\lambda}^c$. So (26) implies $\lambda^* = \overline{\lambda}^c$. Hence, (7) implies that q^* satisfies $R'_o(\overline{\lambda}^c) = \frac{s}{q^*}$. □

Thus, Step I rules out $\lambda^* = 0$, and Steps II - IV establish the uniqueness of the equilibrium with $\lambda^* > 0$. ■

7.2 Proofs for the PU scenario

Claim 4. *(i) $R_u(\lambda, \beta_{\hat{\lambda}})$ is twice differentiable (in λ and $\hat{\lambda}$), and for $\hat{\lambda} > 0$ it is strictly concave in λ.*

(ii)
$$\xi(\lambda) = \int_0^1 \frac{1}{2}\left[1 - G(v)\right]^2 \lambda e^{-\lambda(1-G(v))} dv \equiv \frac{1}{2} R_o'(\lambda), \tag{27}$$
and hence it is single-peaked, and continuous, $\xi(0) = 0$, *and* $\xi(\lambda) \to 0$ *for* $\lambda \to \infty$.

Proof of Claim 4: Part (i) The CDF of the first-order statistic (maximum value) given Poisson(λ) arrival is
$$F_{(1)}(t;\lambda) = \Pr(\max v \le t; \lambda) = e^{-\lambda(1-G(t))}.$$
Hence,
$$R_u(\lambda, \beta_{\hat{\lambda}}) = \int_0^1 \beta_{\hat{\lambda}}(v)\, dF_{(1)}(v;\lambda). \tag{28}$$
Substitute into (11) both $P_\lambda(n) = \frac{\lambda^n}{n!} e^{-\lambda}$ and the explicit expression for β_{FPA} from (1), and rearranging by using that $\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} G(x)^n = e^{-\lambda(1-G(x))}$ yields
$$\beta_\lambda(v) = v - \int_0^v e^{-\lambda(G(v)-G(x))} dx. \tag{29}$$
Next, substituting the explicit forms of $F_{(1)}$ and $\beta_{\hat{\lambda}}(v)$ into the RHS of (28),
$$R_u(\lambda, \beta_{\hat{\lambda}}) = \int_0^1 \left[v - \int_0^v e^{-\hat{\lambda}(G(v)-G(x))} dx\right] \lambda g(v) e^{-\lambda(1-G(v))} dv$$
$$= \left[\left(v - \int_0^v e^{-\hat{\lambda}(G(v)-G(x))} dx\right) e^{-\lambda(1-G(v))}\right]_0^1 - \int_0^1 \left[1 - e^{-\hat{\lambda}(G(v)-G(v))} + \hat{\lambda} g(v) \int_0^v e^{-\hat{\lambda}(G(v)-G(x))} dx\right]$$
$$= 1 - \int_0^1 e^{-\hat{\lambda}(1-G(x))} dx - \int_0^1 \left(\int_0^v e^{-\hat{\lambda}(G(v)-G(x))} dx\right) \hat{\lambda} g(v) e^{-\lambda(1-G(v))} dv,$$
where the second equality is obtained from integration by parts. Obviously, $R_u(\lambda, \beta_{\hat{\lambda}})$ is twice differentiable in λ and $\hat{\lambda}$. Concavity in λ, given $\hat{\lambda} > 0$, follows from
$$\frac{\partial^2}{\partial \lambda^2} R_u(\lambda, \beta_{\hat{\lambda}}) = -\int_0^1 \left(\int_0^v e^{-\hat{\lambda}(G(v)-G(x))} dx\right) \hat{\lambda} \left[(1 - G(v))^2\right] g(v) e^{-\lambda(1-G(v))} dv < 0.$$
(For an alternative, direct argument see the footnote.[31])

Part (ii) We have
$$\xi(\lambda) \;\; = \;\; \frac{\partial}{\partial \lambda} R_u(\lambda, \beta_{\hat{\lambda}}) \mid_{\hat{\lambda}=\lambda} = \int_0^1 (1 - G(v)) \left(\int_0^v e^{-\lambda(G(v)-G(x))} dx\right) \lambda g(v) e^{-\lambda(1-G(v))} dv$$

[31] A direct argument: Since $R_u(\lambda, \beta_{\hat{\lambda}})$ is the expected first-order statistic when sampling from the bid distribution induced by $\beta_{\hat{\lambda}}$, Concavity is implied by the concavity of the expected first-order statistic in the (expected) number of samples.

$$
\begin{aligned}
&= \left[-\frac{1}{2}(1-G(v))^2\lambda\left(\int_0^v e^{-\lambda(1-G(x))}dx\right)\right]_0^1 + \frac{1}{2}\int_0^1 (1-G(v))^2\lambda\left(\int_0^v e^{-\lambda(1-G(x))}dx\right)_v' dv \\
&= \frac{1}{2}\int_0^1 (1-G(v))^2\lambda e^{-\lambda(1-G(v))}dv,
\end{aligned}
$$

where the third equality follows from integration by parts. This establishes the first equality in (27); the second equality follows from inspecting (24) in Claim 3 above.

It is immediate from its explicit expression that ξ is continuous, $\xi(0) = 0$, and $\lim_{\lambda\to\infty}\xi(\lambda) = 0$. The single-peakedness follows from the single-peakedness of $R_o'(\lambda)$ that was established by Claim 3. ■

Claim 1: (i) If (λ^*, q^*) and (λ^{**}, q^{**}) are equilibria (for possibly different (c, s) configurations), then

$$
\lambda^{**} > \lambda^* \Longrightarrow \Pi_u(\lambda^{**}, \beta_{\lambda^{**}}, q^{**}) > \Pi_u(\lambda^*, \beta_{\lambda^*}, q^*).
$$

(ii) The maximal PU equilibrium profit is weakly decreasing in c and s.

Proof of Claim 1. The CDF of the third-order statistic is

$$
F_{(3)}(v;\lambda) = e^{-\lambda(1-G(v))} + (1-G(v))\,\lambda e^{-\lambda(1-G(v))} + \frac{1}{2}(1-G(v))^2\lambda^2 e^{-\lambda(1-G(v))}.
$$

We show that the profit is the expectation of the third-order statistic,

$$
R_u(\lambda,\beta_\lambda) - \lambda\xi(\lambda) = \int_0^1 (1-F_{(3)}(v;\lambda))dv.
$$

Combining the previous findings from the proof of Claim 4, $R_u(\lambda,\beta_\lambda) - \lambda\xi(\lambda)$ equals

$$
1-\int_0^1 e^{-\lambda(1-G(x))}dx - \int_0^1\left(\int_0^v e^{-\lambda(G(v)-G(x))}dx\right)\lambda g(v)e^{-\lambda(1-G(v))}dv - \lambda\frac{1}{2}\int_0^1 (1-G(v))^2\lambda e^{-\lambda(1-G(v))}dv.
$$

Evaluating the second term shows

$$
\begin{aligned}
&\int_0^1\left(\int_0^v e^{-\lambda(G(v)-G(x))}dx\right)\lambda g(v)e^{-\lambda(1-G(v))}dv = \int_0^1\left(\int_0^v e^{-\lambda(1-G(x))}dx\right)\lambda g(v)dv \\
&= \left\{-\left[(1-G(v))\,\lambda\int_0^v e^{-\lambda(1-G(x))}dx\right]_0^1 - \int_0^1\left[-(1-G(v))\right]\lambda e^{-\lambda(1-G(v))}dv\right\} \\
&= \int_0^1 \left[(1-G(v))\right]\lambda e^{-\lambda(1-G(v))}dv,
\end{aligned}
$$

which proves the claim.■

7.3 Proofs for the comparison of the PO and PU scenarios

Proposition 3: The boundary $\hat{s}(c)$ has the following characteristics:

1. $\xi\left(\bar{\lambda}^{c}\right) \leq \hat{s}(c) \leq R_{o}'\left(\bar{\lambda}^{c}\right)$;
2. $\hat{s}(c)$ is continuous and strictly increasing on $[0, c^{\#}]$, for some $c^{\#} \in (0, U(\underline{\lambda}_{o}))$, with $\hat{s}(0) = 0$; $\hat{s}\left(c^{\#}\right) = \xi(\lambda^{peak})$;
3. $\hat{s}(c) = \xi(\lambda^{peak})$ for $c \in [c^{\#}, U\left(\lambda^{peak}\right)]$;
4. $\hat{s}(c) = \xi(\lambda^{c})$ for $c \in (U\left(\lambda^{peak}\right), U(0))$.

Proof of Proposition 3: Propositions 1 and 2 imply that Regions I and IV in Figure 11 are above and below $\hat{s}(c)$, respectively. Let $\lambda_{i}^{*}(c,s)$, $q_{i}^{*}(c,s)$, and $\Pi_{i}(c,s)$ respectively denote the equilibrium participation, entry probability, and profit, as functions of (c,s), where $i = o$ stands for the PO scenario and $i = u$ stands for the maximal profit equilibrium of the PU scenario. Propositions 1 and 2 imply that for (c,s) in Regions II and III of Figure 11, $\lambda_{i}^{*}(c,s) > 0$ for $i = o, u$. The discussion surrounding Figures 9 and 10 earlier in this section established that, for $c < U(\underline{\lambda}_{o})$,

$$\begin{aligned} \Pi_{u}(c,s) &> \Pi_{o}(c,s) \text{ if } s \leq \xi\left(\bar{\lambda}^{c}\right); \\ \Pi_{u}(c,s) &< \Pi_{o}(c,s) \text{ if } s \geq R_{o}'\left(\bar{\lambda}^{c}\right). \end{aligned}$$

Therefore, the following discussion focuses on the remaining cases of (c,s) for which

$$c < U(\underline{\lambda}_{o}) \text{ and } \xi\left(\bar{\lambda}^{c}\right) < s < R_{o}'\left(\bar{\lambda}^{c}\right). \tag{30}$$

Claim: At any (c,s) in the interior of the relevant range (30),

$$\frac{\partial \Pi_{o}(c,s)}{\partial c} < 0 \text{ and } \frac{\partial \Pi_{u}(c,s)}{\partial c} = 0; \tag{31}$$

$$\frac{\partial \Pi_{o}(c,s)}{\partial s} = 0 \text{ and } \frac{\partial \Pi_{u}(c,s)}{\partial s} < 0. \tag{32}$$

Proof of Claim: For all (c,s) in the relevant range (30),

$$\Pi_{o}(c,s) = R_{o}\left(\bar{\lambda}^{c}\right) - R_{o}'\left(\bar{\lambda}^{c}\right)\bar{\lambda}^{c}, \tag{33}$$

and

$$\Pi_u(c,s) = R_o(\lambda_u^*) - \xi(\lambda_u^*)\lambda_u^*, \tag{34}$$

where λ_u^* is the larger solution to $\xi(\lambda) = s$. The RHS of (33) is independent of s and is decreasing in c since, by the discussion around (8), it is increasing in $\bar{\lambda}^c$, which in turn is decreasing in c. Hence, $\frac{\partial \Pi_o(c,s)}{\partial s} = 0$ and $\frac{\partial \Pi_o(c,s)}{\partial c} < 0$. The RHS of (34) is independent of c since $\xi\left(\bar{\lambda}^c\right) < s$ implies $\lambda_u^* < \bar{\lambda}^c$; it is also decreasing in s since, by the discussion around (19), it is increasing in λ_u^*, which in turn is decreasing in s. Hence, $\frac{\partial \Pi_u(c,s)}{\partial c} = 0$ and $\frac{\partial \Pi_u(c,s)}{\partial s} < 0$. $\square$

Define $\lambda^{\#}$ to be the unique solution to

$$R_o\left(\lambda^{peak}\right) - \lambda^{peak}\xi(\lambda^{peak}) = R_o\left(\lambda^{\#}\right) - \lambda^{\#}R_o'\left(\lambda^{\#}\right).$$

Observe that the LHS is equal to $\Pi_u\left(0, \xi(\lambda^{peak})\right)$ (by the revenue equivalence (14)), while the RHS is equal to $\Pi_o\left(0, R'\left(\lambda^{\#}\right)\right)$.[32] The LHS is strictly positive since the profit of a PU equilibrium with positive participation is always strictly positive. Therefore, there must be a unique $\lambda^{\#} > \underline{\lambda}_o$ that satisfies the equality. Of course, this equality of profits holds for any c such that $\bar{\lambda}^c \geq \lambda^{\#}$ (or, equivalently, $c \leq U\left(\lambda^{\#}\right)$). In particular, let $c^{\#} \triangleq U\left(\lambda^{\#}\right)$; it follows that:

$$\Pi_u\left(c^{\#}, \xi(\lambda^{peak})\right) = \Pi_o\left(c^{\#}, R'\left(\lambda^{\#}\right)\right). \tag{35}$$

We are now ready to derive the following segments of the boundary curve $\hat{s}(c)$. First, (35) and (31) imply

$$\Pi_u(c,s) > \Pi_o(c,s) \text{ for all } c > c^{\#} \text{ and } s \leq \xi(\lambda^{peak}). \tag{36}$$

Equation (36) implies that $\hat{s}(c) = \xi(\lambda^{peak})$ is the boundary between Regions II and III for $c \geq c^{\#}$, as claimed.

Second, (31) and (35) imply

$$\Pi_u\left(c, \xi(\lambda^{peak})\right) < \Pi_o\left(c, \xi(\lambda^{peak})\right) \text{ for all } c < c^{\#}, \tag{37}$$

and since at $s = \xi(\lambda^c)$, $\lambda_o^* = \lambda_u^* = \bar{\lambda}^c$ and $q_o^* < q_u^* = 1$, we also have

$$\Pi_u(c, \xi(\lambda^c)) > \Pi_o(c, \xi(\lambda^c)) \text{ for all } c < c^{\#}. \tag{38}$$

[32]The $c = 0$ argument of Π_o and Π_u in this sentence just stands for a sufficiently small c such that $\overline{\lambda}^c$ does not constrain the profit maximizing choices with the relavant values of s.

Therefore, for any $c \in \left(0, c^{\#}\right)$, inequalities (37) and (38), and the continuity of Π_o and Π_u, imply via the intermediate value theorem that there exists some $\hat{s}\left(c\right)$ between $\xi\left(\lambda^c\right)$ and $\xi(\lambda^{peak})$ such that

$$\Pi_u\left(c, \hat{s}\left(c\right)\right) = \Pi_o\left(c, \hat{s}\left(c\right)\right). \tag{39}$$

Then (32) implies that, for $c \in \left(0, c^{\#}\right)$, we have $\Pi_u(c, s) > \Pi_o(c, s)$ if $s < \hat{s}\left(c\right)$ and $\Pi_u(c, s) < \Pi_o(c, s)$ if $s > \hat{s}\left(c\right)$. Thus, $\hat{s}\left(c\right)$ is the boundary between Regions II and III in Figure 11, for $c \in \left(0, c^{\#}\right)$ as well.

Finally, to verify that $\hat{s}$ is continuously increasing in c over $\left(0, c^{\#}\right)$, pick any c' and c'' with $0 \leq c' < c'' \leq c^{\#}$. The fact that (31) and (39) hold for $c = c'$ implies $\Pi_u\left(c'', \hat{s}\left(c'\right)\right) > \Pi_o\left(c'', \hat{s}\left(c'\right)\right)$. This inequality and (32) imply

$$\Pi_u\left(c'', s\right) > \Pi_o\left(c'', s\right) \text{ for all } s \leq \hat{s}\left(c'\right).$$

Thus, for (39) to hold at $c = c''$, it must be that $\hat{s}\left(c''\right) > \hat{s}\left(c'\right)$, as claimed. Finally, the continuity of $\hat{s}\left(c\right)$ follows from that of Π_o and Π_u for (c, s), which satisfies (30). ■

7.4 Proof for the welfare discussion

Claim 5. *(i) For any Λ, there is $\lambda > \Lambda$ such that $U(\lambda) < R_o'(\lambda)$. (ii) There is $\widetilde{\lambda} > \overline{\lambda}$ such that $U(\lambda) \geq R_o'(\lambda)$ for $\lambda \leq \widetilde{\lambda}$, and $U(\lambda) < R_o'(\lambda)$ at least over some interval just above $\widetilde{\lambda}$.*

Proof of Claim 5: Obviously, $R_o(\lambda)$ is the residual surplus not received by the bidders,

$$R_o(\lambda) = T(\lambda) - \lambda U(\lambda).$$

Observe that $R_o(\lambda) \to T(\lambda)$ as $\lambda \to \infty$, since both the expected maximal bid and the expected maximal value approach 1.

Part (i). If there is Λ such that $U(\lambda) > R_o'(\lambda)$ for all $\lambda \geq \Lambda$, then, by (21), for all such λ, $T(\lambda) - R_o(\lambda) > T(\Lambda) - R_o(\Lambda) > 0$, which contradicts the fact that $R_o(\lambda) \to T(\lambda)$ as $\lambda \to \infty$.

Part (ii). By (21),

$$R_o'(\lambda) = -\lambda U'(\lambda) = \lambda \int_0^1 e^{-(1-G(v))\lambda}[1 - G\left(v\right)]^2 dv \tag{40}$$

and

$$U(\lambda) - R_o'(\lambda) = U(\lambda) + \lambda U'(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1 - G\left(v\right)]\left[1 - \left(1 - G\left(v\right)\right)\lambda\right] dv. \tag{41}$$

Therefore,

$$\begin{aligned} R_o''(\lambda) &= -U'(\lambda) - \lambda U''(\lambda) = \int_0^1 e^{-(1-G(v))\lambda}[1-G(v)]^2 dv \\ &\quad -\lambda \int_0^1 e^{-(1-G(v))\lambda}[1-G(v)]^3 dv \\ &= \int_0^1 e^{-(1-G(v))\lambda}[1-G(v)]^2 \left[1-(1-G(v))\lambda\right] dv. \end{aligned} \tag{42}$$

Recall that $R_o'(\lambda)$ is single-peaked and let $\overline{\lambda}$ denote the argument of the peak. Thus, $R_o''(\overline{\lambda}) = 0$, and it follows from (42) that there must be an x such that $(1-G(x))\,\overline{\lambda} = 1$. Hence, the integrand on the right-hand side of (42) is positive for $v > x$ and negative for $v < x$. Therefore,

$$\begin{aligned} 0 &= R_o''(\overline{\lambda}) < \int_0^x e^{-(1-G(v))\overline{\lambda}}[1-G(x)][1-G(v)]\left[1-\left(1-G(v))\overline{\lambda}\right)\right] dv \\ &\quad + \int_x^1 e^{-(1-G(v))\overline{\lambda}}[1-G(x)][1-G(v)]\left[1-\left(1-G(v))\overline{\lambda}\right)\right] dv \\ &= [1-G(x)]\int_0^1 e^{-(1-G(v))\overline{\lambda}}[1-G(v)]\left[1-\left(1-G(v))\overline{\lambda}\right)\right] dv \\ &= [1-G(x)][U(\overline{\lambda}) - R_o'(\overline{\lambda})]. \end{aligned}$$

The first inequality follows from $1-G(x) < 1-G(v)$ for the range $v < x$ where the integrand is negative, and from $1-G(x) > 1-G(v)$ for the range $v > x$ where the integrand is positive; the last equality follows from (41). Therefore, $U(\overline{\lambda}) > R_o'(\overline{\lambda})$. Since U is decreasing and R_o' is increasing for $\lambda < \overline{\lambda}$, it follows that $U(\lambda) > R_o'(\lambda)$ for all $\lambda \leq \overline{\lambda}$. This and Part (i) imply that U and R_o' first intersect at some $\widetilde{\lambda} > \overline{\lambda}$. ■

References

Ashenfelter, O. and Graddy, K. (2003). Auctions and the price of art. *Journal of Economic Literature*, 41(3):763–786.

Boone, A. L. and Mulherin, J. H. (2007). How are firms sold? *The Journal of Finance*, 62(2):847–875.

Boone, A. L. and Mulherin, J. H. (2009). Is there one best way to sell a company? auctions versus negotiations and controlled sales. *Journal of Applied Corporate Finance*, 21(3):28–37.

Bulow, J. and Klemperer, P. (1996). Auctions versus negotiations. *The American Economic Review*, 86(1):180–194.

Burdett, K. and Judd, K. L. (1983). Equilibrium price dispersion. *Econometrica*, 51(4):955.

Crémer, J., Spiegel, Y., and Zheng, C. Z. (2007). Optimal search auctions. *Journal of Economic Theory*, 134(1):226–248.

Gentry, M. and Stroup, C. (2019). Entry and competition in takeover auctions. *Journal of Financial Economics*, 132(2):298–324.

Kohlberg, E. and Mertens, J.-F. (1986). On the strategic stability of equilibria. *Econometrica*, 54(5):1003–1037.

Krishna, V. (2009). *Auction theory.* Academic press.

Lauermann, S. and Wolinsky, A. (2017). Bidder solicitation, adverse selection, and the failure of competition. *American Economic Review*, 107(6):1399–1429.

Lauermann, S. and Wolinsky, A. (2022). A common-value auction with state-dependent participation. *Theoretical Economics*, 17(2):841–881.

Levin, D. and Smith, J. L. (1994). Equilibrium in auctions with entry. *American Economic Review*, 84(3):585–599.

Liu, Q., Mierendorff, K., Shi, X., and Zhong, W. (2019). Auctions with limited commitment. *American Economic Review*, 109(3):876–910.

McAfee, P. R. and McMillan, J. (1987). Auctions with entry. *Economics Letters*, 23(4):343–347.

McAfee, P. R. and Vincent, D. (1997). Sequentially optimal auctions. *Games and Economic Behavior*, 18(2):246–276.

Milgrom, P. R. (1979). A convergence theorem for competitive bidding with differential information. *Econometrica*, 47(3):679–688.

Milgrom, P. R. (1987). Auction theory, in ". In *Advances in Economic Theory: Fifth World Congress"(T. Bewley Ed.)*, volume 1, page 32. Cambridge Univ. Press, Cambridge.

Milgrom, P. R. (2004). *Putting auction theory to work*. Cambridge University Press.

Murto, P. and Välimäki, J. (2025). Affiliated common value auctions with costly entry. *Review of Economic Studies*. Forthcoming.

Myerson, R. B. (1998). Population uncertainty and poisson games. *International Journal of Game Theory*, 27(3):375–392.

Samuelson, W. F. (1985). Competitive bidding with entry costs. *Economics letters*, 17(1-2):53–57.

Subramanian, G. (2011). *Dealmaking: The new strategy of negotiauctions*. WW Norton & Company.

Szech, N. (2011). Optimal advertising of auctions. *Journal of Economic Theory*, 146(6):2596–2607.

Wilson, R. (1977). A bidding model of perfect competition. *The Review of Economic Studies*, 44(3):511–518.

Ye, L. (2007). Indicative bidding and a theory of two-stage auctions. *Games and Economic Behavior*, 58(1):181–207.