BARGAINING FOUNDATIONS FOR THE OUTSIDE OPTION PRINCIPLE

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ABSTRACT. We study a bargaining game in which a seller can trade with one of two buyers, who have values h and l ($h > l$). The outside option principle (OOP) predicts that the seller trades with the high-value buyer with probability converging to 1 at a price converging to $\max(h/2, l)$ as players become patient. While this prediction is supported by the Markov perfect equilibrium (MPE), a wide range of trading outcomes may emerge in subgame perfect equilibria (SPEs): in the patient limit, the seller can obtain any price in the interval $[h/2, h]$ (and no other); moreover, allocative inefficiency and costly delay are possible. We propose equilibrium refinements less restrictive than Markov behavior that guarantee trading outcomes consistent with the OOP. One refinement requires that a buyer's relative probability of trade does not increase dramatically following a failed negotiation with that buyer. Another refinement posits that the seller does not approach a buyer hoping that negotiations fail. SPEs satisfying both refinements conform with the OOP (but need not be MPEs). Our benchmark model features strategic matching by the seller. We provide a parallel analysis for the random matching protocol. Under random matching, prices in SPEs may also rise above and fall below l, but have a narrower range. A refinement particular to this protocol that restores the OOP requires that a random mismatch should not impact the seller excessively.

1. INTRODUCTION

The outside option principle asserts that in Rubinstein (1982) style bilateral bargaining, an outside option leads to a departure from the usual division only when it is binding: if it improves a player's payoff, then the payoff must be equal to the value of the outside option. This principle is intuitively appealing, widely applied, and regarded as an important insight of non-cooperative bargaining theory. However, when the outside option is endogenous, subgame perfect equilibria (SPEs) in standard bargaining models do not all conform with the principle. In this paper, we map the range of SPE outcomes in two such models and explore equilibrium refinements that reduce the range of multiplicity, with the goal of reinstating the outside option principle as the unique asymptotic prediction of refined SPEs.

We investigate the simplest version of the outside option principle in which the value of the outside option is *endogenously* determined via bilateral bargaining between a "seller" who has one

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indivisible good and two potential trading partners, "buyer h" and "buyer l," who value the good at h and l, respectively ($h > l > 0$).[1] In every round (before trade has occurred), the seller chooses to bargain with one of the two buyers, and nature randomly selects (with equal probability) the seller or the chosen buyer to propose a price. If the opponent accepts the proposal, trade takes place at the proposed price, and the game ends. If the proposal is rejected, the process is repeated in the next round. The three players have the same discount factor δ.

We begin the analysis of the game by identifying its *Markov perfect equilibria* (MPEs)—SPEs in which the seller's and buyers' actions in every round depend only on moves by nature and player actions in that round. A simple result in Manea (2018) shows that all MPEs induce the same distribution over trading outcomes, and in every MPE trade occurs without delay. In "the" MPE,[2] buyer l's payoff converges to 0 for $\delta \to 1$, and hence the option of trading with buyer l has an endogenous asymptotic value of l for the seller. If $l \leq h/2$, then the outside option is not sufficiently valuable to improve the seller's bargaining position with buyer h, and the seller trades exclusively with buyer h at expected price $h/2$ in the MPE. If $l > h/2$, then for high δ the outside option is binding, and the seller exercises it with probability converging to 0 as δ goes to 1.

Thus, in the MPE, the seller trades with buyer h with probability 1, or probability converging to 1, at a price converging to $\max(h/2, l)$ as $\delta \to 1$. We refer to this prediction as the *outside option principle* (OOP). This intuitive prediction emerges robustly from this bargaining protocol and a range of others (including the random matching protocol we discuss later), *albeit restricting attention to MPEs*.

What can we say about the potentially wider set of non-Markov SPEs in our bargaining model? We find that SPEs are very permissive.[3] Should we then simply *confine* attention to MPEs? This seems unappealing without further theoretical justification. Indeed, non-Markov strategies are constantly invoked both in theory (e.g., folk theorems) and applications of dynamic games (e.g., industrial organization, international trade).

We show that the primary restriction of SPEs on trading outcomes is that the seller should obtain an expected price of at least $h/2$. In particular, when $l > h/2$, there exist SPEs that yield an expected price lower than the seller's "outside option" l—as low as $h/2$—and allocate the good to either buyer. This point was also made by Rubinstein and Wolinsky (1990) in a setting with

[1]In this market, even if trade with the high-value buyer h is presumptive, the seller has the *option* to trade with buyer l, and the value of this option is determined in equilibrium. By contrast, in the early treatment of Binmore (1985), Binmore, Rubinstein and Wolinsky (1986), Sutton (1986), and Binmore, Shaked and Sutton (1989), outside options were assumed to have *exogenous* values. In a typical model, two players are bargaining, but one can end negotiations and exercise an outside option, which results in an exogenous payoff. Shaked and Sutton (1984) and Rubinstein and Wolinsky (1990) analyze models in which a seller's outside option is endogenously derived in equilibrium from bargaining with other buyers, but focus on the case with homogeneous buyers.

[2]Since MPEs are outcome equivalent, we sometimes drop the qualifier "outcomes" for brevity when referencing unique MPE outcomes.

[3]Herrero (1985) establishes ample equilibrium multiplicity in a version of Rubinstein's (1982) bargaining game in which several symmetric players bargain over how to divide a "dollar." In her model, players rotate in proposing divisions, and a player's proposal is implemented only if all others accept it. Herrero's equilibrium construction does not have a formal analog in our setting with asymmetric players and bilateral agreements.

$h = l$.[4] We provide a related construction here. In our *semi-Markov equilibria*, like in the MPE, the seller randomizes between bargaining with the two buyers and there is immediate agreement in every round. Unlike in the MPE, the seller's randomization between the buyers in each round depends on which buyer was approached in the previous round.

The MPE belongs to the class of semi-Markov equilibria, but there are equilibria in this class that are more favorable to buyers and less favorable to the seller than the MPE. Any price in the interval $[h/2, l]$ is asymptotically attainable in semi-Markov equilibria as $\delta \to 1$. Prices below the OOP prediction arise due to extreme *interdependence* in the seller's sequential choice of bargaining partner: for high δ, if the seller approaches a buyer in a round and negotiations fail, she approaches the same buyer in the next round with probability close to 1. These equilibrium expectations reduce competition between buyers and hurt the seller. An interesting feature of these equilibria is that in every round except the first, there is a buyer whom the seller approaches with probability close to 0, but if that buyer has a disagreement with the seller, he is almost sure to trade in the next round.

Furthermore, we construct SPEs *with threat of delay* in which the price is above l, and may be arbitrarily close to h. In this class of equilibria, buyer h accepts an unattractive offer from the seller because rejecting it results in delay, during which the seller unsuccessfully bargains with buyer l before returning to buyer h. In the delay phase, the seller's payoffs *increase* after successive rounds of failed negotiations with buyer l. The threat of delaying trade with buyer h is credible because if the seller attempts to approach buyer h earlier, then play reverts to a less profitable equilibrium (e.g., the MPE).

As noted, the OOP is compelling and has been used in both theoretical and applied work. Is there a "minimal" set of equilibrium behaviors that characterize every possible divergence from the asymptotic predictions of the OOP? Remarkably, the two types of non-stationary behavior at the core of our non-Markov equilibrium constructions are in a sense comprehensive—ruling them out leads to sharp predictions consistent with the OOP.

Informed by our non-MPE equilibrium constructions, we are led to investigate two *refinements*. The first refinement requires that in every subgame starting at the beginning of a round t, buyer h's relative probability of trade (computed as the ratio of probabilities of trade for buyer h and buyer l in the subgame, where probabilities of future trade are discounted by the factor δ for every round of delay) does not increase dramatically—by a factor greater than some arbitrary $M \geq 1$—in round $t + 1$ in the event buyer h has a disagreement with the seller in round t.[5] For large M, this refinement rules out only extreme instances of non-stationarity in trading probabilities. The second refinement postulates that the seller should not approach a buyer hoping that negotiations fail: any potential increase in the seller's payoff following a disagreement should not be large enough to compensate her for the incurred delay.

[4]Some of our results do not cover the case $h = l$, which is well understood from that paper.

[5]An analogous condition is imposed for relative probabilities of trade with buyer l, but only for consistency, as it is not used in the proofs of our main results.

We find these refinements intuitively plausible, but see no need to be dogmatic on this score. In applications, one or both refinements might be plausible, and researchers can decide which ones seem reasonable: How often does the seller return to a disagreeing buyer? Is a disagreement good or bad news for the seller? From a more neutral perspective, one can simply view the refinements as fundamental structural properties of equilibria that deviate from the OOP predictions.

Our main result establishes that SPEs for discount factors δ converging to 1 that satisfy both refinements conform with the OOP: as $\delta \to 1$, the seller trades with buyer h with limit *discounted* probability 1 at a limit price of $p^* := \max(h/2, l)$. Semi-Markov equilibria with asymptotic payoffs below l satisfy the second refinement, and SPEs with threat of delay satisfy the first, proving that both refinements are needed for the conclusion. We show that the first refinement implies that the seller's asymptotic profit is *at least* p^*, but the second refinement *does not* by itself imply that the seller's profit is *at most* p^*.

Clearly, MPEs satisfy both refinements. Although the two refinements imply asymptotic MPE outcomes, there exist SPEs in the semi-Markov class that satisfy the refinements and are not MPEs. We also present an SPE that satisfies both refinements in which arbitrarily long delay is possible, but consistent with our result, delay occurs with low probability.

In our benchmark bargaining model, there is *strategic matching*: in every round the seller chooses which buyer she negotiates with. We conduct a parallel analysis of another common matching protocol—*random matching*—under which in every round the seller is randomly matched to negotiate with one of the two buyers, and negotiations within a match proceed according to the random-proposer protocol. Like in the benchmark model, all MPEs in the model with random matching are outcome equivalent, and limit outcomes as $\delta \to 1$ conform to the OOP predictions.

Rubinstein and Wolinsky (1990) contrast SPE outcomes under the two matching protocols in the case $h = l$. They show that under strategic matching, asymptotic SPE seller payoffs cover the wide range $[l/2, l]$. Surprisingly, they find that under random matching, all SPEs are outcome equivalent to the MPE, generating an asymptotic payoff of l for the seller and 0 for either buyer. Our analysis demonstrates that the sharp variation in the predictive power of SPEs in the two matching models is special to the edge case $h = l$.

Considering the case $h > l$ reveals more subtle differences between the two matching protocols. When the outside option is binding ($l \geq h/2$), both protocols generate the same high asymptotic SPE payoff h for the seller, but produce starkly different predictions at the lower end. We prove that under random matching, $h/4 + l/2$ is an asymptotic lower bound on the seller's payoff in all SPEs, so for $l > h/2$, the seller's asymptotic payoff cannot get as low as it does under strategic matching ($h/4 + l/2 > h/2$). When $l < h/2$, asymptotic seller profits are $h/2$ in all SPEs under random matching whereas they cover the interval $[h/2, h]$ in SPEs under strategic matching.

A reasonable conjecture is that the seller's lowest SPE payoff cannot decline as we increase h. Combined with Rubinstein and Wolinsky's uniqueness result for $h = l$, this conjecture would imply that the seller's asymptotic payoff cannot fall below l in the random matching model. Surprisingly, this conclusion turns out to be false: for any $h \in (l, 2l)$, we construct SPEs where the

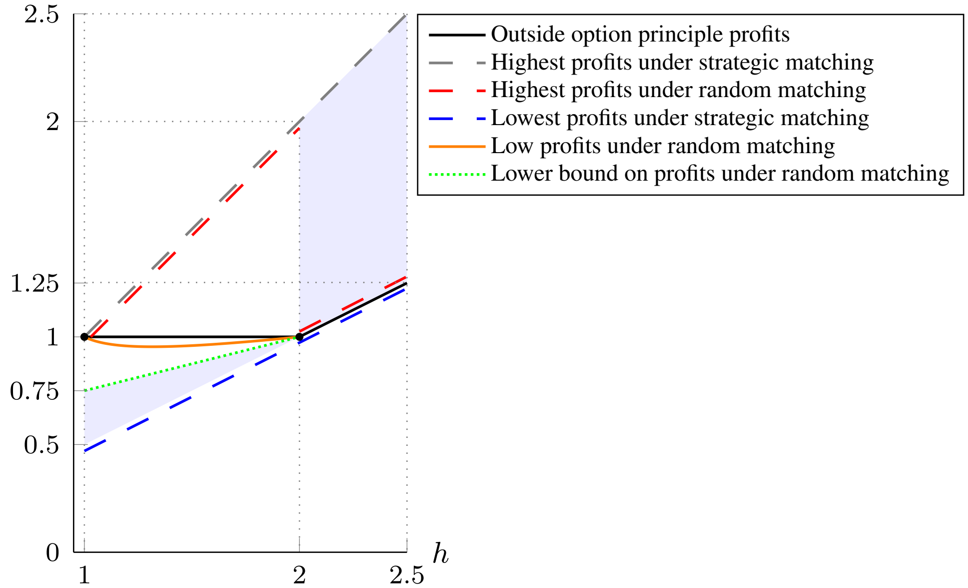

FIGURE 1. Extremal asymptotic equilibrium profits under the two matching protocols for $l = 1$ and $h \in [1, 2.5]$

seller gets an asymptotic payoff of $(4h + l - \sqrt{12h^2 - 12hl + l^2})/4$, which is smaller than l (but greater than $h/4+l/2$). As in the case of the strategic matching protocol, we formulate equilibrium refinements informed by our constructions of extremal SPEs under random matching that narrow down the set of SPE predictions in the direction of the OOP.

Figure 1 illustrates the seller's extremal (asymptotic) SPE profits under the two matching protocols relative to the OOP prediction when we normalize $l = 1$ and vary h in $[1, 2.5]$. For $h \in [1, 2]$, the upper envelope of seller profits attainable in SPEs is h in both models. However, for $h > 2$, profits under strategic matching sweep the interval $[h/2, h]$, whereas under random matching they are uniquely given by $h/2$. Distinctly from strategic matching, the upper envelope for random matching is neither increasing everywhere, nor continuous in h. For $h \in [1, 2]$, the lower envelope under strategic matching is $h/2$, while under random matching it lies between the line $h/4 + 1/2$ and the curve $(4h + 1 - \sqrt{12h^2 - 12h + 1})/4$. Despite this significant difference in lower envelopes, for $h \in (1, 2)$, there exist SPEs in either model in which seller profits are above and below the OOP prediction $\max(h/2, l) = 1$. As emphasized above, in the model with random matching, the lower envelope is not increasing in buyer h's value: the seller's lowest asymptotic SPE profit is $l = 1$ for both $h = 1$ and $h = 2$ (see the two enlarged points in the figure), and is strictly lower than 1 between these endpoints. The shaded areas in the figure represent seller profits that are attainable in SPEs under strategic matching, but are proven to not be attainable under random matching.

The rest of the paper is organized as follows. Section 2 describes the model with strategic matching and presents some preliminary results. In Sections 3 and 5, we construct the semi-Markov equilibria and the equilibria with threat of delay, respectively. Sections 4 and 6 introduce the first and the second refinement, respectively. Sections 7 through 9 analyze the model with random matching, and Section 10 provides concluding remarks.

2. The Model with Strategic Matching and Preliminary Results

A seller has an indivisible good that two buyers are interested in acquiring (or more generally, the seller needs to contract with one of two buyers for a service). The seller has 0 value for the good, and buyers have positive values $h > l$. We refer to the two buyers as buyer h and buyer l, respectively. There is complete information about each player's valuation for the good.

The seller bargains with buyers individually according to the following protocol. In every round $t = 1, 2, \ldots$, the seller strategically chooses (or "approaches") one buyer $k \in \{h, l\}$ to bargain with. Then, either the seller or buyer k is selected randomly (with probability $1/2$) to propose a price.[6] The player receiving the offer may (1) accept the offer, in which case the seller and buyer k trade the good at the proposed price, and the game ends; or (2) reject the offer, in which case the game proceeds to round $t + 1$, when the seller gets a new opportunity to approach one of the two buyers. Players have a common discount factor $\delta \in (0, 1)$ per round: if the seller trades with buyer k in round t at price p, then the seller's payoff is $\delta^{t-1}p$, and buyer k's payoff is $\delta^{t-1}(k - p)$ (the other buyer gets payoff 0). There is perfect information about all past bargaining rounds.

This is our benchmark model, and it entails *strategic matching* by the seller (in contrast to *random matching*, which we analyze in the second part of the paper). We study the subgame perfect equilibria (SPEs) of this model with a focus on equilibrium outcomes as players become patient or the time between bargaining rounds becomes short, which is captured by the limit $\delta \to 1$. To avoid measure theoretic distractions, we confine attention throughout to behavior strategies that have finite support after every history.

This non-cooperative game provides a natural framework for understanding how the value of outside options and the allocation of the good emerge endogenously in equilibrium via bilateral bargaining. We next review a result showing that when attention is restricted to Markov perfect equilibria, this framework produces the intuitive predictions of the OOP.

The Markov Perfect Equilibrium. In a *Markov perfect equilibrium (MPE)*, each of the three players behaves the same way in every bargaining round following any history of disagreements. That is, the seller's and buyers' actions in a given round depend only on moves by nature and player actions in that round, and are independent of calendar time and the history of disagreements in previous rounds. Proposition 1 in Manea (2018) implies that an MPE always exists and that all MPEs are outcome equivalent.[7] Let π_h^* and π_l^* denote the probabilities with which the seller approaches buyers h and l, respectively, in every round, and v_s^*, v_h^* and v_l^* denote the expected

[6] This bargaining protocol has been considered previously by Rubinstein and Wolinsky (1990), Abreu and Manea (2012a, 2024) and Manea (2018).

[7] The MPE is essentially unique: behavior is pinned down in all subgames except those in which the seller has just approached a buyer whom she is not supposed to approach under the MPE strategies.

payoffs of the seller, buyer h and buyer l, respectively, at the beginning of every round prior to which trade has not occurred in the MPE.

If $\pi_h^* = 1$, then standard computations imply that $v_s^* = v_h^* = h/2$ and $v_l^* = 0$. The seller should not find it profitable to deviate to bargaining with buyer l and offer him a price slightly below l (which buyer l would accept in equilibrium given that $v_l^* = 0$), while rejecting any offer from l. This requires that $1/2 \times l + 1/2 \times \delta h/2 \leq h/2$, or equivalently $l \leq h(1-\delta/2)$. Conversely, if $l \leq h(1-\delta/2)$, then there exists an MPE in which the seller trades exclusively with buyer h. In the Online Appendix, we explain why this condition precludes an MPE with $\pi_h^* < 1$. In this case, the outside option of trading with buyer l is *not binding* in the MPE: $v_s^* = h/2$.

In the remaining case $l > h(1-\delta/2)$, we have that $\pi_h^* < 1$. As both π_h^* and π_l^* are positive (the case $\pi_h^* = 0$ is easily ruled out), the seller's expected payoff conditional on approaching either buyer is v_s^*. When player k is selected to propose to player k', player k offers a *utility*[8] of $\delta v_{k'}^*$ to player k', and k' accepts the offer. Thus, payoffs and bargaining probabilities solve the following system of equations:

$$\begin{aligned} v_s^* &= \frac{1}{2}(k - \delta v_k^*) + \frac{1}{2}\delta v_s^*, \forall k \in \{h, l\} \\ v_k^* &= \pi_k^* \Big[\frac{1}{2}\delta v_k^* + \frac{1}{2}(k - \delta v_s^*)\Big], \forall k \in \{h, l\} \\ \pi_h^* + \pi_l^* &= 1. \end{aligned}$$

We provide analytical formulae for the solution in the Online Appendix. In this case, the outside option of trading with buyer l is *binding* in the MPE: $v_s^* > h/2$.

Note that if $l \leq h/2$, then $l \leq h(1-\delta/2)$ for all $\delta \in (0,1)$, so the outside option is not binding in the MPE for any δ. However, if $l > h/2$, then $l > h(1-\delta/2)$ for δ in the non-empty interval $(2(1-l/h), 1)$, which means that the outside option is binding in the MPE for sufficiently high δ.

The asymptotic properties of the MPE for $\delta \to 1$ in the case $l > h/2$ can be derived without solving the system of equations above explicitly. Since the seller has the option to approach buyer h and make an acceptable offer arbitrarily close to δv_h^*, and reject buyer h's offer, securing a continuation payoff of v_s^* in the MPE, we have that

$$v_s^* \geq \frac{1}{2}(h - \delta v_h^*) + \frac{1}{2}\delta v_s^*.$$

It follows that $(2-\delta)v_s^* + \delta v_h^* \geq h$, and hence $\liminf_{\delta\to 1}(v_s^* + v_h^*) \geq h$. However, the total sum of payoffs cannot exceed h, so $v_s^* + v_h^* \leq h$ for every δ. Then, $v_s^* + v_h^*$ converges to h as δ goes to 1. As $l < h$, this is possible only if π_l^* and v_l^* converge to 0 as δ goes to 1. Thus, trade is *asymptotically efficient* in the MPE. If for some $\delta \in (0,1)$, we have that $v_s^* > l$, then the seller does not trade with buyer l in the MPE, which implies that $\pi_h^* = 1$ and $v_s^* = h/2 < l$, a contradiction. Consequently, $v_s^* \leq l$ for all δ. The equilibrium constraint related to the seller approaching buyer l analogous to

[8] As in many bargaining models, it is often convenient to describe an offer in terms of the utility of the player receiving it rather than the price; every offer in the price space has an implicit utility for the receiver and vice versa.

the one for buyer h displayed above implies that

$$v_s^* \geq \frac{1}{2}(l - \delta v_l^*) + \frac{1}{2}\delta v_s^*,$$

and a similar argument leads to $\liminf_{\delta \to 1}(v_s^* + v_l^*) \geq l$. As $\lim_{\delta \to 1} v_l^* = 0$ and $v_s^* \leq l$ for all δ, it follows that $\lim_{\delta \to 1} v_s^* = l$. We conclude that when $l > h/2$, the seller trades with buyer h with limit probability 1 at limit price l in the MPE as δ goes to 1. Exercising the option of trading with buyer l with small probability is sufficient for the seller to drive up the price buyer h pays for the good from the equal split $h/2$ to the *endogenous* limit value l of the outside option of trading with buyer l as $\delta \to 1$. Therefore, MPEs of the bargaining game conform with the asymptotic predictions of the *outside option principle*.[9] Similar arguments deliver the same conclusion for MPEs under other bargaining protocols.[10]

The arguments above rely heavily on the premise of Markov behavior, which generates a stationary strategic environment. We will construct several SPEs involving non-Markov strategies where asymptotic outcomes diverge significantly from the OOP. We first establish that $h/2$ is a lower bound on the expected price in every SPE.

A Coarse Lower Bound on Profits. Our first result shows that in every SPE, the seller's expected payoff is at least $h/2$. Since $h/2$ is the seller's expected payoff in a setting in which she can bargain only with buyer h (with a proposer selected randomly in every round), the interpretation of this result is that giving the seller the opportunity to trade with buyer l as an alternative to trading with buyer h cannot hurt her. The proof of this and subsequent results can be found in the Appendix.

Proposition 1. *In every SPE for any discount factor* $\delta \in (0, 1)$, *the seller's expected payoff is at least* $h/2$.

The lower bound from Proposition 1 is tight. The discussion from the previous subsection shows that the MPE achieves the lower bound if $l \leq h(1 - \delta/2)$. In the next section, we construct a class of SPEs for the case $l > h(1 - \delta/2)$ that yield seller payoffs in the interval $[h/2, l/(2 - \delta)]$.

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[9]In particular, we do not have an equal split of surplus over and above the seller's outside option, which would amount to a price of $(h + l)/2$, as one might expect from the (cooperative) perspective of the Nash (1950) bargaining solution, where outside options define "disagreement payoffs." That conclusion holds for a different model, in which the seller does not *deliberately* exercise her outside option, but rather the outside option is *triggered randomly*, with an exogenous small probability, after every rejection (Binmore, Shaked and Sutton 1989).

[10]These include the random matching protocol of Rubinstein and Wolinsky (1990) that we analyze later in the paper, and the protocol of Elliott and Nava (2019) and Talamas (2019, 2020), under which in every round a player is randomly recognized as the proposer and strategically chooses a bargaining partner. The asymptotic conclusions do not apply, however, to the protocol studied by Chatterjee, Dutta, Ray and Sengupta (1993). Under their protocol, when a buyer rejects an offer from the seller, the buyer becomes the proposer in the next round; in particular, this buyer can hold up trade between the seller and the other buyer. Given this inherent constraint on the seller's ability to induce competition without delay, conditional on approaching buyer h, the seller cannot hope to get a payoff above $h/2$.

3. SEMI-MARKOV SPEs WITH ASYMPTOTIC PRICES IN $[h/2, l]$

Suppose that the outside option is binding for high δ, i.e., $l > h/2$. We construct a class of SPEs that minimally extends the MPE from a one-state to a two-state system. The construction places no constraint on the seller's selection of bargaining partner in the first round. Behavior after the first round belongs to either "state h" or "state l," and is characterized by corresponding probabilities (π_h, π_l). In state $k \in \{h, l\}$, the seller approaches buyer k with probability π_k, and the other buyer with complementary probability $1 - \pi_k$. In every state, the seller is indifferent between the two buyers, and trades with probability 1 conditional on approaching either buyer. Following *any* disagreement with buyer k in the first round or in either state, play transitions to state k.

The construction will ensure that the seller has the same expected payoff v_s in both states. Let v_k denote buyer k's expected payoff in state k. In each state as well as in the first round, strategies prescribe that every player k offers $\delta v_{k'}$ to player k', and player k' accepts offers greater than or equal to $\delta v_{k'}$, while rejecting smaller offers. Thus, the following payoff equations must hold for $k \in \{h, l\}$:

$$\begin{aligned} v_s &= \frac{1}{2}(k - \delta v_k) + \frac{1}{2}\delta v_s \\ v_k &= \pi_k \Big[\frac{1}{2}\delta v_k + \frac{1}{2}(k - \delta v_s)\Big], \end{aligned}$$

where the equation for v_k reflects the assumption that in state k the seller approaches the other buyer and trades with him with probability $1 - \pi_k$. The equations for v_s with $k \in \{h, l\}$ implicitly capture the indifference condition $h - \delta v_h = l - \delta v_l$ required for the seller's randomization between the two buyers in each state. Fixing the value of v_s, we can solve for v_k and π_k for $k \in \{h, l\}$:

$$\begin{aligned} v_k &= \frac{k - (2-\delta)v_s}{\delta} \\ \pi_k &= \frac{k - (2-\delta)v_s}{\delta(k - v_s)}. \end{aligned}$$

We need to check that $\pi_k \in [0, 1]$ for $k \in \{h, l\}$. Noting that for a fixed parameter v_s, the expression for π_k is strictly increasing in k, the binding conditions are $\pi_l \geq 0$ and $\pi_h \leq 1$, which are equivalent to

$$\begin{gathered} l - (2-\delta)v_s \geq 0 \\ h - (2-\delta)v_s \leq \delta(h - v_s), \end{gathered}$$

respectively. These conditions boil down to

$$v_s \in \left[\frac{h}{2}, \frac{l}{2-\delta}\right]. \tag{1}$$

The range of possible v_s is non-empty if and only if $\delta \geq 2(1 - l/h)$. Given the assumption that $h/2 < l$, we have that $2(1 - l/h) < 1$. Thus, an SPE with the structure described here exists for a non-empty interval of discount factors $\delta \in [2(1 - l/h), 1)$.

For the range of v_s from (1), we have that $v_h, v_l \geq 0$, and the constructed strategies deliver the desired payoffs. It can be easily checked that no player has a profitable one-shot deviation because the equation for v_k implies that $\delta v_s + \delta v_k \leq k$ for $k \in \{h, l\}$. The one-shot deviation principle implies that the constructed strategies constitute an SPE.

For $\delta \geq 2(1 - l/h)$, the MPE is a special case of this construction corresponding to $\pi_h + \pi_l = 1$, which means that the seller approaches each buyer k with constant probability π_k in both states. An MPE is memoryless and, in effect, defined by a single probability π_h^* ($\pi_l^* = 1 - \pi_h^*$). In the more general class of constructed equilibria, trading outcomes in subgames depend on the buyer who was most recently approached by the seller, but on no further detail of the history. For this reason, we call them *semi-Markov equilibria*.

For fixed $\delta > 2(1 - l/h)$, π_h and π_l decrease strictly (and continuously) as we increase the parameter v_s in the feasible interval given by (1). They start above their corresponding MPE values at the left endpoint $v_s = h/2$ (in this case, $\pi_h = 1$ and $\pi_l > 0$) and end below at the right endpoint $v_s = l/(2 - \delta)$ (in this case, $\pi_h < \pi_h^*$ and $\pi_l = 0$). It follows that the MPE payoff v_s^* lies strictly within the feasible range.

When $\pi_h + \pi_l > 1$, semi-Markov equilibria favor buyers in the sense that v_h and v_l are greater and v_s is smaller than the corresponding payoffs in the MPE. When $\pi_h + \pi_l < 1$, the opposite is true. Of course, each individual buyer is comparatively favored or disfavored in a semi-Markov equilibrium relative to the MPE conditional upon being approached in the first round, but this is also true in an unconditional sense if in the first round the seller approaches the two buyers with the MPE probabilities.[11]

In the extreme case $v_s = h/2$, we have $v_h = h/2$ and $\pi_h = 1$. By Proposition 1, this family of semi-Markov SPEs yield the minimum seller payoff over all SPEs. If the seller chooses to bargain with buyer h in the first round with probability 1, then she continues to bargain with him regardless of the history of past disagreements with him. In equilibrium, the seller trades with buyer h in the first round at terms identical to those obtained in the unique SPE of a two-player bargaining game between the seller and buyer h. This equilibrium yields the maximum equilibrium payoff $h/2$ for buyer h. If instead the seller approaches buyer l in the first round with probability 1, we obtain an SPE that delivers expected payoff $l - h/2$ to buyer l, which is also the maximum asymptotic payoff buyer l can achieve in equilibrium.

At the other extreme, we have $v_s = l/(2 - \delta)$ and $\pi_l = 0$. For $\delta > 2(1 - l/h)$, the corresponding semi-Markov SPEs generate a greater payoff for the seller than the MPE, but the difference vanishes in the limit as δ goes to 1, where both payoffs converge to l.

[11]Buyers' payoffs in the construction depend on the (unconstrained) probabilities with which the seller randomizes between them in the first round. If the seller approaches buyer k with probability q in the first round, then buyer k's equilibrium payoff is

$$q\Big[\frac{1}{2}\delta v_k + \frac{1}{2}(k - \delta v_s)\Big].$$

Except for the special case of the MPE, $\pi_h + \pi_l \neq 1$, so v_k can match buyer k's actual equilibrium payoff for at most one k.

Fix any $v_s \in [h/2, l)$ and let $\delta \to 1$. Since the seller's asymptotic payoff under the MPE is $l > v_s$, for high enough δ, the corresponding semi-Markov equilibria favor buyers relative to the MPE. For any $k \in \{h, l\}$, π_k converges to 1, and v_k converges to $k - v_s$ as $\delta \to 1$. If the seller approaches buyer l in the first round, then following a disagreement with buyer l, the seller does not threaten to switch to buyer h with significant probability, but instead continues to bargain with buyer l with probability close to 1. By contrast, in the MPE, buyer l trades with asymptotic probability 0 and obtains an asymptotic payoff of 0.[12] For buyer h, this effect is less stark: the seller returns to h following rejection with probability above the corresponding MPE probability ($\pi_h > \pi_h^*$), but both π_h and π_h^* converge to 1 as δ goes to 1. Nevertheless, the difference is consequential as v_h converges to $h - v_s$, while v_h^* converges to the smaller value $h - l$.

In the family of semi-Markov SPEs for a fixed $v_s \in [h/2, l)$, as $\delta \to 1$ the seller is unable to exploit effectively the competition between buyers because she is *loyal to a fault* to any buyer she engages with. Conditional on being approached by the seller, each buyer is almost certain to trade, and these expectations reduce competition and hurt the seller. We refer to any family of SPEs associated with some $v_s \in [h/2, l)$ and δ in a neighborhood of 1 as *buyer-favoring* semi-Markov SPEs.

Rubinstein and Wolinsky (1990) briefly consider the strategic matching protocol in their Proposition 4.ii (they call it "voluntary" matching) and construct a related family of equilibria for a setting with multiple buyers who have identical valuations.[13] In the Online Appendix, we extend their construction to our setting with two buyers who have different valuations.

Semi-Markov equilibria highlight a connection between each buyer's probability of trade and the buyer's bargaining power in subgames, which we explore further in the next section.

4. Refinement for Asymptotic Prices at Least l

In buyer-favoring semi-Markov equilibria, the seller oscillates between extreme loyalty (despite failure to trade) to each of the two buyers. For instance, if the seller approaches buyer l in round $t-1$ and trade does *not* occur, the seller approaches buyer h with low probability $1-\pi_l$ (close to 0 for high δ) in round t. However, conditional on buyer h being approached and trade not occurring in round t, the seller approaches buyer h again—and trades with him—with high probability π_h (close to 1 for high δ) in round $t+1$. The refinement below seeks to rule out this highly non-stationary equilibrium structure.

The refinement is parameterized by a constant $M \geq 1$, and is weaker for larger M. Proposition 2 and subsequent results are valid for any $M \geq 1$, but our interpretation of the refinement implicitly

[12]As we will see in the proof of forthcoming Proposition 2, if a buyer's maximal asymptotic payoff is strictly positive, then the probability of trade along any corresponding sequence of equilibria (that attain the maximal asymptotic payoff) must converge to 1. In this sense, trading with asymptotic probability 1 is a necessary condition for a buyer to achieve a positive asymptotic payoff.

[13]Rubinstein and Wolinsky discuss results for this setting due to Binmore (1985) and Shaked (working paper from 1987 published in 1994), which show that equilibrium multiplicity and deviations from the OOP prediction depend on whether the buyers or the seller can reject an offer and respond with a counteroffer before a match dissolves.

presumes large M. For large M, the refinement requires that failure to reach an agreement with buyer k *does not dramatically increase* the relative probability of eventual trade with buyer k. We state this requirement in terms of discounted probabilities of trade. A buyer's *discounted probability of trade* in a subgame starting at date t is given by $\sum_{t' \geq t} \delta^{t'-t} q_{t'}$, where $q_{t'}$ denotes the probability that the seller trades with the buyer at date t' in the subgame.

R1(M) (A buyer's relative probability of trade does not explode after a disagreement). *Let x_k denote the discounted probability of trade with buyer $k \in \{h, l\}$ in any subgame at the start of round t, and x'_k denote the discounted probability of trade with buyer $k \in \{h, l\}$ at the start of round $t+1$ after any disagreement with buyer h (on or off the equilibrium path) in round t. If $x_l > 0$, then*[14]

$$M \frac{x_h}{x_l} \geq \frac{x'_h}{x'_l}. \tag{2}$$

An analogous condition holds for disagreements with buyer l (with the corresponding ratios inverted).

The MPE satisfies R1(M) for every $M \geq 1$, since under the MPE, the (discounted) probability of trade with a given buyer is history-independent and constant. By contrast, any family of buyer-favoring semi-Markov equilibria delivering a fixed payoff smaller than l to the seller violates R1(M) for sufficiently high δ. To see this, consider a subgame where play starts in state l, so the seller approaches buyer l with probability π_l in the first round. The relative probability of trade with buyer h in this subgame is $(1 - \pi_l)/\pi_l$. However, if the seller approaches buyer h in the first round of the subgame, and the meeting results in disagreement, then play transitions to state h, in which the relative probability of trade with buyer h is $\pi_h/(1 - \pi_h)$. Since $(1 - \pi_l)/\pi_l$ goes to 0 and $\pi_h/(1 - \pi_h)$ goes to ∞ as $\delta \to 1$, R1(M) is violated for any fixed $M \geq 1$ when δ is close to 1.

It turns out that R1(M) implies an asymptotic lower bound of l for seller payoffs. The proof reveals that requiring R1(M) only for disagreements with buyer h is sufficient for this result, but we impose the refinement for both buyers for consistency.

Proposition 2. *Fix $M \geq 1$. In any family of SPEs for discount factors $\delta \in (0, 1)$ that satisfy R1(M), buyer l's payoff converges to 0 as $\delta \to 1$, and $\liminf_{\delta \to 1}$ of the seller's payoff is at least l.*

To sketch the proof, fix $M \geq 1$ and let $\bar{v}_l(\delta)$ and $\underline{v}_s(\delta)$ denote the supremum of buyer l's payoff and the infimum of the seller's payoff, respectively, in the game with discount factor δ under SPEs satisfying R1(M); define $\bar{v}_l = \limsup_{\delta \to 1} \bar{v}_l(\delta)$ and $\underline{v}_s = \liminf_{\delta \to 1} \underline{v}_s(\delta)$. Note that $\bar{v}_l = 0$ implies that $\underline{v}_s \geq l$ since one available strategy to the seller is to repeatedly approach buyer l and reject all offers from l until she is selected to make an offer, at which point she can offer an

[14]Inequality (2) is logically violated when $x_l > 0, x'_l = 0$ and $x'_h > 0$ (in this case, the left-hand side takes a finite value, while the right-hand side is interpreted to be infinity). Instances with $x'_l = x'_h = 0$ are assumed to satisfy (2). (However, there can be no such instance in any *SPE*: in the corresponding subgame, trade would happen with probability zero and the seller would get payoff zero, contradicting Proposition 1.)

acceptable price arbitrarily close to l as δ goes to 1. Hence, it is sufficient to argue that $\bar{v}_l = 0$. To this end, we prove that $\bar{v}_l > 0$ leads to a contradiction. Suppose that $\bar{v}_l > 0$, and consider a sequence of SPEs satisfying R1(M) for discount factors going to 1 in which buyer l's payoff converges to $\bar{v}_l$. The claims below apply to this sequence of SPEs. A key step in the argument shows that the discounted probability of trade with buyer l must converge to 1. This allows us to conclude that the seller's asymptotic payoff in the sequence of SPEs must be $\underline{v}_s = l - \bar{v}_l$. As the discounted probability of trade with buyer h converges to 0, we leverage R1(M) to argue that buyer h's probability of trade and payoff converge to 0 following his rejection of any seller offer. It follows that the seller can obtain a payoff arbitrarily close to h far enough along the sequence of SPEs if she approaches buyer h and is selected to make the offer. For this not to generate a profitable deviation for the seller, it must be that the seller's asymptotic payoff in the sequence of SPEs under consideration is h, contradicting the conclusion that it should be $\underline{v}_s = l - \bar{v}_l < l$.

Propositions 1 and 2 have the following immediate implication.

Corollary 1. *In any family of SPEs for discount factors $\delta \in (0, 1)$ that satisfy R1(M) for a fixed $M \geq 1$, $\liminf_{\delta \to 1}$ of the seller's payoff is bounded below by $\max(h/2, l)$.*

Recall that $\max(h/2, l)$ is the price predicted by the OOP. While R1(M) shields the seller from asymptotic profits below $\max(h/2, l)$, it does not preclude higher asymptotic profits, as the next example demonstrates.

5. SPEs with (Threat of) Delay

We showcase a class of SPEs satisfying R1(M) in which the seller obtains asymptotic payoffs above l, and arbitrarily close to h. In this class of equilibria, the seller is supposed to trade with buyer h in the first round, but if buyer h rejects the seller's equilibrium offer, then trade with buyer h is delayed for $T \geq 1$ rounds (we will discuss the flexibility in specifying T later), during which the seller bargains unsuccessfully with buyer l. By contrast, when the seller rejects buyer h's offer, trade with buyer h is expected to take place on the same terms in the next round.

Let v_s and v_h denote the expected payoffs of the seller and buyer h, respectively, in this construction. In the first round, the seller approaches buyer h with probability 1. If selected to propose, the seller offers a payoff $\delta^{T+1} v_h$ to buyer h, and buyer h accepts only offers greater than or equal to $\delta^{T+1} v_h$. Buyer h offers δv_s to the seller, and the seller accepts only offers greater than or equal to δv_s. If the seller deviates and chooses to bargain with buyer l in the first round, play enters the delay phase described below.

First-round rejections by the seller and buyer h are handled *asymmetrically*. If the seller rejects an offer, second-round play follows the strategies prescribed for the first round. However, if buyer h rejects an offer, the seller is supposed to bargain unsuccessfully with buyer l for T rounds before returning to buyer h. During this *delay phase*, as long as the seller has bargained only with buyer l, she continues approaching buyer l in each of the T rounds. In every match with buyer l in the delay phase, the proposer selected by nature offers price l, and the responder rejects the equilibrium offer; in the t-th encounter with buyer l, the seller accepts only offers higher than $\delta^{T-t+1}v_s$, while buyer l accepts only price offers lower than l. If the seller chooses to bargain with buyer h in one of the T rounds, then on the first occasion this happens play switches to MPE strategies. When the delay phase is completed (after T rounds of disagreement with buyer l), play continues according to first-round strategies.

For the proposal and acceptance rules under the prescribed strategies to be optimal, we are led to define (v_s, v_h) as the solution to the following system of equations:

$$v_s = \frac{1}{2}(h - \delta^{T+1}v_h) + \frac{1}{2}\delta v_s$$
$$v_h = \frac{1}{2}(h - \delta v_s) + \frac{1}{2}\delta^{T+1}v_h.$$

The system has a unique solution, with v_s given by

$$v_s = \frac{1-\delta^{T+1}}{2-\delta-\delta^{T+1}}h$$

and $v_h = h - v_s$.

For the constructed strategies to form an SPE, the seller and buyer l should not have incentives to make acceptable offers to each other during the T rounds of delay following a rejection by buyer h. Since the seller expects a discounted payoff of $\delta^T v_s$ in the first stage of the delay phase (and greater payoffs in later stages), and buyer l expects 0 payoff at all stages of this phase, a sufficient condition for these incentives is that $\delta^T v_s \geq l$. The seller should also not have an incentive to deviate to bargaining with buyer h at any stage of the delay phase. Such deviations trigger MPE play, which is not profitable for the seller if $\delta^T v_s \geq v_s^*$. The seller does not have an incentive to trade with buyer l if she approaches him in the first round if $\delta^T v_s \geq l$. Remaining incentives for one-shot deviations are immediately verified. Thus, the constructed strategies constitute an SPE if $\delta^T v_s \geq \max(v_s^*, l)$.[15]

Let $p^* = \max(h/2, l)$ denote the limit of v_s^* when δ goes to 1. For a fixed $T \geq 1$, L'Hospital's rule implies that v_s converges to $(T+1)/(T+2)h$ as $\delta \to 1$; the same is true about $\delta^T v_s$. Thus, the constructed strategies constitute an SPE for high δ if $(T+1)/(T+2)h > \max(p^*, l) = \max(h/2, l)$. As $(T+1)/(T+2)h > h/2$, the sufficient condition becomes $(T+1)/(T+2)h > l$. Since $l < h$, this condition is satisfied for T above a threshold. In the Appendix, we check that this SPE satisfies R1(M) for any $M \geq 1$.

The equilibrium construction above generates asymptotic seller payoffs $(T+1)/(T+2)h$ for integers T such that $(T+1)/(T+2)h > l$. In the Appendix, we adjust the construction to obtain SPEs for high δ in which the seller's payoff is any $\tilde{v}_s \in [p^*, h)$ (this implies the existence of a sequence of SPEs that also attains the asymptotic upper bound h for seller payoffs).

SPEs with asymptotic delay. Consider the SPE induced by the equilibrium constructed above in the subgame following a rejection by buyer h in the first round. In this SPE, trade with buyer h takes

[15]Note that this construction does not require $l > h/2$, so it is not necessarily the case that $v_s^* \leq l$.

place after T rounds of delay. For fixed T, this delay is not asymptotically costly ($\lim_{\delta\to 1} \delta^T = 1$). However, by making T a function of δ, we can obtain SPEs with arbitrarily long and asymptotically costly delay.[16] To achieve this, it is sufficient to specify a length of delay $T(\delta)$ for every discount factor δ such that $\lim_{\delta\to 1} \delta^{T(\delta)} < 1$ and $\lim_{\delta\to 1} \delta^{T(\delta)} v_s > p^*$. Fix $\varepsilon \in (0, 1 - p^*/h)$, and let $T(\delta)$ be the largest integer T such that $\delta^T > 1 - \varepsilon$. Then, $\lim_{\delta\to 1} T(\delta) = \infty$ and $\lim_{\delta\to 1} \delta^{T(\delta)} = 1 - \varepsilon$. Moreover, $\lim_{\delta\to 1} v_s = h$ and $\lim_{\delta\to 1} \delta^{T(\delta)} v_s = (1 - \varepsilon)h$. Since $\varepsilon < 1 - p^*/h$, we have that $\lim_{\delta\to 1} \delta^{T(\delta)} v_s > p^*$, as desired. In the resulting SPEs, delay in trading with buyer h costs an asymptotic share ε of the available surplus. By varying ε in the interval $(0, 1 - p^*/h)$, we generate SPEs in this family in which the seller's asymptotic payoff takes any value in the interval (p^*, h), while both buyers' asymptotic payoffs are 0.

SPEs with asymptotic allocative inefficiency and trade in every round. Consider again the SPE induced by the construction above in which the seller trades with buyer h after T rounds of delay for a fixed T. We can modify this SPE so that the seller trades with buyer l with some positive probability x independent of δ. In the new equilibrium, in the first round, the seller first approaches buyer l and offers him price l, which buyer l accepts with probability $2x$. If buyer l is the proposer, buyer l offers price 0, and the seller accepts only offers of $\delta^T v_s$ or better. If the seller offers price l or buyer l is the proposer in the first round, and agreement is not reached, play conforms to the SPE under consideration. If the seller offers a price different from l to buyer l or approaches buyer h in the first round, strategies follow MPE behavior. For the seller not to have an incentive to trigger MPE play in the first round for high δ, we choose x such that her asymptotic payoff under the constructed strategies is greater than her asymptotic MPE payoff, i.e., $xl + (1 - x)(T + 1)/(T + 2)h > p^*$, which holds for small x and large T.

6. Additional Refinement Implying the Outside Option Principle

In the equilibria from the previous section, the seller threatens to delay trade with buyer h. The threat is credible because during the delay phase, the seller's expected payoff constantly improves following disagreements with buyer l, while trying to cut the delay short by approaching buyer h is interpreted as a plan to switch to the MPE, which is less profitable for the seller. This motivates the following refinement, which requires that in equilibrium the seller's payoff does not increase significantly in any subgame following a failure to trade; specifically, the seller's profit may improve after a disagreement, but not enough to compensate her for the trading delay. When this condition is violated, the seller approaches a buyer in equilibrium *hoping that negotiations fail* (or being indifferent about their success).

[16] Haller and Holden (1990) and Fernandez and Glazer (1991) observed that significant delay is also possible under complete information in a two-player bargaining game if one player has the option to "strike" in each round. In markets that involve larger numbers of participants and allow for richer dynamics, disagreement may persist in some matches—even in the context of MPEs—due to the (endogenous) evolution of outside options (e.g., Manea (2011, 2017), Abreu and Manea (2012b), Elliott and Nava (2019), Talamas (2019, 2020), Dilme (2023)).

R2 (Disagreement is not beneficial to the seller). *If v_s is the seller's payoff in any subgame starting at the beginning of round t, and v'_s is the seller's payoff at the beginning of round $t+1$ following any disagreement in round t on or off the equilibrium path (v'_s may depend on play in round t), then $v_s > \delta v'_s$.*

The SPEs with threat of delay do not satisfy R2 because in every round of the delay phase, the seller's payoff is equal to her discounted payoff in the event of a disagreement with buyer l. We next show that R2, in conjunction with R1(M), is sufficient to deliver the OOP asymptotically. One cannot drop either refinement in the statement of this result. The family of buyer-favoring semi-Markov SPEs (Section 3) for a fixed seller payoff $v_s < l$ and $\delta \in (0,1)$ satisfies R2, but not R1(M) for any finite M. The family of SPEs with threat of delay (Section 5) satisfies R1(M) for any $M \geq 1$, but not R2. The result demonstrates that the types of non-stationary behavior highlighting the two main equilibrium constructions are in a way exhaustive—shutting them down via the two refinements guarantees the asymptotic OOP predictions.

Proposition 3. *In any family of SPEs for discount factors $\delta \in (0,1)$ that satisfy R1(M) for a fixed $M \geq 1$ and R2, the seller's payoff converges to $\max(h/2, l)$, buyer h's payoff converges to $\min(h/2, h-l)$, and the discounted probability of trade with buyer h converges to 1 as $\delta \to 1$.*

Whereas R1(M) implies that the asymptotic price is at least $\max(h/2, l)$, *R2 does not by itself imply that the asymptotic price is at most* $\max(h/2, l)$. Indeed, in the Appendix we construct a family of SPEs for the case $l > h/2$ that satisfy R2 and yield prices close to h for high δ (by Proposition 3, this family of SPEs violates R1(M) for any M). However, in the simpler case $l < h/2$, every SPE that satisfies R2 for a fixed δ is outcome equivalent to the corresponding MPEs and yields a seller payoff of $h/2 = \max(h/2, l)$.[17]

We now provide a summary of the proof of Proposition 3. For $k \in \{h, s\}$, let $\underline{v}_k(\delta)$ and $\bar{v}_k(\delta)$ denote the infimum and supremum payoff of player k, respectively, in all SPEs satisfying R1(M) and R2 for the game with discount factor $\delta \in (0,1)$, and define $\underline{v}_k = \liminf_{\delta \to 1} \underline{v}_k(\delta)$ and $\bar{v}_k = \limsup_{\delta \to 1} \bar{v}_k(\delta)$.

The main step of the proof establishes that $\underline{v}_h \geq \min(h/2, h-l)$. By Corollary 1, we have that $\underline{v}_s \geq \max(h/2, l)$. Then, $\bar{v}_s \leq h - \underline{v}_h$ and $\bar{v}_h \leq h - \underline{v}_s$ lead to $\underline{v}_s = \bar{v}_s = \max(h/2, l)$ and $\underline{v}_h = \bar{v}_h = \min(h/2, h-l)$. It follows that in any family of SPEs satisfying the two refinements, the seller's payoff converges to $\max(h/2, l)$ and buyer h's payoff converges to $\min(h/2, h-l)$ for $\delta \to 1$, which is possible only if the seller trades with buyer h with a limit discounted probability of 1.

[17]In the proof of Proposition 3, we show that the seller's payoff does not exceed l in any SPE satisfying R2 in which the seller bargains with buyer l with positive probability in the first round. By Proposition 1, the seller's payoff is at least $h/2$ in every SPE. If $l < h/2$, it follows that in every SPE satisfying R2, the seller must bargain exclusively with buyer h in all subgames. Standard two-player bargaining arguments then imply that every SPE that satisfies R2 is outcome equivalent to the MPEs (strategies are not pinned down off the equilibrium path in subgames in which the seller approaches buyer l and need not be stationary in such SPEs).

To prove that $\underline{v}_h \geq \min(h/2, h-l)$, it is useful to consider a sequence of discount factors $\delta \to 1$ such that $\underline{v}_h(\delta)$ converges to $\underline{v}_h$ as $\delta \to 1$, and additionally one of the following conditions holds for all δ in the sequence: (1) $\underline{v}_h(\delta)$ is attained as the limit of buyer h's payoffs in a sequence of SPEs for discount factor δ that satisfy the two refinements, in which the seller approaches buyer h with probability 1 in the first round of the game; (2) analogous statement to condition (1) with "probability 1" replaced by "probability less than 1."

In the first case, standard bargaining inequalities imply that $\underline{v}_h(\delta) \geq h/2$ for δ in the sequence. As $\lim_{\delta\to 1} \underline{v}_h(\delta) = \underline{v}_h$, it follows that $\underline{v}_h \geq h/2 \geq \min(h/2, h-l)$.

In the (more interesting) second case, there exists a sequence of δ's converging to 1 and associated SPEs σ^δ satisfying the two refinements under which the seller approaches buyer l with positive probability in the first round of the game, and buyer h's payoff converges to $\underline{v}_h$ as $\delta \to 1$.

Let $v_s(\delta)$ denote the seller's payoff under σ^δ. We argue via R2 that $v_s(\delta) \leq l$. Since under σ^δ the seller approaches buyer l in the first round with positive probability, the seller's expected payoff conditional on approaching l should be $v_s(\delta)$. However, no equilibrium agreement with buyer l yields a payoff greater than l for the seller, so $v_s(\delta) \leq ql + (1-q)\delta v'_s(\delta)$, where q denotes the conditional probability of an agreement with buyer l in the first round, and $v'_s(\delta)$ denotes the expected value of the seller's payoff after an (possibly stochastic) equilibrium disagreement with buyer l. As σ^δ satisfies R2, we have that $\delta v'_s(\delta) < v_s(\delta)$. It is then easy to show that $v_s(\delta) \leq l$. Two conclusions follow. Since $v_s(\delta) \geq h/2$ (Proposition 1), this case can arise only if $l \geq h/2$. Since $\liminf_{\delta\to 1} v_s(\delta) \geq l$ (Proposition 2), $\lim_{\delta\to 1} v_s(\delta) = l$.

Let $x_h(\delta)$ and $x_l(\delta)$ be the discounted probability of trade with buyers h and l, respectively, under σ^δ. To complete the proof, we argue that $x_h(\delta) \to 1$ as $\delta \to 1$. If $x_h(\delta) \to 1$, then $\lim_{\delta\to 1} v_s(\delta) = l$ implies that buyer h's payoff under σ^δ converges to $h-l$ as $\delta \to 1$, and thus $\underline{v}_h = h-l$. Finally, $h-l = \min(h/2, h-l)$ because $l \geq h/2$ in this case.

We are left to show that $x_h(\delta) \to 1$ as $\delta \to 1$. To obtain a contradiction, suppose that $x_h(\delta)$ does not converge to 1 for $\delta \to 1$. Then, there exists a subsequence of δ going to 1 for which $x_h(\delta)$ and $x_l(\delta)$ converge to limits $x_h(1) < 1$ and $x_l(1)$, respectively.

If $x_l(1) > 0$, we obtain a contradiction via R1(M) as follows. The fact that σ^δ satisfies R1(M) implies that the discounted probability of trade with buyer h under σ^δ following a failed negotiation with him is asymptotically bounded above by some $q < 1$ as δ goes to 1. Furthermore, buyer h's payoff in any agreement with the seller is asymptotically bounded above by $h-l$ because $\underline{v}_s \geq l$. Hence, buyer h's asymptotic payoff under σ^δ following any first-round rejection by buyer h is bounded above by $q(h-l)$. Consequently, by approaching buyer h in the first round and offering him a price slightly below $h - q(h-l)$ (which the buyer accepts under σ^δ for sufficiently high δ), the seller can obtain an asymptotic payoff arbitrarily close to $1/2(h - q(h-l)) + 1/2\underline{v}_s$. Since $\underline{v}_s \geq l$, this payoff is greater than l, contradicting the conclusion that $\lim_{\delta\to 1} v_s(\delta) = l$.

If instead $x_l(1) = 0$, then we show that $x_h(1) < 1$ implies that $\lim_{\delta\to 1} v_s(\delta) < l$, which again contradicts $\lim_{\delta\to 1} v_s(\delta) = l$. This part of the argument is by far the most involved and entails a delicate classification of the different ways that the seller may ultimately trade with either type of buyer after a history of failed transactions. (Note that our refinements do not guarantee immediate trade—indeed, later in this section we discuss an SPE that satisfies both refinements and exhibits delay in every bargaining round with positive probability.) In particular, it is helpful to anchor buyer h's trades to the last round the seller has unsuccessfully bargained with buyer l, so that we can use the following implication of R2. If the seller approaches buyer l with positive probability at the beginning of a subgame under σ^δ, then the seller's expected payoff following any sequence of τ consecutive disagreements with buyer h in that subgame does not exceed l/δ^τ. The recursive use of R2 is key to obtaining payoff bounds that yield a contradiction in this case. We refer the reader to the proof for details.

The formulation of R2 is directly motivated by the SPEs with threat of delay, and R2 clearly rules out the possibility that disagreement and delay occur with probability 1 in any round. However, R2 also rules out potential benefits to the seller following a disagreement with a buyer who deviates from his own equilibrium strategy. Our proof of Proposition 3 involves applications of R2 for disagreements both on and off the equilibrium path. Footnote 35 highlights a step in the proof of Proposition 3 where implications of R2 for disagreements possibly off the equilibrium path are invoked.

We conclude this section with two remarks. The first emphasizes that our refinements are not so strong as to exclude non-Markov SPEs, and are even consistent with trading delay. The second discusses alternatives to R2.

Non-Markov SPEs satisfying R1(M) and R2. Let $\Sigma(\delta)$ be the set of SPEs satisfying R1(M) for some $M \geq 1$ and R2. When $l < h/2$, footnote 17 implies that $\Sigma(\delta)$ consists only of SPEs that are outcome equivalent to the MPEs. In the more interesting case $l > h/2$, the set $\Sigma(\delta)$ contains two rich classes of non-Markov SPEs when $\delta > 2(1 - l/h)$: the first is a subset of semi-Markov SPEs, and the second consists of SPEs in which there is positive probability of delay. In Section 3, we showed that for $\delta > 2(1 - l/h)$, there exists a range of semi-Markov SPEs with $\pi_h + \pi_l < 1$ yielding the continuum of payoffs $(v_s^*, l/(2 - \delta)]$ for the seller. Every such SPE satisfies R1(M) for any $M \geq 1$ as long as the seller approaches each buyer k with probability at least π_k in the first round. To see this, note that under these SPEs, there is immediate agreement in every round (so a buyer's discounted probability of trade in a subgame is simply the probability with which the seller approaches that buyer in the first stage of the subgame), and following any disagreement with either buyer, the probability of trade with that buyer either remains the same or decreases. Furthermore, all semi-Markov SPEs satisfy R2 because in these SPEs the seller's payoff is constant across subgames. It follows that $\Sigma(\delta)$ contains SPEs covering the entire interval $(v_s^*, l/(2 - \delta)]$ of seller payoffs when $\delta > 2(1 - l/h)$. In the Appendix, we demonstrate by construction that in this case $\Sigma(\delta)$ also contains SPEs that exhibit *arbitrarily long delay* with positive probability. However, the discounted probability of trade with buyer h in this construction converges to 1 as $\delta \to 1$, a necessary feature according to Proposition 3.

Alternatives to R2. As mentioned above, R1(M) and R2 do not imply that agreement is reached without delay in every round. In fact, the property of immediate agreement in every subgame (starting at the beginning of a round) has powerful implications. Indeed, we can prove a version of Proposition 3 that replaces R2 with the immediate agreement property. We do not highlight this characterization as we consider R2 to be a more primitive refinement.

The delay phase in our equilibrium construction from the previous section seems not to be "renegotiation proof." There are several notions of renegotiation proofness, but perhaps the most common starting point is the requirement that equilibrium continuation values should not be Pareto ranked (e.g., Bernheim and Ray 1989, Farrell and Maskin 1989). In this basic sense, the equilibria with threat of delay are not renegotiation proof since any "scheduled" disagreement with buyer l Pareto improves players' payoffs. Note that R2 concerns only payoff comparisons for the seller, and is neither weaker nor stronger than renegotiation proofness in its basic form. Whether a notion of renegotiation proofness suitable for our bargaining environment can replace R2 is a question worth pursuing in future research.

7. The Model with Random Matching

Although the strategic matching protocol seems natural in the context of a model with a single seller and multiple asymmetric buyers, it is worthwhile to investigate whether equilibrium multiplicity issues arise under other protocols, and whether intuitive refinements narrow down their "perfect" equilibrium predictions. An alternative protocol that has been widely considered in the literature is *random matching*.[18] In our setting, the random matching protocol entails that in every round the seller and buyer k are matched with *exogenous* probability p_k, where $p_h + p_l = 1$. As in the original model, when the seller is matched with buyer k, either player gets the opportunity to make an offer with probability $1/2$. We consider the symmetric version of this protocol with $p_h = p_l = 1/2$.

We first observe that in the model with random matching all MPEs are outcome equivalent, and that the MPEs in this model also conform asymptotically with the OOP when δ goes to 1.[19] In particular, when $l < h/2$, for high enough δ, every match with buyer l results in disagreement, and the first match with buyer h leads to an agreement. In this case, the seller and buyer h expect a common payoff of $h/(4 - 2\delta)$, reflecting delay caused by matches with buyer l (whereas the corresponding payoff in the model with strategic matching is exactly $h/2$ for every δ). When $l \geq h/2$, in a match with buyer h an agreement is reached with probability 1, while in a match with buyer l an agreement is reached with a positive probability that converges to 0 as $\delta \to 1$. The prices at which trade takes place with either buyer converge to l as $\delta \to 1$. In this case, for sufficiently high δ, the seller's expected payoff in the MPE is l/δ, while buyer l's is exactly 0; buyer h's payoff

[18]This protocol seems more suitable for modeling bargaining in markets with a large number of buyers and sellers.

[19]The exact offers being made and the probability with which certain offers are accepted in a match with buyer l are not pinned down in MPEs when there is a positive probability of disagreement with buyer l. However, all MPEs generate the same expected payoffs and probability of trade with each buyer, and as before we simply refer to the unique MPE outcome as "the MPE."

converges to $h - l$ as $\delta \to 1$. Hence, the MPEs for the game with random matching also conform asymptotically with the OOP.

As mentioned earlier, Rubinstein and Wolinsky (1990) explore both the strategic and the random matching protocols in a setting with multiple buyers, all of which have the same value l. In the model with random matching, they show that the MPE is also the unique SPE (see their Proposition 4.i). For the convenience of the reader, we provide a (shorter) proof of this result in the Online Appendix.

It may be easily checked that the counterpart of Proposition 1 for the model with random matching is that the seller's payoff is greater than or equal to $h/(4 - 2\delta)$ for every discount factor δ. Hence, $h/2$ is an asymptotic lower bound on the seller's payoff as $\delta \to 1$. It follows that if $l < h/2$, then for sufficiently high δ, the seller never trades with buyer l in any SPE. In this case, all SPEs must be outcome equivalent (to the MPE), and correspond to a two-player bargaining game in which the seller and buyer h are matched to bargain with probability $1/2$ in every round (in the complementary event, the round elapses without trade), in which both the seller and buyer h obtain expected payoffs of $h/(4 - 2\delta)$. Therefore, when $l < h/2$, every SPE under random matching generates (asymptotic) profits of $h/2$ for the seller and buyer h, and 0 for buyer l. Recall that even for $l < h/2$, the range of SPE profits under strategic matching is the entire interval $[h/2, h]$. To understand the divergence between these predictions, note that the construction of SPEs with threat of delay leverages the fact that the strategic matching protocol allows the seller to *avoid matching* with buyer h for any length of time. This type of threat to buyer h is not available under random matching, where the seller is exogenously matched with buyer h in every round with positive probability.

We have argued that in the simple cases $h = l$ and $l < h/2$, all SPEs for high δ are outcome equivalent to the MPE. We next establish that the range of SPE outcomes is significantly richer when $h > l \geq h/2$.

8. Extreme SPE Payoffs Under Random Matching

Here we explore the range of SPE payoffs for the seller and the buyers in the game with random matching. As in the bargaining model with strategic matching, the random matching protocol supports a wide range of trading outcomes if $h > l \geq h/2$. For brevity, we highlight the extremal SPE payoffs although intermediate values are also attainable. In particular, we identify the maximum payoff of the seller and the minimum payoff of each buyer over all SPEs, and show that these extreme payoffs converge to h for the seller and 0 for either buyer. While the underlying equilibrium constructions are distinct from those we developed for strategic matching, the asymptotic equilibrium payoffs most favorable to the seller are the same.

What about least favorable SPE outcomes for the seller? In a market where both buyers have value l, Rubinstein and Wolinsky's uniqueness result establishes that the MPE is the only SPE in the random matching model. A reasonable conjecture is that increasing the value of one of the buyers from l to $h \in (l, 2l)$ in such a market could not possibly make the seller worse off or the low-valuation buyer better off in *any* SPE. This conjecture would then imply that under random matching, the seller's asymptotic payoff cannot fall below l, and buyer l's asymptotic payoff must be 0 in any family of SPEs. We show that this is not the case—and hence the conjecture is false—by constructing SPEs for the case $h \in (l, 2l)$ in which the seller gets an asymptotic payoff below l.

This construction requires fundamentally new ideas since the random matching protocol does not sustain interdependence in consecutive matches like the corresponding construction of semi-Markov SPEs. Recall that in buyer-favoring semi-Markov SPEs, after a round of failed negotiations with buyer l, the seller favors buyer l by trading with him in the next round with probability close to 1 in equilibrium. In the state favoring buyer l in the new equilibrium construction, the seller trades with small probability when matched with buyer h, and with probability 1 when matched with buyer l, inducing a similar trading pattern as a *time average*. To minimize the probability of agreement in every match with buyer h, we design discontinuous acceptance rules that incentivize the proposer to lowball the responder with an offer that is accepted with small probability instead of making a generous offer that would guarantee acceptance. In general, constructions of extremal SPEs under random matching are more involved due to the seller's lack of agency in the matching process.

Can the seller's asymptotic payoff under random matching get as low as $h/2$ when the outside option is asymptotically binding ($l > h/2$) as it does under strategic matching? Interestingly, the answer is negative. In the Appendix, we prove that $h/4 + l/2$ ($> h/2$) is an asymptotic lower bound on the seller's payoff in all SPEs under random matching.

Best SPE for the seller and worst SPE for buyers. Under the retained assumption that $h > l \geq h/2$, we first construct an SPE that yields the highest payoff $\bar{v}_s$ for the seller. Not surprisingly, this construction involves an SPE that generates the lowest equilibrium payoff $\underline{v}_h$ for buyer h in a subgame. The construction relies on the following modification of the MPE, which we call the *quasi-MPE*. In the MPE, conditional on being matched with buyer l, trade occurs with positive probability, but this probability converges to 0 as $\delta \to 1$. In the quasi-MPE, play follows the MPE strategies after all histories except that *in the first round*, conditional on a match with buyer l, the selected proposer offers a payoff equal to the discounted MPE payoff of the responder, and the responder accepts this offer *with probability 1*.

In the *best SPE for the seller*, play begins in "state $\bar{s}$," in which the seller expects payoff $\bar{v}_s$. Following any first-round rejection by buyer h, play transitions to "state $\underline{h}$," in which buyer h expects payoff $\underline{v}_h$. In state $\bar{s}$, if the seller is matched with buyer l in the first round, then no agreement is reached: buyer l accepts only price offers smaller than or equal to l, the seller accepts only price offers greater than or equal to $\delta\bar{v}_s$, and both players make unacceptable offers. If the seller is matched with buyer h in the first round, then when selected to propose, the seller offers a payoff of $\delta\underline{v}_h$, and buyer h accepts only offers greater than or equal to this amount. If selected to make the offer, buyer h offers $\delta\bar{v}_s$ to the seller, and the seller accepts only offers greater than or equal to $\delta\bar{v}_s$. As mentioned earlier, when buyer h rejects an offer, play transitions to state $\underline{h}$. Following any other disagreement, play remains in state $\bar{s}$. In state $\underline{h}$, if the seller is matched with buyer l in the first round, then play follows the quasi-MPE strategies thereafter. However, if the seller is matched with buyer h in the first round, then play reverts to state $\bar{s}$.

For the constructed strategies to generate the desired payoffs and form an SPE, $(\bar{v}_s, \underline{v}_h)$ must solve the following system of equations:

$$\bar{v}_s = \frac{1}{4}(h - \delta\underline{v}_h) + \frac{3}{4}\delta\bar{v}_s \tag{3}$$

$$\underline{v}_h = \frac{1}{4}(h - \delta\bar{v}_s) + \frac{1}{4}\delta\underline{v}_h. \tag{4}$$

The system has the unique solution

$$\bar{v}_s = \frac{(2-\delta)h}{8 - 8\delta + \delta^2} \tag{5}$$

$$\underline{v}_h = \frac{2(1-\delta)h}{8 - 8\delta + \delta^2}. \tag{6}$$

As $\delta \to 1$, these payoffs converge to h for the seller and 0 for buyer h.

Given the definition of $(\bar{v}_s, \underline{v}_h)$, this construction generates an SPE whenever $\delta\bar{v}_s \geq l$ (so that the seller and buyer l do not have incentives to make acceptable offers to each other), which is the case for δ sufficiently close to 1 since $\lim_{\delta\to 1}\delta\bar{v}_s = h > l$. In the Online Appendix, we prove that for high enough δ, $\bar{v}_s$ is the seller's maximum payoff among all SPEs, and $\underline{v}_h$ is buyer h's minimum payoff among all SPEs. The construction also yields the lowest SPE payoff (0) for buyer l.

SPEs favorable to buyers and unfavorable to the seller. We now turn to SPEs that generate low payoffs for the seller. For $l = h/2$, the MPE delivers asymptotic payoff $l = h/2$ to the seller, which is her lowest possible asymptotic profit in all SPEs given the universal asymptotic lower bound $h/2$ discussed earlier. The construction below assumes that $h > l > h/2$.

We construct two interlinked SPEs that generate a common low payoff $\underline{v}_s$ for the seller and high payoffs for the corresponding buyer. One of the SPEs delivers a high payoff $\bar{v}_h$ to buyer h and payoff 0 to buyer l, while the other delivers a high payoff $\bar{v}_l$ to buyer l and a low payoff v_h to buyer h. We show that v_h is identical to $\underline{v}_h$, buyer h's minimum SPE payoff identified in the previous subsection. We conjecture that these constructions yield the minimum payoff for the seller and the maximum payoff for each buyer over *all* SPEs, and the notation $(\underline{v}_s, \bar{v}_h, \bar{v}_l)$ is meant to be suggestive.

In this construction, behavior may be in "state" $\bar{h}, \bar{l}$ or s', and certain deviations trigger transitions to the best SPE for the seller (described in the previous subsection), which for convenience we refer to as "state $\bar{s}$."[20] In state $k \in \{\bar{h}, \bar{l}\}$, buyer k expects a high SPE payoff $\bar{v}_k$, while the other buyer receives his lowest SPE payoff. In both states $\bar{h}$ and $\bar{l}$, the seller expects the same low payoff $\underline{v}_s$, which converges to $(4h + l - \sqrt{12h^2 - 12hl + l^2})/4$ as $\delta \to 1$; this limit lies in the

[20]The full description of this SPE involves three additional states corresponding to state $\underline{h}$, the initial state of the quasi-MPE, and a state describing behavior in the MPE.

interval $(h/4 + l/2, l)$. As explained in the opening of the section, the fact that in this equilibrium construction buyer l's asymptotic payoff is positive—and correspondingly the seller's asymptotic profit is below l—means that the seller's lowest SPE asymptotic profit does not respond in the expected monotonic fashion to an increase in a buyer's value.

In constructing equilibria of a bargaining game, a central concern is deterring non-equilibrium offers which, if accepted, *end* the game, precluding punishment of the deviator(s). A non-equilibrium offer can be punished only if it is declined, and this is achieved most effectively if the rejection of such offers triggers the best SPE for the rejecting player. In other words, a non-equilibrium offer "signals" a transition to the most favorable SPE for the responder.[21] Our equilibrium constructions embody this signaling mechanism in subgames associated with the "wrong" match (buyer h in state $\bar{l}$, and buyer l in state $\bar{h}$), while relying on familiar recursion in subgames associated with the right match.

We will determine the payoffs $(\underline{v}_s, \bar{v}_h, v_h, \bar{v}_l)$ consistent with the prescribed strategies after we describe behavior in each state. In *state* $\bar{l}$, the three players expect payoffs $(\underline{v}_s, v_h, \bar{v}_l)$, and play proceeds as follows. For any proposer-responder pair selected by nature, the proposer offers $\delta\underline{v}_s$ to the seller, δv_h to buyer h, and $\delta\bar{v}_l$ to buyer l. In a match with buyer l, the equilibrium offer is accepted with probability 1; the responder rejects lower offers and accepts higher offers. In a match with buyer h, when responding to an offer: (i) buyer h accepts the seller's equilibrium offer with probability θ, accepts offers greater than $\delta\bar{v}_h$ with probability 1, and rejects all other offers; and (ii) the seller accepts buyer h's equilibrium offer with probability θ', accepts offers greater than $\delta\bar{v}_s$ with probability 1, and rejects all other offers. The probabilities θ and θ' will be specified shortly; for now, we note that both converge to 0 as $\delta \to 1$. Following a disagreement between the seller and either buyer, play continues in state $\bar{l}$ with the following exceptions. If buyer h rejects an offer different from the seller's equilibrium offer, play switches to state $\bar{h}$, while if the seller rejects an offer different from buyer h's equilibrium offer, play switches to state $\bar{s}$.

State $\bar{l}$ behavior has a standard stationary structure when the seller is matched with buyer l. However, if the seller is matched with buyer h, then maximizing buyer l's payoff entails returning to the initial state with high probability while providing incentives for the proposer to make the "right" offer, and for the responder to reject the offer with appropriate probability. In the favorable scenario in which the seller is selected to propose in a match with buyer h, she is guaranteed a payoff arbitrarily close to $h - \delta\bar{v}_h$. An efficient way to deliver this minimal "bribe" to the seller in terms of the dual objective of returning to the initial state with high probability and increasing total available surplus via opportunistic trade with the high valuation buyer is given by equation (10) below: $\theta(h-\delta v_h)+(1-\theta)\delta\underline{v}_s = h-\delta\bar{v}_h$. The seller trades with buyer h at the highly favorable price $h - \delta v_h$ (we will argue that $v_h = \underline{v}_h$), but with low probability θ. Buyer h is indifferent between accepting and rejecting the equilibrium offer, as needed to justify his randomization. Equation (10) implicitly defines the minimal probability θ with which buyer h should accept the seller's

[21] This interplay between best and worst equilibria is reminiscent of Abreu (1986).

demanding equilibrium offer in a recursive environment where the seller receives the bribe $h - \delta\bar{v}_h$ in expectation, and a rejection of the equilibrium offer leads to the same unfavorable state.[22] When buyer h proposes, the seller randomizes and incentives are maintained in an analogous manner by ensuring that buyer h's on-path conditional expected payoff is $h - \delta\bar{v}_s$ (where $\bar{v}_s$ represents the seller's highest SPE payoff characterized earlier).

In *state* $\bar{h}$, the three players expect payoffs $(\underline{v}_s, \bar{v}_h, 0)$, and play proceeds as follows. In a match with buyer h, the seller offers $\delta\bar{v}_h$ to h, and buyer h offers $\delta\underline{v}_s$ to the seller. The equilibrium offers and better ones are accepted, and worse ones are rejected. In a match with buyer l, the seller demands a price of l and buyer l rejects, while buyer l proposes a price of 0 and the seller rejects. The seller accepts only price offers greater than $\delta\bar{v}_s$, and buyer l accepts only offers that give him a utility greater than $\delta\bar{v}_l$; hence, the seller does not trade with buyer l in equilibrium. Following a disagreement between the seller and either buyer, play continues in state $\bar{h}$ with the following exceptions. If the seller rejects an offer different from buyer l's equilibrium offer, play switches to state $\bar{s}$. If buyer l rejects any non-equilibrium offer, play switches to state $\bar{l}$. Finally, if buyer l rejects the seller's equilibrium offer—as prescribed in this state—play switches to state s'.

State s' completes the equilibrium construction by delivering a "bribe" to the seller that has expected *discounted* value $l - \delta\bar{v}_l$ in exchange for having passed on an agreement with buyer l when she is selected to propose to buyer l.[23] Given the goal of maximizing buyer h's payoff, this bribe is most efficiently delivered in the *future* via exclusive trade with buyer h.[24] Indeed, if $\delta h/(2-\delta) \geq l$ (which holds for high δ given that $h > l$), then the maximum attainable total (expected future) surplus under the random matching protocol following an initial match with buyer l is $\delta h/(2-\delta)$, and is achieved by waiting for the first opportunity to trade with buyer h. Hence, maximizing buyer h's payoff subject to providing the seller's expected bribe is perfectly aligned with efficient surplus creation.

We are led to prescribe that in *state* s' the seller is rewarded for making an unacceptable offer to buyer l in state $\bar{h}$ with the expected payoff equivalent of the bribe $(l - \delta\bar{v}_l)/\delta$, which leaves buyer h with the rest of the total expected discounted surplus, $h/(2-\delta) - (l - \delta\bar{v}_l)/\delta$ (and buyer l with 0). To attain these payoffs, play in state s' is as follows. In a match with buyer l, behavior and state transitions are identical to those in state $\bar{h}$ except that a rejection of either player's equilibrium offer leads back to state s' (in equilibrium, a match with buyer l leads to disagreement). When nature

[22]The seller cannot obtain acceptance with probability 1 by a modest sweetening of the offer to buyer h because such a deviation signals a transition to the best equilibrium for buyer h. Then, buyer h should accept only non-equilibrium price offers below $h - \delta\bar{v}_h$, which is precisely the seller's conditional expected payoff on the equilibrium path via (10).

[23]Since $\bar{v}_l$ is the highest available equilibrium payoff for buyer l, the seller's expected payoff conditional on being selected to propose to buyer l should be at least $l - \delta\bar{v}_l$. By contrast, when buyer l is selected to propose to the seller, if the seller expects her most attractive equilibrium continuation payoff $\delta\bar{v}_s > l$, it is not necessary to provide buyer l with positive utility in the current match or in the future.

[24]The seller does not have an incentive to demand the bribe from buyer l because such an attempt would be perceived as a signal that play will shift to state $\bar{l}$, in which buyer l expects payoff $\delta\bar{v}_l$. Conversely, buyer l is prevented from making an acceptable offer to the seller via the equilibrium expectation that any such offer signals a transition to the best equilibrium for the seller (we show that $\delta\bar{v}_s > l$).

selects the seller to propose to buyer h, behavior and state transitions are specified as in state $\bar{h}$ (in equilibrium, the seller obtains payoff $h - \delta\bar{v}_h$). If nature selects buyer h to propose to the seller, then the buyer offers price v_s', and the seller accepts this equilibrium offer, where v_s' solves

$$\frac{l - \delta\bar{v}_l}{\delta} = \frac{1}{2-\delta}\left(\frac{1}{2}(h - \delta\bar{v}_h) + \frac{1}{2}v_s'\right). \tag{7}$$

The seller accepts offers greater than $\delta\bar{v}_s$, and rejects all other offers different from v_s'. In equilibrium, trade occurs only with buyer h, generating an expected payoff for the seller given by the right-hand side of (7). Play continues in state s' unless the seller rejects a non-equilibrium offer from buyer h, in which case it switches to state $\bar{s}$, or the seller rejects buyer h's equilibrium offer, in which case it switches to state $\bar{h}$.

We are left to define $(\underline{v}_s, v_h, \bar{v}_h, \bar{v}_l, \theta, \theta')$. The prescribed equilibrium behavior implies the following payoff equations for the two buyers:

$$\bar{v}_h = \frac{1}{4}(h - \delta\underline{v}_s) + \frac{1}{2}\delta\bar{v}_h + \frac{1}{4}\left(\delta\frac{h}{2-\delta} - (l - \delta\bar{v}_l)\right) \tag{8}$$

$$\bar{v}_l = \frac{1}{4}(l - \delta\underline{v}_s) + \frac{1}{4}\delta\bar{v}_l + \frac{1}{4}(1-\theta)\delta\bar{v}_l + \frac{1}{4}(1-\theta')\delta\bar{v}_l. \tag{9}$$

The probabilities θ and θ' are chosen to satisfy

$$h - \delta\bar{v}_h = \theta(h - \delta v_h) + (1-\theta)\delta\underline{v}_s \tag{10}$$

$$h - \delta\bar{v}_s = \theta'(h - \delta\underline{v}_s) + (1-\theta')\delta v_h. \tag{11}$$

Moreover, buyer h's payoff in state $\bar{l}$ must solve

$$v_h = \frac{1}{4}(\theta\delta v_h + (1-\theta)\delta v_h) + \frac{1}{4}(\theta'(h - \delta\underline{v}_s) + (1-\theta')\delta v_h). \tag{12}$$

We will argue that the seller's payoff in both states $\bar{h}$ and $\bar{l}$ solves

$$\underline{v}_s = \frac{1}{4}(h - \delta\bar{v}_h) + \frac{1}{4}(l - \delta\bar{v}_l) + \frac{1}{2}\delta\underline{v}_s. \tag{13}$$

Substituting (11) into (12) and simplifying yields a linear equation in v_h that is equivalent to equation (4) defining $\underline{v}_h$. Therefore, $v_h = \underline{v}_h$. This identity holds because in both states $\underline{h}$ and $\bar{l}$, the seller trades with buyer l when they are matched, and buyer h has the same expected payoffs when matched with the seller and selected as proposer or responder, respectively, although these payoffs are achieved via sure trade in state $\underline{h}$ and via randomization in state $\bar{l}$.

In the Appendix, we establish that for sufficiently high δ, the system of equations has a (unique) solution with $\underline{v}_s, \bar{v}_h, \bar{v}_l \geq 0$ and $\theta, \theta' \in [0,1]$. We argue that both θ and θ' converge to 0 at rate $1-\delta$ as $\delta \to 1$. The exact limit values of $\theta/(1-\delta)$ and $\theta'/(1-\delta)$ determine buyer l's positive limit payoff in state $\bar{l}$. We find that as $\delta \to 1$, $\bar{v}_l$ converges to $\bar{v}_l(1) := (3l + \sqrt{12h^2 - 12hl + l^2} - 4h)/4 \in (0, h-l)$, while $\underline{v}_s$ and $\bar{v}_h$ converge to $l - \bar{v}_l(1)$ and $h - l + \bar{v}_l(1)$, respectively.

Also in the Appendix, we verify that the construction delivers the same payoff $\underline{v}_s$ to the seller in both states $\bar{h}$ and $\bar{l}$. In either state, the seller expects a payoff of $\delta\underline{v}_s$ when one of the buyers proposes (even though the equilibrium proposal is not accepted in all cases). In state $\bar{l}$, trade with

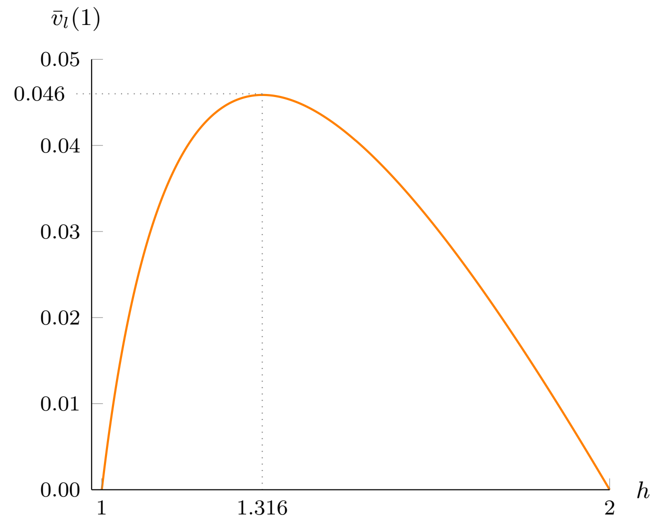

FIGURE 2. Buyer l's limit payoffs in SPEs favorable to him for $l = 1$ and $h \in (1, 2)$

buyer l takes place at price $l - \delta\bar{v}_l$ when the seller proposes, while a match with buyer h generates an expected payoff of $\theta(h - \delta v_h) + (1-\theta)\delta\underline{v}_s = h - \delta\bar{v}_h$ (via (10)) for the seller in the event she is the proposer. In state $\bar{h}$, trade with buyer h takes place at price $h - \delta\bar{v}_h$ when the seller proposes, while the seller receives an expected discounted payoff of $l - \delta\bar{v}_l$ via the transition to state s' when buyer l rejects her offer in equilibrium.

The recursive structure of states $\bar{h}$ and s' is almost identical, with the main difference being that buyer h offers the seller price $\delta\underline{v}_s$ in state $\bar{h}$, but the higher price v'_s in state s'. Indeed, we have that $v'_s > \delta\underline{v}_s$ because the seller's target expected payoff $(l - \delta\bar{v}_l)/\delta$ in state s' is greater than her expected payoff $\underline{v}_s$ in state $\bar{h}$ (since $l - \delta\bar{v}_l > \delta\underline{v}_s$). In state s', buyer h is disciplined to make the required offer v'_s via the threat than any other offer signals a transition to the unfavorable state $\bar{s}$ for him ($h - v'_s > h - \delta\bar{v}_s$), whereas the seller is happy to accept the price v'_s because rejecting it leads to state $\bar{h}$ ($v'_s > \delta\underline{v}_s$). In the Appendix, we confirm the three requisite inequalities. We also verify the other two main inequalities precluding profitable (one-shot) deviations: $h - \delta\bar{v}_h > \delta\underline{v}_s$ and $\delta\bar{v}_s > l$.

To visualize buyer l's most favorable limit payoff $\bar{v}_l(1)$ in (state $\bar{l}$ of) this construction, fix $l = 1$ and assume that $h \in (1, 2)$. Figure 2 plots $\bar{v}_l(1)$ as a function of h. This limit payoff goes to 0 as $l \to 1$ and $l \to 2$, and achieves a maximum of $1/4 - \sqrt{6}/12 \approx 0.046$ (which represents approximately 4.6% of buyer l's value) for $h = 1/2 + \sqrt{2/3} \approx 1.316$. Since buyer l's asymptotic payoff is 0 in every family of SPEs when $l \in \{h/2, h\}$,[25] we conclude that buyer l's payoff in the equilibrium most favorable to him is not monotonic in h.[26]

[25]For $l = h$, this is a consequence of Rubinstein and Wolinsky's result, while for $l = h/2$, this follows from the general lower bound $h/2$ on asymptotic seller profits.

[26]If we alternatively fix h and vary l in $[h/2, h]$, we reach the conclusion that neither buyer l's limit payoff $\bar{v}_l(1)$ nor the share $\bar{v}_l(1)/l$ of his own value he appropriates in his preferred equilibrium is monotonic in l.

9. Refinements for Random Matching

As was the case for the strategic matching protocol, a single refinement yields an asymptotic payoff of 0 for buyer l and an asymptotic lower bound of l for the seller's payoff. This refinement is a symmetric version of R2 introduced earlier, and for this reason we label it R2'.

R2' (Disagreement does not improve payoffs). *If v_s and v_k are the payoffs of the seller and buyer $k \in \{h, l\}$, respectively, in any subgame starting at the beginning of some round t, and v'_s and v'_k are the corresponding payoffs at the beginning of round $t+1$ following any disagreement with buyer k in round t (both payoffs may depend on play in round t of the subgame), then $v_s \geq v'_s$ and $v_k \geq v'_k$.*

R2' requires that neither party involved in a disagreement has future payoffs *increase* (but the buyer not involved in the disagreement might well experience a payoff increase). This version of R2 is stronger than the original in that it applies symmetrically to the seller and both buyers. However, in limiting payoff increases to a disagreeing buyer, R2' is closer in spirit to R1(M). Proposition 2 established that R1(M) is sufficient for an asymptotic lower bound of l on seller profits in the model with strategic matching. R2' delivers the corresponding result for random matching.[27]

Proposition 4. *In any family of SPEs of the bargaining game with random matching for discount factors $\delta \in (0,1)$ that satisfy R2', buyer l's payoff converges to 0 as $\delta \to 1$, and $\liminf_{\delta \to 1}$ of the seller's payoff is at least l.*

A key feature of the equilibria that generate the asymptotic payoff $\underline{v}_s(1) < l$ is that rejections by the seller or buyer k of some non-equilibrium offers are "rewarded" by transitions to the rejector's preferred equilibrium. These high rewards are triggered in a context in which on-path pre-match payoffs are low, $\underline{v}_s$ and $\underline{v}_k$ in state $\bar{k}'$ where $\{k, k'\} = \{h, l\}$, respectively, leading to robust violations of R2'.

We now turn to a novel refinement that is particular to the random matching protocol. Recall that in the MPE, every match with buyer h leads to trade with probability 1. When $l < h/2$, a match with buyer l results in no trade for sufficiently high δ, and when $l \geq h/2$, a match with buyer l results in trade with a positive probability that converges to 0 as $\delta \to 1$. In either case, successive random matches with buyer l might occur (and end in disagreement) before the seller is matched and trades with buyer h. Thus, under random matching, the probability of encountering buyer l is excessive when $l \geq h/2$, and even more starkly so when $l < h/2$. The strategic matching protocol precludes such redundant matches in the MPE. Indeed, the MPE under strategic matching has the property that if the seller chooses to bargain with a specific buyer with positive probability

[27] The proof of Proposition 2 uses only implications of R1(M) for buyer h's relative probability of trade following a disagreement with buyer h. Analogously, the proof of Proposition 4 relies only on implications of R2' for buyer h's payoffs.

in equilibrium, then trade occurs with conditional probability 1, so there are no "wasted" matches or delay.

This discussion reveals that the random matching protocol *strips the seller of agency* in the selection of buyers and may mechanically induce delay. To mitigate this issue, we introduce a refinement requiring that, in equilibrium, the "bad luck" of being matched with the wrong buyer does not impact the seller's utility excessively when we account for her patience.

R3 (Random mismatch is not worse than delay). *For every $k \in \{h, l\}$, if v_s^k is the seller's payoff in any subgame in which the seller is matched to bargain with buyer k in round t (before nature selects the proposer), and v_s is the seller's payoff prior to the matching at the beginning of round t, then $v_s^k \geq \delta v_s$.*

The random matching protocol "forces" a match with each buyer $k \in \{h, l\}$ with probability $1/2$. R3 posits that if a match with buyer k is undesirable for the seller *in equilibrium*, the impact should be limited to at most the cost of one round of delay relative to the *ex ante* pre-match expectations. This implies that the seller's expected payoff cannot vary excessively in response to nature's draw of the match (as it actually does in our constructions of extremal SPEs). Note that R3 places no restriction on the inter-temporal evolution of seller payoffs, and is thus not directly linked to stationarity.

To see that the MPE satisfies both refinements, let v_k^* denote player k's MPE payoff. The MPE satisfies R2' because in the MPE each player's payoff is the same at the beginning of every round. To check that the MPE satisfies R3, note that in the event the seller is matched with any buyer k, she can prevent an agreement in equilibrium by making an unreasonable offer (e.g., demanding price h) and rejecting any offer. Such a strategy delivers a discounted payoff of δv_s^* to the seller at the time of the match with buyer k. Therefore, the seller's equilibrium payoff in the event she is matched with buyer k must be at least δv_s^*. The conclusion follows from the fact that the seller's pre-match utility in the MPE is v_s^*.

Our final result establishes that the two refinements introduced here jointly imply the asymptotic predictions of the OOP in the model with random matching.

Proposition 5. *In any family of SPEs of the bargaining game with random matching for discount factors $\delta \in (0, 1)$ that satisfy R2' and R3, the seller's payoff converges to $\max(h/2, l)$, buyer h's payoff converges to $\min(h/2, h-l)$, and the discounted probability of trade with buyer h converges to 1 as $\delta \to 1$.*

The conclusion is trivially true when $l < h/2$ because, as argued in Section 7, in this case all SPEs are outcome equivalent to the MPE for high δ. When $l \geq h/2$, for high δ, the quasi-MPE satisfies R3 but violates R2'. Indeed, the quasi-MPE satisfies R3 because the seller's pre and post match payoffs in every round—including the first—are identical to the corresponding payoffs in the MPE, and the MPE satisfies R3. Quasi-MPEs for high δ violate R2' because they yield limit payoff $(h-l)/2$ for buyer h, but a first-round disagreement with buyer h leads to a subgame in which they yield the greater (MPE) limit payoff $h-l$ for buyer h.

While the quasi-MPEs deliver limit payoff l to the seller, they lead to an agreement with buyer l with probability $1/2$ and generate limit payoff $(h-l)/2$ for buyer h as $\delta \to 1$. In the Online Appendix, we showcase SPEs building on quasi-MPEs that continue to satisfy R3 (while violating R2') and yield limit payoffs different from l for the seller. We also construct SPEs that satisfy R2' but violate R3 and generate asymptotic outcomes inconsistent with the OOP. Therefore, neither refinement alone is sufficient for the conclusion of Proposition 5.

10. CONCLUSION

Many dynamic strategic models display a vast multiplicity of "perfect" equilibria, and these have been elaborately documented in the extensive "folk theorem" literature. From the point of view of predictive economics, this multiplicity is problematic. For repeated games and other types of dynamic games susceptible to substantial multiplicity, there have been few attempts to propose selection criteria.[28]

Since Rubinstein's (1982) seminal paper, it has been typical to model non-cooperative bargaining as unfolding over an infinite horizon. Rubinstein showed that his two-player model has a unique SPE, and uniqueness has become a traditional desideratum in the theory of non-cooperative bargaining. But even with complete information, once we move beyond the two-player case, uniqueness of SPEs is not guaranteed.

In this paper, we investigated the robustness of the classic *outside option principle* as an equilibrium prediction. We considered bilateral bargaining between a seller and two potential buyers with valuations $h > l$. We analyzed two bargaining protocols: in our benchmark model, the seller strategically chooses which buyer to negotiate with in every bargaining round, while in the other the seller is randomly matched with one of the buyers for negotiations in every round.

Both protocols generate a unique MPE outcome, but less was known about the range of possible SPE trading outcomes. One of the contributions of the paper is to explore this range in both models. In doing so, we developed several equilibrium constructions that characterize the set of prices at which trade can occur in SPEs. We found that a wide range of prices may arise. Under strategic matching, in the limit as players become patient, trade can occur at any price in the interval $[h/2, h]$ and no other. In the model with random matching, in the interesting case when the outside option is binding ($l > h/2$), trade can take place at prices arbitrarily close to h and also prices below l, but not as low as $h/2$. Moreover, substantial allocative efficiency and costly delay are possible.

The OOP posits that as players become patient, trade occurs without costly delay and with the high-value buyer with limit probability 1 at limit price $\max(h/2, l)$. As discussed above, for both protocols, there exist SPEs that deviate from the OOP along all these dimensions. While

[28]Exceptions include Abreu and Pearce (2007, 2025). We have in mind here complete information games. There is, of course, a large literature on refinements of sequential equilibrium via restrictions on plausible beliefs about player types in settings with incomplete information.

the MPE delivers the intuitive OOP predictions, the underlying assumption of stationary behavior is unsatisfactory absent further justification. Indeed, in other contexts (e.g., repeated oligopoly), history dependent strategies are a staple even in stationary environments and are used to rationalize cooperative behavior and other outcomes considered more appealing than MPE outcomes.

Our goal was to identify refinements less restrictive and more interpretable than stationarity that are sufficient for the OOP. For the strategic matching protocol, the first refinement requires that buyer h's relative probability of trade in any subgame does not increase dramatically in the event the seller chooses to bargain with buyer h and agreement is not reached. We showed that this simple refinement eliminates the seller's potential vulnerability in an SPE relative to the OOP prediction when $l > h/2$, guaranteeing that the price is asymptotically bounded below by l in the patient limit. The second refinement postulates that a seller never approaches a buyer hoping that negotiations fail. Our main result for the strategic matching protocol establishes that SPEs satisfying both refinements conform asymptotically with the OOP predictions (although they do not imply Markov behavior or immediate agreement).

Similarly, for the random matching protocol, we introduced two refinements that together yield the OOP predictions. One of them has a similar flavor to the second refinement for the strategic matching model. The key new refinement, which is particular to the random matching protocol, requires that seller payoffs within a bargaining round do not vary excessively (relative to discounting costs) depending on which buyer she is exogenously matched with. This refinement does not *per se* place any restrictions on the evolution of seller payoffs across bargaining rounds.

Our refinements are intuitively appealing and have substantial cutting power in the bargaining problem we analyzed. The refinements are by design, context, and indeed, protocol specific. They are not general-purpose refinements that can be applied to any game or refinements derived from epistemic considerations.[29] While such refinements are valuable in their universality, this feature is a double edged sword, and one edge is likely to be blunt in many applications. We have instead looked for refinements tailored to the model in question, and behaviorally plausible in context. This approach may prove fruitful in other dynamic settings.[30]

[29]In a different approach, Sabourian (2004) and Gale and Sabourian (2005) provide foundations for competitive equilibrium predictions in a class of market games via a refinement positing that players prefer less complex strategies. In their bargaining game with no discounting, the complexity refinement implies Markov behavior, which in turn implies competitive equilibrium outcomes. Due to the absence of discounting, MPEs in their game for the three-player market we consider are consistent with *any* price in the interval $[l, h]$, and are not directly related to the OOP. Whether the introduction of discounting combined with a complexity refinement reinstates the OOP is a question for future research. There is an older tradition of using equilibrium refinements to rule out implausible beliefs about player types in bargaining games with incomplete information (e.g., Rubinstein 1985, Admati and Perry 1987).

[30]Extending the analysis to more general models of bargaining in networks is a logical next step. However, the OOP does not yet have a generally accepted counterpart in settings with multiple buyers and sellers—there is variation in predictions across various models (see Manea (2016) for a survey). Furthermore, multiplicity of MPEs is common in many of the leading models in this literature (e.g., Sabourian (2004), Gale and Sabourian (2005), Abreu and Manea (2012b), Manea (2017), Elliott and Nava (2019)), and the appropriate counterpart of the OOP in these richer settings is likely a set-valued solution concept.

APPENDIX

Proof of Proposition 1. Let $\underline{v}_s$ be the infimum of the seller's payoff and $\bar{v}_h$ be the supremum of buyer h's payoff in all SPEs for the game with discount factor δ. Standard bargaining arguments imply that

$$\underline{v}_s \geq \frac{1}{2}\delta\underline{v}_s + \frac{1}{2}(h - \delta\bar{v}_h) \tag{14}$$

$$\bar{v}_h \leq \frac{1}{2}\delta\bar{v}_h + \frac{1}{2}(h - \delta\underline{v}_s). \tag{15}$$

The first inequality reflects a lower bound on seller payoffs in every SPE derived from the strategy whereby the seller chooses to bargain with buyer h in the first round, makes an acceptable offer to buyer h arbitrarily close to $\delta\bar{v}_h$, and rejects buyer h's offer to obtain a payoff of at least $\underline{v}_s$ in the continuation subgame. The second inequality is a consequence of the following upper bound on buyer h's payoff in every SPE:

$$\max\left(\delta\bar{v}_h, \frac{1}{2}\delta\bar{v}_h + \frac{1}{2}\max(h - \delta\underline{v}_s, \delta\bar{v}_h)\right).$$

Indeed, in the event the seller does not bargain with buyer h in the first round of an SPE, buyer h's expected continuation payoff does not exceed $\bar{v}_h$. In the event she does, the seller becomes the proposer with probability $1/2$, in which case buyer h's payoff does not exceed $\delta\bar{v}_h$,[31] and buyer h becomes the proposer with probability $1/2$, in which case either the seller accepts an offer of at least $\delta\underline{v}_s$, leaving buyer h with a payoff of at most $h - \delta\underline{v}_s$, or rejects the buyer's offer, resulting in a continuation payoff of at most $\bar{v}_h$ for the buyer.

If either of the maxima in the expression displayed above is achieved by $\delta\bar{v}_h$, then it must be that $\bar{v}_h \leq 0$. However, $\bar{v}_h > 0$ because buyer h receives a positive payoff in the MPE, a contradiction. This argument leads to inequality (15).

We can derive an upper bound on $\bar{v}_h$ from inequality (15), which we then substitute on the right-hand side of inequality (14) to obtain an inequality equivalent to $\underline{v}_s \geq h/2$ (this is intuitive as inequalities (14) and (15) are familiar from a two-player bargaining setting in which the seller can trade only with buyer h; in that setting, in the unique SPE, both players receive payoffs of $h/2$). □

Proof of Proposition 2. Fix $M \geq 1$, and let $\bar{v}_l$ be the $\limsup_{\delta\to 1}$ of buyer l's supremum payoff and $\underline{v}_s$ be the $\liminf_{\delta\to 1}$ of the seller's infimum payoff in the game with discount factor δ under SPEs satisfying R1(M).

For every $\varepsilon > 0$, there exists $\underline{\delta} < 1$ such that in every SPE for the game with any discount factor $\delta > \underline{\delta}$ that satisfies R1(M), buyer l's payoff is smaller than $\bar{v}_l + \varepsilon$. As R1(M) is a "recursive property," buyer l's payoff following any history in which he just rejected the seller's offer is smaller than $\bar{v}_l + \varepsilon$ in such SPEs; this implies that buyer l should accept an offer of $\bar{v}_l + \varepsilon$ (or more)

[31] In equilibrium, buyer h must accept any offer above $\delta\bar{v}_h$, which makes it non-optimal for the seller to offer more than $\delta\bar{v}_h$ to the buyer. Hence, buyer h either receives at most $\delta\bar{v}_h$ from an agreement with the seller or expects a discounted payoff not exceeding $\delta\bar{v}_h$ after a disagreement.

in any round. A strategy available to the seller is to approach buyer l in every round and reject all offers from l until she is selected to make an offer, at which point she offers $\bar{v}_l + \varepsilon$ to l. In every SPE for $\delta > \underline{\delta}$ that satisfies R1(M), buyer l should accept such offers, so this strategy generates an asymptotic payoff of $l - \bar{v}_l - \varepsilon$ for the seller. It follows that $\underline{v}_s \geq l - \bar{v}_l - \varepsilon$, which given the arbitrary choice of $\varepsilon > 0$ implies that $\underline{v}_s \geq l - \bar{v}_l$.

We will argue that $\bar{v}_l = 0$. Consider a family of SPEs σ^δ for a sequence of δ going to 1 that satisfy R1(M), in which buyer l's payoff converges to $\bar{v}_l$ as $\delta \to 1$. Assume by possibly passing to a subsequence that as $\delta \to 1$, the seller's payoff under σ^δ converges to $\tilde{v}_s$, the discounted probability $q(\delta)$ with which the seller makes an offer that buyer l accepts in the entire game under σ^δ converges to q, and the discounted probability $q'(\delta)$ with which buyer l makes an offer that the seller accepts in the game under σ^δ converges to q'.

Fix $\varepsilon > 0$. Since for δ sufficiently close to 1 in the sequence, buyer l should accept an offer of $\bar{v}_l + \varepsilon$ or more from the seller after any history under σ^δ, it is not optimal for the seller to offer more than $\bar{v}_l + \varepsilon$ in any round. Hence, for high δ, any agreement reached with buyer l under σ^δ when the seller proposes yields a payoff of at most $\bar{v}_l + \varepsilon$ for buyer l. Furthermore, for high δ, the seller should reject offers lower than $\underline{v}_s - \varepsilon$ from buyer l following any history under σ^δ, so any agreement reached when buyer l proposes yields a payoff of at most $l - \underline{v}_s + \varepsilon$ for l. It follows that for sufficiently high δ, buyer l's expected payoff under σ^δ does not exceed $q(\delta)(\bar{v}_l + \varepsilon) + q'(\delta)(l - \underline{v}_s + \varepsilon)$. As $q(\delta) \to q, q'(\delta) \to q'$ and buyer l's payoff under σ^δ converges to $\bar{v}_l$ for $\delta \to 1$, it follows that $\bar{v}_l \leq q(\bar{v}_l + \varepsilon) + q'(l - \underline{v}_s + \varepsilon)$. Taking the limit $\varepsilon \to 0$ in the latter inequality, we get that

$$\bar{v}_l \leq q\bar{v}_l + q'(l - \underline{v}_s).$$

As $\underline{v}_s \geq l - \bar{v}_l$ is equivalent to $l - \underline{v}_s \leq \bar{v}_l$, it follows that $\bar{v}_l(1 - q - q') \leq 0$.

Suppose, by way of contradiction, that $\bar{v}_l > 0$. Then, $\bar{v}_l(1-q-q') \leq 0$ combined with $q+q' \leq 1$ leads to $q + q' = 1$. Hence, the discounted probability with which the seller trades with buyer l under σ^δ converges to 1 as $\delta \to 1$. Since an agreement with buyer l generates total payoffs of l, and the seller's and buyer l's payoffs under σ^δ converge to $\tilde{v}_s$ and $\bar{v}_l$, respectively, as $\delta \to 1$, it follows that $\tilde{v}_s + \bar{v}_l = l$. Therefore, $\tilde{v}_s \leq \underline{v}_s$.

For any fixed $\varepsilon > 0$, the seller can deviate from σ^δ by approaching buyer h in the first round and proposing a price of $h - \varepsilon$ when chosen to make an offer, rejecting any offer from buyer h, and following σ^δ thereafter. Since σ^δ satisfies R1(M) and the discounted probability of trade for buyers h and l under σ^δ converge to 0 and 1, respectively, as $\delta \to 1$, buyer h's discounted probability of trade if he rejects the seller's offer under this deviation converges to 0 as $\delta \to 1$ (see footnote 14). Hence, for high enough δ, buyer h should accept the price offer $h - \varepsilon$ under σ^δ. For this deviation not to be profitable for the seller for high δ, it must be that

$$\tilde{v}_s \geq \frac{1}{2}(h - \varepsilon) + \frac{1}{2}\underline{v}_s. \tag{16}$$

Taking the limit $\varepsilon \to 1$, it follows that

$$\tilde{v}_s \geq \frac{1}{2}h + \frac{1}{2}\underline{v}_s \geq \frac{1}{2}h + \frac{1}{2}\tilde{v}_s,$$

which leads to $\tilde{v}_s \geq h$. This contradicts $\tilde{v}_s = l - \bar{v}_l < l \leq h$.

We conclude that $\bar{v}_l = 0$, and hence $\underline{v}_s \geq l - \bar{v}_l = l$. □

The SPE with threat of delay from Section 5 satisfies R1(M) for any $M \geq 1$. R1(M) is satisfied for buyer h in subgames where play is isomorphic to prescribed behavior in the first round or in the T stages of delay because the (discounted) probability of trade with buyer l in these subgames is 0, and in this case the refinement has no bite. R1(M) is satisfied for buyer l in any such subgame since buyer l trades with discounted probability 0 in the subgame, and if the seller bargains with buyer l unsuccessfully at the beginning of the subgame, then buyer l continues to trade with probability 0,[32] while buyer h trades with positive probability in every subgame that starts at the beginning of a round (the corresponding relative trading probabilities for buyer l in the statement of R1(M) are both 0). R1(M) is satisfied in subgames in which MPE play is to be followed because the MPE satisfies R1(M).

Proof of Proposition 3. Fix $M \geq 1$. For $k \in \{h, s\}$, let $\underline{v}_k(\delta)$ and $\bar{v}_k(\delta)$ be the infimum and supremum payoff of player k, respectively, in all SPEs satisfying R1(M) and R2 for the game with discount factor $\delta \in (0,1)$, and define $\underline{v}_k = \liminf_{\delta \to 1} \underline{v}_k(\delta)$ and $\bar{v}_k = \limsup_{\delta \to 1} \bar{v}_k(\delta)$. Corollary 1 implies that $\underline{v}_s \geq \max(h/2, l)$.

We will use the following implications of R2. In every SPE σ for a discount factor δ that satisfies R2 in which the seller approaches buyer l in the first round of the game with positive probability, the seller's expected payoff does not exceed l. Moreover, for such a σ, the seller's expected payoff at the beginning of round $t+1$ following any history of t disagreements does not exceed l/δ^t. To prove the first claim, let v_s denote the seller's payoff under an SPE σ with the properties above. Since the seller approaches buyer l in the first round with positive probability under σ, the seller's expected payoff conditional on approaching l should also be v_s. As no equilibrium agreement with buyer l can yield a payoff greater than l for the seller, we have that $v_s \leq ql + (1-q)\delta v'_s$, where q denotes the conditional probability of an agreement with buyer l in the first round, and v'_s denotes the expected value of the seller's payoff after an (possibly stochastic) equilibrium disagreement with buyer l. As σ satisfies R2, we have that $\delta v'_s < v_s$. If $q = 0$, then $v_s = \delta v'_s$, a contradiction. Hence, $q > 0$. Since $v_s \leq ql + (1-q)v_s$ implies that $q(l - v_s) \geq 0$, we conclude that $v_s \leq l$. The second claim follows inductively from the first and the observation that σ induces an SPE that satisfies R2 in every subgame.

Suppose—as we prove below—that $\underline{v}_h \geq \min(h/2, h-l)$. As the sum of the three players' payoffs in the game with discount factor δ cannot exceed h, it must be that $\bar{v}_s(\delta) \leq h - \underline{v}_h(\delta)$. Applying $\limsup_{\delta \to 1}$ to the previous inequality, we get that $\bar{v}_s \leq h - \underline{v}_h$, and hence $\bar{v}_s \leq h -$

[32]It is important for this conclusion that a deviation by the seller to bargaining with buyer l in the first round triggers the delay phase (as opposed to MPE play).

$\min(h/2, h-l) = \max(h/2, l)$. Similarly, we have that $\bar{v}_h \leq h - \underline{v}_s$, and $\underline{v}_s \geq \max(h/2, l)$ leads to $\bar{v}_h \leq \min(h/2, h-l)$. Since $\bar{v}_s \geq \underline{v}_s$ and $\bar{v}_h \geq \underline{v}_h$, we conclude that $\underline{v}_s = \bar{v}_s = \max(h/2, l)$ and $\underline{v}_h = \bar{v}_h = \min(h/2, h-l)$. Therefore, in every sequence of SPEs satisfying the two refinements for $\delta \to 1$, the seller gets a limit payoff of $\max(h/2, l)$, and buyer h gets a limit payoff of $\min(h/2, h-l)$, leaving buyer l with a limit payoff of 0. This is possible only if the seller trades with buyer h with a limit discounted probability of 1. Thus, to complete the proof, we need to show that $\underline{v}_h \geq \min(h/2, h-l)$.

For every δ, at least one of the following conditions must hold: (1) $\underline{v}_h(\delta)$ is attained as the limit of buyer h's payoffs in a sequence of SPEs for discount factor δ that satisfy the two refinements, in which the seller approaches buyer h with probability 1 in the first round of the game; (2) analogous statement to condition (1) with "probability 1" replaced by "probability less than 1." Moreover, there exists a sequence of discount factors $\delta \to 1$ such that $\underline{v}_h(\delta)$ converges to $\underline{v}_h$ as $\delta \to 1$, and additionally condition (1) or condition (2) is uniformly satisfied for every δ in the sequence. We prove that $\underline{v}_h \geq \min(h/2, h-l)$ by considering these two possibilities: cases 1 and 2 below assume the existence of sequences of δ satisfying conditions (1) and (2), respectively.

Case 1. For every δ in the sequence corresponding to this case, the payoff $\underline{v}_h(\delta)$ is the limit of buyer h's payoffs in a sequence of SPEs satisfying the two refinements in which the seller approaches buyer h with probability 1 in the first round of the game. Then, standard bargaining arguments imply that

$$\underline{v}_h(\delta) \geq \frac{1}{2}\delta \underline{v}_h(\delta) + \frac{1}{2}(h - \delta \bar{v}_s(\delta)). \tag{17}$$

Assume by way of contradiction that $\underline{v}_h < \min(h/2, h-l)$. Then, for sufficiently high δ in the sequence, we have that $h - \delta \underline{v}_h(\delta) > l$, so the payoff the seller can receive from having an offer accepted by one of the buyers is bounded above by $h - \delta \underline{v}_h(\delta)$. Again, standard arguments lead to[33]

$$\bar{v}_s(\delta) \leq \frac{1}{2}\delta \bar{v}_s(\delta) + \frac{1}{2}(h - \delta \underline{v}_h(\delta)). \tag{18}$$

If we group the terms $\bar{v}_s(\delta)$ in (18) to derive an upper bound on $\bar{v}_s(\delta)$ that depends on $\underline{v}_h(\delta)$, and then substitute the bound in the right-hand side of (17), we obtain an inequality equivalent to $\underline{v}_h(\delta) \geq h/2$. Since $\underline{v}_h$ is the limit of $\underline{v}_h(\delta)$ for $\delta \to 1$ in the sequence, it follows that $\underline{v}_h \geq h/2 \geq \min(h/2, h-l)$, a contradiction.

Case 2. For every δ in the sequence corresponding to this case, there exists an SPE σ^δ satisfying the two refinements in which the seller approaches buyer l with positive probability in the first round of the game, and buyer h receives a payoff $v_h(\delta) \in [\underline{v}_h(\delta), \underline{v}_h(\delta) + 1 - \delta)$. Then, $\lim_{\delta \to 1} v_h(\delta) = \underline{v}_h$.

[33] We have that

$$\bar{v}_s(\delta) \leq \frac{1}{2}\delta \bar{v}_s(\delta) + \frac{1}{2}\max(h - \delta \underline{v}_h(\delta), l, \delta \bar{v}_s(\delta)),$$

and the maximum in the expression above is achieved by the term $h - \delta \underline{v}_h(\delta)$. Indeed, we argued that $h - \delta \underline{v}_h(\delta) > l$, and it must also be that $h - \delta \underline{v}_h(\delta) > \delta \bar{v}_s(\delta)$ because otherwise the displayed inequality leads to $\bar{v}_s(\delta) \leq 0$. However, $\bar{v}_s(\delta) > 0$ since the MPE for discount factor δ satisfies the two refinements and yields a positive payoff for the seller.

Let $v_s(\delta)$ denote the seller's payoff under σ^δ. By the claim related to R2 from the preamble of the proof, $v_s(\delta) \leq l$. If $h/2 > l$, this yields a contradiction with Proposition 1. It follows that Case 2 can arise only if $l \geq h/2$, which we assume for the remainder of the proof. Since $\liminf_{\delta\to 1} v_s(\delta) \geq l$ by Proposition 2, it follows that $\lim_{\delta\to 1} v_s(\delta) = l$ (for δ in the sequence).

Let $x_h(\delta)$ and $x_l(\delta)$ be the discounted equilibrium probability of trade with buyers h and l, respectively, under σ^δ. We will show that $x_h(\delta) \to 1$ as $\delta \to 1$. If $x_h(\delta) \to 1$, then $\lim_{\delta\to 1} v_s(\delta) = l$ implies that buyer h's payoff under σ^δ converges to $h - l$ as $\delta \to 1$. Hence, $\underline{v}_h = h - l = \min(h/2, h - l)$.

We are left to prove that $x_h(\delta) \to 1$ as $\delta \to 1$. To obtain a contradiction, suppose that $x_h(\delta)$ does not converge to 1 as $\delta \to 1$. Then, there exists a subsequence of δ's where $x_h(\delta)$ and $x_l(\delta)$ converge to limits $x_h(1) < 1$ and $x_l(1)$, respectively, for $\delta \to 1$. We refer to this subsequence as "the sequence" in what follows.

Assume first that $x_l(1) > 0$. Since σ^δ satisfies R1(M), if $x'_h(\delta)$ and $x'_l(\delta)$ denote the discounted probabilities of trade with buyers h and l, respectively, following a disagreement with buyer h, then $x'_l(\delta) > 0$ and

$$M\frac{x_h(\delta)}{x_l(\delta)} \geq \frac{x'_h(\delta)}{x'_l(\delta)} \geq \frac{x'_h(\delta)}{1 - x'_h(\delta)}.$$

Letting $\tilde{M} = Mx_h(1)/x_l(1)$, this leads to $\limsup_{\delta\to 1} x'_h(\delta) \leq \tilde{M}/(\tilde{M}+1) := q < 1$. Since $\underline{v}_s \geq l$, buyer h's payoff in any agreement with the seller is asymptotically bounded above by $h-l$. Hence, buyer h's asymptotic payoff under σ^δ in the subgame following any first-round rejection by buyer h does not exceed $q(h - l)$.

However, in this case the seller would have a profitable deviation from σ^δ for sufficiently high δ in the sequence: she can approach buyer h, offer slightly more than $q(h - l)$ to buyer h so that the buyer accepts the offer under σ^δ, and reject the buyer's offer (following σ^δ thereafter). This deviation yields an asymptotic seller payoff of at least $1/2 \times (h - q(h - l)) + 1/2 \times l$, which is greater than l, contradicting the conclusion that $\lim_{\delta\to 1} v_s(\delta) = l$.

Assume now that $x_l(1) = 0$. We will show that in this case, $x_h(1) < 1$ implies that $\lim_{\delta\to 1} v_s(\delta) < l$, which again contradicts $\lim_{\delta\to 1} v_s(\delta) = l$.

Fix a finite positive integer T. To develop an upper bound on the seller's payoff under σ^δ for δ in the sequence, we classify the seller's trades as follows:

(1) trades with buyer h after histories in which the seller approached buyer h a total of $\tau \leq T$ times, the seller was the proposer each of the τ times, and the buyer accepted the seller's τ-th offer (these histories may also include prior disagreements with buyer l);

(2) trades with buyer h or l in *subgames* following histories in which the seller approached buyer h at least $\tau \leq T$ times, and the seller was the proposer the first $\tau - 1$ times, while buyer h was the proposer the τ-th time (there may be disagreements with buyers h and l after the τ-th approach to buyer h before trade occurs);

(3) trades with buyer h following histories in which the seller approached buyer h for a total of $\tau \geq T + 1$ times, and the seller was the proposer the first T times;

(4) trades with buyer l that are not of type (2).

For every non-terminal history of play ω, let $t(\omega)$ denote the calendar time at which nature or a player takes the next action after history ω. Let Ω^s_τ denote the set histories that arise with positive probability under σ^δ in which the seller approached buyer h for a total of τ times, and the seller was selected as the proposer every time she approached buyer h, ending with the τ-th selection of the seller as proposer. Define the set of histories Ω^h_τ analogously *except* that the τ-th time the seller approaches buyer h, the buyer is selected as proposer for the first time at the end of the history.[34] Finally, let μ be the probability distribution over histories generated by σ^δ.

For every $\omega \in \Omega^s_\tau$, we argue that the seller's expected payoff under σ^δ in the subgame starting at date $t(\omega)$ after observing the first $t(\omega) - 1$ rounds of history ω does not exceed l/δ^τ. If the seller never approaches buyer l during ω, then $t(\omega) = \tau$. Since the seller approaches buyer l with positive probability in the first round of the game under σ^δ, the claim related to R2 from the preamble of the proof implies that the seller's expected payoff in the subgame starting at date $t(\omega)$ after observing the first $t(\omega) - 1$ rounds of ω does not exceed $l/\delta^{t(\omega)-1}$, which is smaller l/δ^τ. If instead the seller approaches buyer l during ω, let $t_l(\omega)$ denote the last date at which this happens. Then, the seller approaches buyer h at dates $t_l(\omega) + 1, \ldots, t(\omega)$ under ω. By definition, $\omega \in \Omega^s_\tau$ implies that the seller approaches buyer h a total of τ times along the history ω, so it must be that $t(\omega) - t_l(\omega) \leq \tau$. Another consequence of $\omega \in \Omega^s_\tau$ is that the seller approaches buyer l with positive probability at date $t_l(\omega)$ in the subgame following the truncation of history ω up to that date. As in the preceding case, R2 implies that the seller's expected payoff under σ^δ at date $t(\omega)$ after observing rounds $t_l(\omega), \ldots, t(\omega) - 1$ of history ω in this subgame—i.e., the seller's payoff after observing the first $t(\omega) - 1$ rounds of history ω—does not exceed $l/\delta^{t(\omega)-t_l(\omega)}$, which is in turn bounded above by l/δ^τ.

An analogous argument shows that for any $\omega \in \Omega^h_\tau$, the seller's expected payoff in every subgame starting at date $t(\omega) + 1$ after history ω and *any* follow-up play at date $t(\omega)$ does not exceed $l/\delta^{\tau+1}$.

In the next four paragraphs, we develop bounds on the seller's payoffs under σ^δ deriving from each of the four types of trades enumerated above. Let $\varepsilon = 1/2^{T+1}$.

Consider a trade of type **(1)** at time $t(\omega)$ following a history $\omega \in \Omega^s_\tau$ for $\tau \leq T$. Since σ^δ and the equilibria it induces in any subgame satisfy R1(M), Proposition 2 implies that for sufficiently high δ, the seller's expected payoff under σ^δ exceeds $l - \varepsilon$ in any subgame. Let $\theta(\omega, p)$ denote the probability that buyer h accepts an offer of price p from the seller at date $t(\omega)$ after the history ω. Offering price p and rejecting buyer h's offer in round $t(\omega)$ should not be a profitable deviation for the seller. Since as argued above, the seller's expected payoff under σ^δ at the beginning of round $t(\omega)$ is at most l/δ^τ, and the seller obtains a continuation payoff of at least $l - \varepsilon$ after a

[34]Given the maintained assumption that behavior strategies have finite support after every history, the sets Ω^s_τ and Ω^h_τ are countable, but not necessarily finite because they may include histories in which the seller approaches buyer l an arbitrary number of times.

disagreement with buyer h, it must be that

$$\theta(\omega,p)p+(1-\theta(\omega,p))\delta(l-\varepsilon)\leq l/\delta^{\tau}.$$

Since $\tau \leq T$, if δ is sufficiently high, this leads to

$$\theta(\omega,p)p\leq(1/\delta^{\tau}-\delta)l+(1-\theta(\omega,p))\delta\varepsilon+\delta\theta(\omega,p)l\leq\theta(\omega,p)l+2\varepsilon.$$

Every trade of type **(2)** takes place in a *subgame* induced by a history $\omega \in \Omega_\tau^h$ (not necessarily in the first round of the subgame) for $\tau \leq T$. Let $x_h^\omega(\delta)$ and $x_l^\omega(\delta)$ denote the discounted probabilities of trade with the two buyers in this subgame (relative to the beginning of the subgame). In the subgame, the sum of expected payoffs of the three players under σ^δ is $x_h^\omega(\delta)h+x_l^\omega(\delta)l$. Since buyer h is the proposer in the first round of the subgame, and we argued above via R2 that the seller's continuation payoff under σ^δ at date $t(\omega)+1$ after history ω and *any* rejection by the seller at date $t(\omega)$ does not exceed $l/\delta^{\tau+1}$,[35] buyer h's payoff in the subgame is at least $h-\delta l/\delta^{\tau+1}$. It follows that the seller's expected payoff in the subgame is at most $x_h^\omega(\delta)h+x_l^\omega(\delta)l-(h-l/\delta^\tau)$. This expression is bounded above by $x_h^\omega(\delta)l/\delta^\tau+x_l^\omega(\delta)l$, which is smaller than or equal to $x_h^\omega(\delta)l/\delta^T+x_l^\omega(\delta)l$ because $\tau \leq T$. Note that the subgame arises under σ^δ with probability $\mu(\omega)$.

We will argue that trades of type **(3)** have probability at most $1/2^T$. The probability of such trades does not exceed the probability of reaching a subgame in Ω_T^s under σ^δ, which is $\mu(\Omega_T^s)$ in notation introduced earlier. Consider a history $\omega \in \Omega_{T-1}^s$. Let $\Omega_T^s(\omega)$ denote the subset of histories in Ω_T^s that emerge with positive probability under σ^δ following ω. For $\omega' \in \Omega_T^s(\omega)$, let ω'_- denote the truncation of history ω' omitting its last component, in which nature selects the seller to make an offer (ω'_- ends with the seller's T-th approach of buyer h at date $t(\omega')$). Since the seller becomes the proposer at $t(\omega')$ following history ω'_- with probability $1/2$, the probability of reaching $\omega' \in \Omega_T^s(\omega)$ from ω under σ^δ is $\mu(\omega'_-|\omega)/2$. Then,

$$\mu(\Omega_T^s(\omega))=\sum_{\omega'\in\Omega_T^s(\omega)}\frac{1}{2}\mu(\omega'_-|\omega)\mu(\omega)\leq\frac{1}{2}\mu(\omega),$$

where the inequality is a consequence of $\sum_{\omega'\in\Omega_T^s(\omega)}\mu(\omega'_-|\omega)\leq 1$. Then,

$$\mu(\Omega_T^s)=\sum_{\omega\in\Omega_{T-1}^s}\mu(\Omega_T^s(\omega))\leq\sum_{\omega\in\Omega_{T-1}^s}\frac{1}{2}\mu(\omega)=\frac{1}{2}\mu(\Omega_{T-1}^s).$$

It follows that $\mu(\Omega_T^s)\leq 1/2^T$. Therefore, the probability of a trade of type (3) is at most $1/2^T$. The seller obtains a price of at most h from buyer h in each trade of this type.

Finally, the discounted probability of trades of type **(4)** can be computed as

$$x_l(\delta)-\sum_{\tau=1}^{T}\sum_{\omega\in\Omega_\tau^h}\mu(\omega)\delta^{t(\omega)-1}x_l^\omega(\delta).$$

The seller obtains a price of at most l from buyer l in each trade of this type.

[35] Implications of R2 for disagreements off the equilibrium path are essential in this step.

Putting together the bounds on seller payoffs and trading probabilities for each of the four types of trades implies the following bound on the seller's expected payoff $v_s(\delta)$ under σ^δ for high enough δ:

$$v_s(\delta) \le \sum_{\tau=1}^{T} \sum_{\omega\in\Omega_\tau^s} \sum_{p\in\mathcal{P}(\omega)} \delta^{t(\omega)-1}\mu(\omega,p)\left(\theta(\omega,p)l + 2\varepsilon\right) +$$

$$\sum_{\tau=1}^{T} \sum_{\omega\in\Omega_\tau^h} \delta^{t(\omega)-1}\mu(\omega)\left(x_h^\omega(\delta)\frac{l}{\delta^T} + x_l^\omega(\delta)l\right) + \frac{1}{2^T}h + \left(x_l(\delta) - \sum_{\tau=1}^{T} \sum_{\omega\in\Omega_\tau^h} \delta^{t(\omega)-1}\mu(\omega)x_l^\omega(\delta)\right)l,$$

where $\mathcal{P}(\omega)$ denotes the finite set of prices the seller offers with positive probability at date $t(\omega)$ after history $\omega \in \Omega_\tau^s$, and (ω, p) denotes the history formed by ω followed up by the seller offering price p.[36]

Using the formulae for the discounted probabilities $x_h^1(\delta)$ and $x_h^2(\delta)$ of type (1) and type (2) trades with buyer h, respectively,

$$\begin{aligned} x_h^1(\delta) &= \sum_{\tau=1}^{T} \sum_{\omega\in\Omega_\tau^s} \sum_{p\in\mathcal{P}(\omega)} \delta^{t(\omega)-1}\mu(\omega,p)\theta(\omega,p) \\ x_h^2(\delta) &= \sum_{\tau=1}^{T} \sum_{\omega\in\Omega_\tau^h} \delta^{t(\omega)-1}\mu(\omega)x_h^\omega(\delta), \end{aligned}$$

and canceling out the terms involving $x_l^\omega(\delta)$, the bound on seller payoffs becomes

$$v_s(\delta) \le x_h^1(\delta)l + 2\varepsilon \sum_{\tau=1}^{T} \sum_{\omega\in\Omega_\tau^s} \sum_{p\in\mathcal{P}(\omega)} \delta^{t(\omega)-1}\mu(\omega,p) + x_h^2(\delta)\frac{l}{\delta^T} + \frac{1}{2^T}h + x_l(\delta)l.$$

For $\tau \ge 1$, we have that

$$\sum_{\omega\in\Omega_\tau^s} \sum_{p\in\mathcal{P}(\omega)} \delta^{t(\omega)-1}\mu(\omega,p) \le \sum_{\omega\in\Omega_\tau^s} \sum_{p\in\mathcal{P}(\omega)} \mu(\omega,p) = \sum_{\omega\in\Omega_\tau^s} \mu(\omega) = \mu(\Omega_\tau^s) \le 1.$$

Since $\varepsilon = 1/2^{T+1}$ and $x_h^1(\delta) + x_h^2(\delta)/\delta^T \le x_h(\delta)/\delta^T$, it follows that

$$v_s(\delta) \le \frac{x_h(\delta)}{\delta^T}l + \frac{T}{2^T} + \frac{1}{2^T}h + x_l(\delta)l.$$

Taking the limit $\delta \to 1$ (for δ in the sequence) in the inequality above, and noting that $\lim_{\delta\to1} x_h(\delta)/\delta^T = x_h(1)$ and $\lim_{\delta\to1} x_l(\delta) = 0$, we obtain

$$\lim_{\delta\to1} v_s(\delta) \le x_h(1)l + \frac{T}{2^T} + \frac{1}{2^T}h.$$

As T can be chosen to be arbitrarily large in the arguments above and $x_h(1) < 1$, it follows that $\lim_{\delta\to1} v_s(\delta) < l$, a contradiction with $\lim_{\delta\to1} v_s(\delta) = l$, which completes the proof. $\square$

[36]We suppress the dependence of the sets Ω_τ^s, Ω_τ^h and $\mathcal{P}(\omega)$ and of the probabilities μ and θ on δ (and σ^δ).

SPEs that satisfy R2, but not R1(M) for any M with asymptotic profits in (l, h). Assume that $l > h/2$ and fix $\delta > 2(1 - l/h)$, so that in the MPE the seller approaches both buyers with positive probability. Consider an SPE σ under which buyer h has an expected payoff $v_h \leq v_h^*$. We can construct an SPE in which the seller trades with buyer h in the first round and expects payoff $v_s := (h - \delta v_h)/(2 - \delta)$ as follows. In the first round, the seller approaches buyer h, and trade is supposed to happen: the seller offers δv_h to buyer h, and buyer h accepts only offers of δv_h or higher; buyer h offers δv_s to the seller, and the seller accepts only offers of δv_s or higher. If buyer h rejects any first-round offer from the seller, then play proceeds according to σ in the second round. If the seller rejects any offer from buyer h in the first round, then play restarts using the first-round strategies specified above. If the seller deviates to bargaining with buyer l in the first round, then play follows MPE strategies.

The definition of v_s implies that the seller's expected payoff from the first-round agreement under the constructed strategies is v_s. Note that $v_h \leq v_h^*$ implies that $v_s = (h - \delta v_h)/(2 - \delta) \geq (h - \delta v_h^*)/(2 - \delta) = v_s^*$. It follows that the seller does not have an incentive to deviate to bargaining with buyer l in subgames in which play is isomorphic to the first round. We can easily verify the other incentives to conclude that the constructed strategy profile constitutes an SPE.

Since in the case under consideration $\lim_{\delta \to 1} v_h^* = h - l$, this construction demonstrates that the availability of a family of SPEs for discount factors $\delta \in (0, 1)$ that yield an asymptotic payoff $\tilde{v}_h < h - l$ for buyer h as $\delta \to 1$ implies the existence of a family of SPEs in which the seller efficiently trades with buyer h in the first round and holds down buyer h to an asymptotic payoff of $\tilde{v}_h$ to effectively obtain the "take-it-or-leave-it" asymptotic profit $h - \tilde{v}_h$.

We now specialize the construction above to obtain SPEs that satisfy R2 and yield prices in the interval (l, h) for high δ. Fix $q \in (0, 1)$ and let σ^q be the SPE that differs from the MPE in that the seller approaches buyer h with probability q, and buyer l with probability $1 - q$ in the first round, but is identical to the MPE in all other subgames. Buyer h's asymptotic payoff under σ^q is $q(h - l)$, which is smaller than $h - l$. The construction above then delivers an SPE $\tilde{\sigma}^q$ in which rejections by buyer h are punished by continuation play according to σ^q, and the seller obtains an asymptotic profit of $h - q(h - l)$. As we vary q in the interval $(0, 1)$, asymptotic profits under this construction cover the interval (l, h).

We can easily verify that the SPE $\tilde{\sigma}^q$ corresponding to high discount factors δ satisfies R2. For high δ, the SPE satisfies R2 in subgames where play is isomorphic to first-round strategies because the seller's equilibrium payoff converges to $h - q(h - l)$ as $\delta \to 1$, and the seller's payoff after a disagreement is either unchanged or becomes v_s^* (we have that $h - q(h - l) > l = \lim_{\delta \to 1} v_s^*$). In all other subgames, the seller gets payoff v_s^*, and continues to get the same payoff after a disagreement.

The SPE $\tilde{\sigma}^q$ does not satisfy R1(M) for any given $M \geq 1$ when δ is close to 1 because under $\tilde{\sigma}^q$ the relative trading probability for buyer h in a subgame following a rejection by buyer h in the first round is $q/(1 - q)$, but if there is another disagreement with buyer h in the second round, it changes to π_h^*/π_l^*, which diverges to ∞ for $\delta \to 1$.

Asymptotic lower bound on SPE seller payoffs under random matching. Fix a discount factor δ, and let $\bar{v}_s$ denote the supremum of seller payoffs and $\underline{v}_k$ the infimum of buyer $k \in \{h, l\}$ payoffs in SPEs for the discount factor δ. Arguments by now familiar lead to the following inequalities:

$$\underline{v}_s \geq \frac{1}{4}(h - \delta\bar{v}_h) + \frac{1}{4}(l - \delta\bar{v}_l) + \frac{1}{2}\delta\underline{v}_s \tag{19}$$

$$\bar{v}_l \leq \frac{1}{4}(l - \delta\underline{v}_s) + \frac{3}{4}\delta\bar{v}_l \tag{20}$$

$$\bar{v}_h \leq \frac{1}{4}(h - \delta\underline{v}_s) + \frac{1}{2}\delta\bar{v}_h + \frac{1}{4}\left(\delta\frac{h}{2-\delta} - (l - \delta\bar{v}_l)\right). \tag{21}$$

The last term on the right-hand side of (21) represents an upper bound on buyer h's payoff in the event that the seller is matched to buyer l and is selected to make the offer. When $h/(2-\delta) \geq l$,[37] the maximum total surplus that can be created in any subgame is achieved by trading only when the seller is matched with buyer h, and has an expected discounted value of $h/(2-\delta)$. Then, conditional on the seller being matched with buyer l and being selected to make the offer, the maximum total surplus that can be generated by trading under the random matching protocol is the amount $h/(2-\delta)$ obtained with one round of delay: $\delta h/(2-\delta)$. Since the seller's payoff in this subgame is at least $l - \delta\bar{v}_l$, the highest payoff buyer h can expect in this event is $\delta h/(2-\delta) - (l - \delta\bar{v}_l)$.

Deriving an upper bound on $\bar{v}_l$ as a function of $\underline{v}_s$ from (20), which we then use to obtain an upper bound on $\bar{v}_h$ as a function of $\underline{v}_s$ via (21), and then substituting both of these bounds in (19), we get that

$$\underline{v}_s \geq \frac{(4-3\delta)h + (4-2\delta)l}{4(2-\delta)^2},$$

whenever $h/(2-\delta) \geq l$, or equivalently $\delta \geq 2l/(h+l)$. We conclude that if $h > l$, then

$$\liminf_{\delta\to 1} \underline{v}_s \geq \frac{h}{4} + \frac{l}{2}.$$

Applying $\limsup_{\delta\to 1}$ to (20) and (21), we obtain that

$$\limsup_{\delta\to 1} \underline{v}_l \leq \frac{l}{2} - \frac{h}{4} \;\&\; \limsup_{\delta\to 1} \underline{v}_h \leq \frac{3h}{4} - \frac{l}{2}.$$

Proofs of claims about SPEs favorable to buyers and unfavorable to the seller. Recall that this equilibrium construction presumes that $h > l > h/2$. Solving for $(\underline{v}_s, \bar{v}_h)$ in the linear system of equations formed by (8) and (13), we obtain that

$$\underline{v}_s = \frac{2(1-\delta)h + (2-\delta)(l - \delta\bar{v}_l)}{(2-\delta)(4-3\delta)} \tag{22}$$

$$\bar{v}_h = \frac{h - l + \delta\bar{v}_l}{4-3\delta}. \tag{23}$$

[37] This argument works for high δ when $h > l$. For $h = l$, Rubinstein and Wolinsky's (1990) SPE uniqueness result implies a tighter lower bound of h on the seller's asymptotic profits.

Deriving θ and θ' from (10) and (11), respectively, and substituting the expressions for $\underline{v}_s, \bar{v}_s, \bar{v}_h, \underline{v}_h$, we get

$$\theta = \frac{h - \delta\bar{v}_h - \delta\underline{v}_s}{h - \delta\underline{v}_h - \delta\underline{v}_s} = \frac{2(1-\delta)(4-3\delta)(8-8\delta+\delta^2)h}{\Delta} \tag{24}$$

$$\theta' = \frac{h - \delta\underline{v}_h - \delta\bar{v}_s}{h - \delta\underline{v}_h - \delta\underline{v}_s} = \frac{4(1-\delta)(4-3\delta)(2-\delta)^2 h}{\Delta}, \tag{25}$$

where $\Delta = (64 - 176\delta + 180\delta^2 - 78\delta^3 + 11\delta^4)h - \delta(2-\delta)(8-8\delta+\delta^2)(l - \delta\bar{v}_l)$.

Note that (9) is equivalent to

$$4\bar{v}_l = \frac{l - \delta\bar{v}_l - \delta\underline{v}_s}{1-\delta} - \frac{\theta + \theta'}{1-\delta}\delta\bar{v}_l, \tag{26}$$

and $l - \delta\bar{v}_l - \delta\underline{v}_s$ simplifies to

$$l - \delta\bar{v}_l - \delta\underline{v}_s = \frac{2(1-\delta)((4-2\delta)(l-\delta\bar{v}_l) - \delta h)}{(2-\delta)(4-3\delta)}.$$

Multiplying both sides of (26) by $\Delta/2$, we obtain a quadratic equation of the form $a(\delta)\bar{v}_l^2 + b(\delta)\bar{v}_l + c(\delta) = 0$, where $a(\delta),\ b(\delta)$ and $c(\delta)$ are continuous functions of δ with $a(1) = 4,\ b(1) = 8h - 6l$, and $c(1) = (h-l)(h-2l)$. The quadratic $a(1)\bar{v}_l^2 + b(1)\bar{v}_l + c(1)$ has positive leading coefficient and changes sign between $\bar{v}_l = 0$ ($(h-l)(h-2l) < 0$) and $\bar{v}_l = h - l$ ($(13h-12l)(h-l) > 0$), and hence its largest root

$$\bar{v}_l(1) = \frac{-4h + 3l + \sqrt{12h^2 - 12hl + l^2}}{4}$$

belongs to the interval $(0, h-l)$. It follows that for sufficiently high δ, equation (26) has a solution $\bar{v}_l \in (0, h-l)$ that converges to $\bar{v}_l(1)$ as $\delta \to 1$. Then, this value of $\bar{v}_l$ along with the values of $\underline{v}_s, \bar{v}_h, \theta$ and θ' computed by substituting $\bar{v}_l$ in (22)-(25) solve the system of equations (8)-(13) for sufficiently high δ. It is easy to check that for δ close to 1, we have that $\bar{v}_l, \underline{v}_s, \bar{v}_h > 0$. As Δ converges to $h - l + \bar{v}_l(1)$ for $\delta \to 1$, it follows that θ and θ' converge to 0 from above for $\delta \to 1$, so $\theta, \theta' \in (0,1)$ for δ close to 1, as desired.

To confirm that the seller receives payoff $\underline{v}_s$ in state $\bar{l}$ under the constructed strategies, note that her payoff v_s^l in this state must solve the following linear equation:

$$v_s^l = \frac{1}{4}(\theta(h - \delta\underline{v}_h) + (1-\theta)\delta v_s^l) + \frac{1}{4}(\theta'\delta\underline{v}_s + (1-\theta')\delta v_s^l) + \frac{1}{4}(l - \delta\bar{v}_l) + \frac{1}{4}\delta\underline{v}_s.$$

Equations (10) and (13) imply that $v_s^l = \underline{v}_s$ is the unique solution.

Similarly, the seller's payoff v_s^h in state $\bar{h}$ solves the linear equation

$$v_s^h = \frac{1}{4}(h - \delta\bar{v}_h) + \frac{1}{4}\delta\underline{v}_s + \frac{1}{4}\delta\frac{l - \delta\bar{v}_l}{\delta} + \frac{1}{4}\delta v_s^h,$$

which has the unique solution $v_s^h = \underline{v}_s$ due to (13).

When the solution $(\underline{v}_s, \bar{v}_h, \bar{v}_l, \theta, \theta')$ is "in range," we have that $\bar{v}_l, \Delta > 0$, so (24) implies that $h - \delta\bar{v}_h - \delta\underline{v}_s > 0$, while (26) implies that $l - \delta\bar{v}_l - \delta\underline{v}_s > 0$.

Clearly, for high δ, we have that $h - v'_s > h - \delta\bar{v}_s$ because v'_s and $\bar{v}_s$ converge to $l - \bar{v}_l(1) < l$ and $h > l$ as $\delta \to 1$, respectively, so $v'_s < l < \delta\bar{v}_s$.

We are left to argue that $v'_s > \delta\underline{v}_s$. Equation (13) leads to

$$\underline{v}_s = \frac{1}{2-\delta}\left(\frac{1}{2}(h - \delta\bar{v}_h) + \frac{1}{2}(l - \delta\bar{v}_l)\right).$$

As $l - \delta\bar{v}_l > \delta\underline{v}_s$, we have that $(l - \delta\bar{v}_l)/\delta > \underline{v}_s$. Then, (7) implies that $v'_s > l - \delta\bar{v}_l$, so $v'_s > \delta\underline{v}_s$.

Proof of Proposition 4. Define $\underline{v}_s$ and $\bar{v}_l$ as in the proof of Proposition 2 for SPEs in the bargaining game with the random matching protocol that satisfy R2' (instead of R1(M)). Note that like R1(M), R2' is a recursive property. For every $\varepsilon > 0$, a strategy available to the seller is to reject all offers, demand price h from buyer h and propose a price of $l - \bar{v}_l - \varepsilon$ to buyer l in all rounds. Since buyer l should accept such offers for sufficiently high δ in SPEs satisfying R2', this strategy generates an asymptotic payoff of at least $l - \bar{v}_l - \varepsilon$ for the seller. We conclude that $\underline{v}_s \geq l - \bar{v}_l$.

Consider a set of SPEs σ^δ for a sequence of δ going to 1 that satisfy R2' in which buyer l's payoff converges to $\bar{v}_l$ as $\delta \to 1$, and the seller's payoff converges to some $\tilde{v}_s$. If $\bar{v}_l > 0$, then analogous arguments to those for Proposition 2 establish that the discounted probability with which the seller trades with buyer l under σ^δ converges to 1 as $\delta \to 1$. It follows that $\tilde{v}_s + \bar{v}_l = l$ and consequently that $\tilde{v}_s \leq \underline{v}_s$.

In addition, the probability of trade with buyer h under σ^δ converges to 0 as $\delta \to 1$. Hence for any $\varepsilon > 0$, buyer h's expected payoff under σ^δ is lower than εh for sufficiently high δ. Then, the fact that σ^δ satisfies R2' for buyer h implies that following any disagreement with buyer h, buyer h's continuation payoff is smaller than εh. Hence, for high enough δ, buyer h should accept the price offer $h - \varepsilon h$ under σ^δ.

In the first round, the seller can offer a price of $h(1-\varepsilon)$ when matched with buyer h and selected to propose, make an unreasonable demand (e.g., price h) when matched with buyer l and selected to propose, and reject any first-round offer from either buyer, and then follow σ^δ from the second round on. Given that in equilibrium buyer h should accept the seller's price offer $h(1-\varepsilon)$ for high enough δ, for this strategy not to be a profitable deviation from σ^δ for the seller, it must be that

$$\tilde{v}_s \geq \frac{1}{4}h(1-\varepsilon) + \frac{3}{4}\underline{v}_s.$$

As in the proof of Proposition 2, this implies that $\tilde{v}_s \geq h$, generating a contradiction. We conclude that $\bar{v}_l = 0$ and $\underline{v}_s \geq l$. □

Proof of Proposition 5. If $l < h/2$, then every SPE is outcome equivalent to the MPE (see Section 7), and the result clearly holds. For the remainder of the proof, we assume that $l \geq h/2$.

For $k \in \{h, s\}$, let $\underline{v}_k(\delta)$ and $\bar{v}_k(\delta)$ be the infimum and supremum payoff of player k, respectively, in all SPEs satisfying R2' and R3 for the game with random matching for discount factor $\delta \in (0,1)$, and define $\underline{v}_k = \liminf_{\delta\to 1} \underline{v}_k(\delta)$ and $\bar{v}_k = \limsup_{\delta\to 1} \bar{v}_k(\delta)$. By Proposition 4, we have that that $\underline{v}_s \geq l$. Then, the result follows if we show that $\underline{v}_h \geq h - l$.

For every δ, at least one of the following conditions must hold: (1) $\underline{v}_h(\delta)$ is attained as the limit of buyer h's payoffs in a sequence of SPEs for discount factor δ that satisfy the two refinements, in which the seller trades with buyer l with probability 0 in the first round of the game; (2) analogous statement to condition (1) with "probability 0" replaced by "positive probability." Moreover, there exists a sequence of discount factors $\delta \to 1$ such that $\underline{v}_h(\delta)$ converges to $\underline{v}_h$ as $\delta \to 1$, and additionally condition (1) or condition (2) is uniformly satisfied for every δ in the sequence. We prove that $\underline{v}_h \geq h - l$ by considering these two possibilities: cases 1 and 2 below assume the existence of sequences of δ satisfying conditions (1) and (2), respectively.

Case 1. For every δ in the sequence corresponding to this case, the payoff $\underline{v}_h(\delta)$ is the limit of buyer h's payoffs in a sequence of SPEs satisfying the two refinements in which the seller does not trade when matched with buyer l in the first round of the game. Since buyer h can deviate from his first-round strategy to making an offer arbitrarily close to $\delta\bar{v}_s(\delta)$, which the seller should accept in equilibrium, and rejecting any offer from the seller, while following equilibrium play in the probability $3/4$ event in which he rejects the seller's offer or the seller has an equilibrium disagreement with buyer l, we have that

$$\underline{v}_h(\delta) \geq \frac{1}{4}(h - \delta\bar{v}_s(\delta)) + \frac{3}{4}\delta\underline{v}_h(\delta). \tag{27}$$

We also have that

$$\bar{v}_s(\delta) \leq \frac{1}{4}\max(h - \delta\underline{v}_h(\delta), \delta\bar{v}_s(\delta)) + \frac{1}{4}\max(l, \delta\bar{v}_s(\delta)) + \frac{1}{2}\delta\bar{v}_s(\delta).$$

Since the seller's payoff in the MPE for high δ is l/δ when $l \geq h/2$, and the MPE satisfies both refinements, we have that $\bar{v}_s(\delta) \geq l/\delta$, which means that $\max(l, \delta\bar{v}_s(\delta)) = \delta\bar{v}_s(\delta)$. If $\delta\bar{v}_s(\delta) > h - \delta\underline{v}_h(\delta)$ then the inequality displayed above implies that $\bar{v}_s(\delta) \leq 0$, contradicting $\bar{v}_s(\delta) \geq l/\delta$. It follows that $\max(h - \delta\underline{v}_h(\delta), \delta\bar{v}_s(\delta)) = h - \delta\underline{v}_h(\delta)$, which leads to

$$\bar{v}_s(\delta) \leq \frac{1}{4}(h - \delta\underline{v}_h(\delta)) + \frac{3}{4}\delta\bar{v}_s(\delta). \tag{28}$$

If we group the terms $\bar{v}_s(\delta)$ in (28) to derive an upper bound on $\bar{v}_s(\delta)$ that depends on $\underline{v}_h(\delta)$, and then substitute the bound in the right-hand side of (27), we obtain an inequality equivalent to $\underline{v}_h(\delta) \geq h/(4 - 2\delta)$. Since $\underline{v}_h$ is the limit of $\underline{v}_h(\delta)$ for $\delta \to 1$ in the sequence, it follows that $\underline{v}_h \geq h/2 \geq h - l$.

Case 2. For every δ in the sequence corresponding to this case, there exists an SPE σ^δ satisfying the two refinements in which the seller trades with buyer l with positive probability in the first round of the game, and buyer h receives a payoff $v_h(\delta) \in [\underline{v}_h(\delta), \underline{v}_h(\delta) + 1 - \delta)$. Then, $\lim_{\delta\to 1} v_h(\delta) = \underline{v}_h$. Let $v_s(\delta), v_s^h(\delta), v_s^l(\delta)$ denote the seller's expected payoffs under σ^δ at the beginning of the game and conditional on being matched to buyers h and l in the first round, respectively, and $r(\delta)$ the (unconditional) probability that the seller trades with buyer l in the first round under σ^δ. Consider a subsequence of δ converging to 1 for which these variables converge to limits denoted by $v_s(1), v_s^h(1), v_s^l(1)$ and $r(1)$, respectively.

Since $\liminf_{\delta\to 1} v_s(\delta) \geq l$ by Proposition 4, it follows that $v_s(1) \geq l$.

Let $V'_s(\delta)$ denote the set of expected payoffs the seller can obtain in the second round under σ^δ following any first-round disagreement with buyer l. Since σ^δ satisfies R2', we have that $\sup V'_s(\delta) \leq v_s(\delta)$. The fact that σ^δ satisfies R3 implies that $\delta v_s(\delta) \leq v_s^l(\delta)$. Hence,

$$v_s^l(\delta) \leq r(\delta)l + (1 - r(\delta))\delta \sup V'_s(\delta) \leq r(\delta)l + (1 - r(\delta))\delta v_s(\delta) \leq r(\delta)l + (1 - r(\delta))v_s^l(\delta),$$

or equivalently $r(\delta)(l - v_s^l(\delta)) \geq 0$. As $r(\delta) > 0$, it must be that $v_s^l(\delta) \leq l$. Therefore, $v_s(\delta) \leq l/\delta$. Then, $v_s(1) \geq l$ leads to $v_s(1) = l$. Since σ^δ satisfies R3 for every δ in the sequence, we have that $v_s^h(1), v_s^l(1) \geq l$, which along with $l = v_s(1) = (v_s^h(1) + v_s^l(1))/2$ implies that $v_s^h(1) = v_s^l(1) = l$.

Suppose now that $r(1) > 0$, and fix $\varepsilon > 0$. Proposition 4 implies that for sufficiently high δ, the seller's payoff in every subgame under σ^δ is bounded below by $l - \varepsilon$, so buyer h's payoff in every subgame is bounded above by $h - l + \varepsilon$. Hence, it is not optimal for the seller to accept prices below $\delta(l - \varepsilon)$ or offer a price above $h - \delta(h - l + \varepsilon)$ to buyer h in equilibrium. It follows that any transaction with buyer h takes place at an asymptotic price not lower than l, implying that $\bar{v}_h < (1 - r(1))(h - l)$.

Since σ^δ satisfies R2', for sufficiently high δ, buyer h's payoff under σ^δ following any rejection of the seller's offer does not exceed $(1 - r(1))(h - l) + \varepsilon$. It follows that buyer h should accept a payoff offer of $(1 - r(1))(h - l) + \varepsilon$ under σ^δ. However, in this case, for high δ, the seller would have a profitable deviation whereby in the first round she offers payoff $(1 - r(1))(h - l) + \varepsilon$ to buyer h, demands price h from buyer l, and rejects all offers from either buyer, and from the second round on follows σ^δ. This deviation yields an asymptotic payoff of at least $1/4 \times (h - (1 - r(1))(h - l) - \varepsilon) + 3/4 \times l$ for the seller, which is greater than the seller's limit payoff $v_s(1) = l$ for small enough ε.

Suppose insted that $r(1) = 0$. Since σ^δ satisfies R2' and $\lim_{\delta \to 1} v_s(\delta) = l$, for any $\varepsilon > 0$ there exists $\underline{\delta}$ such that the seller's payoff in a subgame following a disagreement with buyer h is smaller than $l + \varepsilon$ for $\delta > \underline{\delta}$ in the sequence. Then, for $\delta > \underline{\delta}$, the seller should accept an offer of $\delta(l + \varepsilon)$ from buyer h, so buyer h's equilibrium payoff under σ^δ satisfies

$$v_h(\delta) \geq \frac{1}{4}\left(h - \delta(l + \varepsilon)\right) + \left(\frac{3}{4} - r(\delta)\right)\delta \underline{v}_h(\delta).$$

Taking the limit $\delta \to 1$ along the sequence, we obtain that

$$\underline{v}_h \geq \frac{1}{4}\left(h - (l + \varepsilon)\right) + \frac{3}{4}\underline{v}_h,$$

which is equivalent to $\underline{v}_h \geq h - (l + \varepsilon)$. As ε is an arbitrary positive number, we conclude that $\underline{v}_h \geq h - l$. $\square$

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