Search Direction: Position Externalities and Position Auction Bias*

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Abstract

We formulate a tractable model of pricing under directed search with heterogeneous firm demands. Demand characteristics drive bids in a position auction and enable us to bridge insights from the ordered search literature to those in the position auction literature. Equilibrium pricing implies that the marginal consumer's surplus decreases down the search order, so consumers optimally follow the firms' position ordering. A firm suffers from "business stealing" by firms that precede it and "search appeal" from subsequent firms. We find rankings that achieve the maximal joint profit or consumer surplus by constructing firm-specific scores. A generalized second price auction for positions endogenizes equilibrium orders and bids are driven by position externalities that impact incremental profit from switching positions. The joint profit maximization order is upheld when firm heterogeneity concerns mostly their mark-up potentials. But the consumer welfare order is robust when firms differ mostly over their potential market sizes.

Keywords: Ordered search, product heterogeneity, position externalities, optimal and equilibrium rankings, generalized second price auction, position auction.

JEL Classification: L13, M37, L65

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1 Introduction

Internet search is guided by ad positions. These slots are allocated through firms' bids. The extant literature on ordered search has not fully integrated the role of bidding for positions, while the literature on position auctions has not gone deep into firm pricing with ordered search. The ordered search literature shows that otherwise symmetric firms expecting to be searched earlier charge lower prices so consumers indeed do start search at those firms (Armstrong, 2017, is an invaluable synthesis and buildout of that literature). By contrast, the research on position auctions focuses on how heterogeneous firms bid to get a favorable position in a directed search order. However, it treats prices as exogenous so it does not address some critical conundrums of existing results on competition with ordered search. How does firm heterogeneity modify pricing to induce consumers to follow the specified search order? Does a firm selling a product which is less appealing to sample drop its price enough to compensate for this disadvantage when it is placed early in the search order? If early firms price low, how can consumers expect them to bid high enough to hold such favorable positions? We propose a comprehensive analysis of these issues while considerably expanding the dimensions of firm heterogeneity that can be accommodated within a tractable model that should be useful for further applications to firm pricing with consumer search under asymmetries.

Whether a firm will bid more than its rivals to be searched early is impacted by position externalities, which are the effects that a firm has on other firms' profits if it changes its position in the search order. There is an obvious traffic externality, or *business stealing* (Chen and He, 2011), which results from a firm moving ahead in the search order siphoning off customers (who stop there) from following firms. But there is a more subtle reverse impact for firms that are demoted: there is less competition due to the reduced number of firms remaining to be searched. To illustrate, consider the symmetric case, where firms searched later charge higher prices. When a firm moves ahead in the search order, firms jumped over charge higher prices because they are one step closer to the end so it is less attractive for consumers to search on.[1] We call this the *search appeal* externality. These externalities are nuanced when firms have different demand profiles. For example, advancing a popular firm hurts those superseded more than advancing a niche firm. Similarly, the search appeal effect is stronger when consumers get a higher expected surplus with the firm. Our setting provides a precise measure of these externalities and how they are affected by the properties of demand for the various products. It is stripped down to basic component parts that capture the complex manner in which these externalities impact pricing, bidding, and firms' and consumers' welfare. We account for multiple dimensions of firm heterogeneity by isolating factors that affect their pricing and factors that affect their sales.

To capture the heterogeneity in pricing we define a *quality* parameter for each product: all other things equal, a higher quality product is sold at a higher price in any given slot. A firm's profit is impacted directly by its product's quality but others are not, so this is a private value dimension. A second dimension of pricing arises through the distribution of valuations for competing products that are positioned later. In our model, the search appeal of those products is precisely measured by the reservation utility associated with sampling the product as characterized in Weitzman (1979). Increasing a firm's search appeal brings down the prices of all firms that are positioned earlier in the search order. We show that the equilibrium price of a firm increases in step with its product's quality and decreases in the cumulative search appeal of all the products positioned later. This pricing behavior yields a higher surplus for the marginal consumer (who is indifferent between buying and searching on) at earlier slots. As a result, it is always optimal for a consumer to follow the directed search order: pricing ensures that this search order is optimal according to the characterization of Weitzman (1979), no matter what the order is.

A firm's sales are characterized by the product's *market potential* which is the probability that a consumer is interested in buying the product. This potential has a direct impact on the firm's profit by determining the quantity sold to consumers who search up to that firm: a

[1] If all prices are the same as for instance in Athey and Ellison (2011), and if the optimal search rule is myopic as in Weitzman (1979), then the number of remaining firms is irrelevant to the consumer's search decision as long as there is at least one left.

firm positioned in a given slot sells more if its product has a higher market potential. Market potential also induces a negative externality on firms farther down in the search order: the higher the market potential of preceding products, the lower the sales of each firm.

Using the parameters of product heterogeneity we can construct product-specific scores to characterize an optimal ordering of firms to maximize either total profit or total consumer surplus. Total profit maximization is achieved by prioritizing products with larger qualities and search appeals and lower market potentials. This mitigates the adverse effect of the search appeal and business stealing externalities by ensuring that the prices of early firms, which sell more, are as high as possible while the sales of later firms, which extract more surplus, are as large as possible. By contrast, consumers like low prices early, at slots where they are more likely to buy, and low probability of purchasing a product positioned late (which extracts a large surplus).

We characterize two dominance relations between products for which the consumers' preferred order is the reverse of that which maximizes total profit. The first, *monotonic inverse demand dominance*, is a special case of first order stochastic dominance applied to the distribution of valuations for different products: it naturally applies when products have different qualities. The second, *niche dominance*, corresponds to a rotation of the inverse demand for different products along the lines of Johnson and Myatt (2006) and is relevant for products with different market potentials. Firms prefer inverse demand dominant or niche dominant products to be first (depending on which order applies) whereas consumers prefer the reverse.

Consider now a position auction with slots going to firms in the order of their bids and each firm paying the bid of the next highest bidder. We analyze such a *generalized second price auction* assuming bidders have complete information. Whether such an auction mechanism can achieve joint surplus maximization for the auction participants (joint profit maximization) is a priori ambiguous due to the two position externalities. We first show that any order can be sustained when product qualities and market potentials are similar enough. However when product heterogeneity concerns product quality, while there is always an equilibrium with qualities ranked in a decreasing order, a ranking that differs too much from this decreasing quality ranking cannot arise in equilibrium if quality differences are too substantial. Hence, joint profit maximization can always be sustained in equilibrium if products with higher qualities dominate products with lower qualities in terms of monotonic inverse demand dominance. This is a generalization of results in previous literature which show this is true in pure private value settings, corresponding in our model to the case where only product qualities differ.[2] Conversely, we show that the preferred order for consumers can never prevail under monotonic inverse demand dominance if qualities are dissimilar enough.

By contrast, when there is sufficient firm heterogeneity in market potentials, the maximal total profit order is no longer an equilibrium if products can be ranked by niche dominance. Indeed, it would require firms with low market potential bid to be early despite their low value for being in a top slot: the value of the additional clicks they can obtain in such positions is low because these clicks are less likely to be converted into purchases. Furthermore the reverse order, where products with large market potential come first, can always be sustained as an equilibrium outcome. This is the order favored by consumers because a product with a high market potential is niche dominated.

Our paper integrates and enriches two streams of literature. An important insight from the analysis of sequential ordered search is that when prices are endogenously chosen by competing sellers there can be equilibria where firms searched earlier are both more attractive to search and earn more profit, so they might be willing to pay for such prominence.[3] A first step forward is made by Armstrong, Vickers, and Zhou (2009),[4] with only one prominent firm and the remaining sellers searched randomly. The setting where all firms are searched in order is explored by Zhou (2011). However, these papers assume i.i.d. consumer tastes for products. We allow for asymmetric match distributions and show that the marginal consumer's surplus is lower at firms searched later, which implies prices increase along the

[2]See Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007)

[3]Previous literature on sequential search and competition focused on random search and is surveyed in Anderson and Renault (2018). The ordered search models dicussed here build on the setting introduced by Wolinsky (1986) and Anderson and Renault (1999) with search and horizontal product differentiation.

[4]See Arbatskaya (2007) for an earlier contribution with homogenous products.

search order when qualities are the same. Asymmetries in product demands are considered in Armstrong, Vickers, and Zhou (2009) and Song (2017). The former allow for heterogeneous product qualities in an extension of their basic setting with only one prominent firm while the latter considers products with asymmetric taste heterogeneity in a duopoly. Our setting allows for multiple dimensions of product heterogeneity and deals with ordered search among any number of competitors. Choi, Dai, and Kim (2018) and Haan, Moraga-Gonzalez, and Petrikaite (2018) study how firms can use posted prices to direct search. Different consumers follow different search orders based on prior information about how much they like the various products but firms are symmetric in aggregate and charge the same price.

The position auctions literature has made valuable progress on the auction side of the slate while suppressing the market competition side.[5] A first group of articles studies the properties of generalized second price auctions with private values but does not account for consumer behavior[6] or surplus. Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007) focus on the existence of an efficient equilibrium, which maximizes the firms' surplus. Gomes and Sweeney (2014) show that such an equilibrium may fail to exist in a sealed bid auction with asymmetric information. In our setting, a joint profit maximizing equilibrium may not exist even though bidders have complete information. Athey and Ellison (2011) use a setting very similar to Chen and He (2011) to look at auctions with asymmetric information and then optimal auction design, while assuming that consumers go on searching until a "need" is fulfilled. This yields a business stealing externality similar to ours. But there is no externality from search appeal because both price and the conversion rate (the probability a consumer buys conditional on reaching the firm) are exogenous. These two papers establish that it is optimal for consumers to search in the order that arises from the auction because firms with a higher probability of meeting a need bid more. By contrast, in our model it is optimal for consumers to search in the order because of the pricing behavior they expect.

We also contribute to the analysis of auctions with externalities where the bidder will-

[5] Although Chen and He (2011) have endogenous prices, all firms end up charging the monopoly price due to a standard mechanism *à la* Diamond (1971).

[6] An exception is Gomes (2014) who endogenizes consumer click behavior in a two-sided market setting.

ingness to pay is determined in equilibrium and depends on which other bidders are likely to win. That literature has considered auctions for a single object (e.g. Jehiel and Moldovanu, 1996a, 1996b) whereas we allow for any number of slots to be allocated.

Section 2 describes our search and competition environment while optimal ranking scores for maximization of total industry profit and consumer surplus are derived in Section 3. We consider when allocation rules such as auctions used on internet platforms might achieve total profit or consumer surplus maximization in Section 4. Section 5 concludes.

2 Market equilibrium

2.1 Competition with ordered search

We first describe a model of oligopolistic competition with ordered consumer search, and find firms' equilibrium prices. There are n firms with Firm i selling product i, with zero production costs. Consumers have unit demand with independent valuations for the n competing products. Let $F_i(v)$ denote the distribution function of a consumer's valuation with product i, $i = 1, ..., n$. We break down $F_i(v)$ into three component parts.

We are thinking of situations in which consumers idiosyncratically either like the product, or they do not, but they have heterogeneous valuations if they like it. For example, a consumer may reject out-of-hand several styles of jacket, but the lowest valuation for a jacket that she will countenance taking home to take up space in the closet can be quite high. Nonetheless, there may be several jackets that could interest her if she knows their details.[7]

Let then the probability of rejecting the product outright (regardless of price) be γ_i. Lower γ_i products are more popular, per se. Second, let q_i be the lowest valuation associated to product i, conditional on it being desired. As we elaborate below, we shall assume that q_i is sufficiently large that all consumers who have some appreciation for a product end up buying it in equilibrium when they come across it. The valuation for product i, v_i, therefore

[7] This set-up contrasts with Athey and Ellison (2011) in which prices are fixed, and with Chen and He (2011) where each consumer has at most one product that could interest her, regardless of prices.

has support $\{0\} \cup S_i$, where $\min S_i = q_i$ and it is further assumed that v_i is bounded: let then $B_i \equiv \max_{v_i \in S_i} v_i - q_i < \infty$. To facilitate the exposition, it is assumed differentiable on S_i, and f_i denotes the corresponding pdf.[8] Assume that for all products $i = 1, \ldots, n$, there exists $a_i > 0$ such that $f_i(q_i) = a_i$.

Distribution functions F_i are common knowledge but neither consumers nor firms know the realizations of v_i. Consumers may however learn these realizations through search. Search is sequential, with cost $s > 0$ per search. Searching a firm reveals both its price and the consumer's valuation of the product searched. Searching a firm is necessary for a consumer to be able to buy its product. As is standard in sequential search models the consumer may always purchase from any previously searched firm at no extra search cost. Buying none of the n products nets an outside value utility of zero.[9]

The timing is that firms simultaneously choose prices and consumers choose their search rules based on match values and prices they have found out so far, the distributions at other firms and the prices expected there. We seek a Perfect Bayesian Equilibrium at which search is ordered, meaning that all consumers follow the same search order. Because we have not specified any systematic difference between the n firms (i.e. the match distribution for Firm $i = 1, ..., n$ can be any distribution satisfying the properties described above), there is no loss of generality in assuming that this order is from Firm 1 to Firm n and we then check that this order is indeed optimal for consumers. In our equilibrium, each firm optimally prices so as to sell to all consumers that reach it with a positive draw for its product. This pricing property means that Firm i renders any consumer drawing q_i with it indifferent between buying from i and searching further. Thus a consumer has zero willingness to pay for any product encountered before Product i, and never goes back (as long as prices are strictly positive, which will hold true in equilibrium). We show below that such an equilibrium exists, provided that for all i q_i is large enough and the density f_i is strictly positive at q_i.

[8] The analysis goes through without assuming the existence of a p.d.f. on S_i and can accommodate any specification of F_i on S_i including atoms in the distribution.

[9] It can be thought of as a continuation value (searching the organic links of a search engine after searching the sponsored links, or purchasing a product off line) for consumers who have searched through all the n firms: equilibrium prices are simply shifted down by the continuation value.

To derive the equilibrium, we engage the powerful results of Weitzman (1979) to describe optimal consumer search. He shows that remaining search options can be ordered by simple myopic reservation values such that a consumer searches the option with the highest reservation value next, or else stops searching if she already holds a utility above the highest value (and buys the best option held or buys nothing). These reservation values are summary statistics for options, which set equal the expected costs and benefits of an additional search. They are therefore determined independently of what has already been discovered.

A version of these reservation values is a key ingredient of pricing analysis. For each product we define Δ_i as per standard search analysis as the valuation net of quality q_i that equates the expected upside gain to the search cost, so

$$\int_{q_i+\Delta_i}^{q_i+B_i} (v - q_i - \Delta_i) dF_i(v) = s. \tag{1}$$

Integrating by parts, $\int_{q_i+\Delta_i}^{q_i+B_i} [1 - F_i(v)]dv = s$. The LHS strictly increases from 0 to $+\infty$ as Δ_i drops from B_i to $-\infty$. Hence a unique Δ_i always exists. Graphically, the value of Δ_i is determined from the value of the critical valuation $q_i + \Delta_i$ for which the area under the demand curve $(1 - F_i(v))$ equals the search cost s. It is illustrated in Figure 1. For interpretation, if the consumer currently held a utility value of $q_i + \Delta_i$ then searching Firm i would be a break-even prospect if it were expected to charge a zero price. With this interpretation, the values $q_i + \Delta_i$, $i = 1, ..., n$, are the reservation values that characterize consumer optimal search behavior if prices are all zero: from the analysis in Weitzman (1979), a consumer should always choose to search next the remaining alternative with the highest reservation utility $q_i + \Delta_i$ or else stop searching if she already holds a higher utility. If prices were equal for all products, consumers would choose to search in the order in which we have indexed the firms only if $q_i + \Delta_i \geq q_{i+1} + \Delta_{i+1}$, $i = 1, ..., n - 1$.

Assume $\Delta_i > 0$. This implies that if a consumer holds valuation q_i with product i and contemplates searching Firm $i+1$, and if price differences were to exactly match base quality differences (i.e., if $p_i - p_{i+1} = q_i - q_{i+1}$), then s is low enough that her expected benefit from searching Firm $i + 1$ would be strictly positive. We assume throughout the paper that all prices are low enough that the consumer does start search.

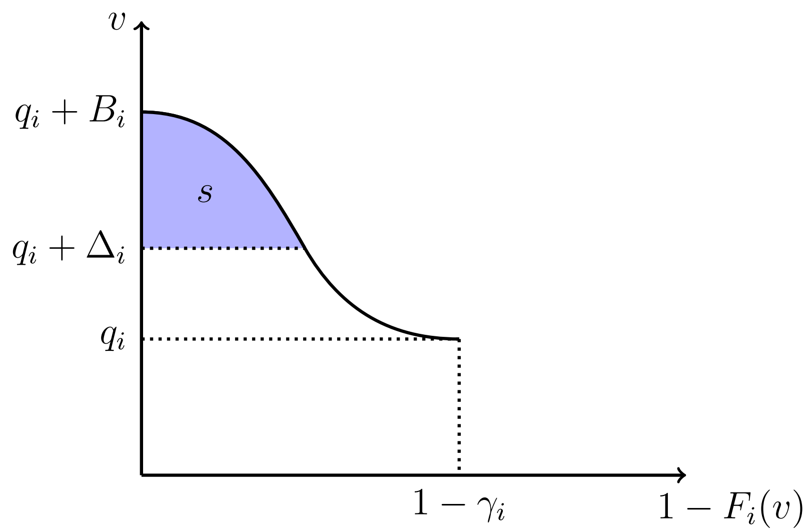

Figure 1: Determination of Δ_i.

For our characterization analysis it is useful that we can vary parameters γ_i, q_i, and Δ_i independently from each other. A change in q_i is merely a shift up or down of the support of strictly positive valuations, S_i, so it can be done independently of the value of γ_i (the probability that the product is not desired.) We show in the Appendix that the LHS of (1) can be rewritten appropriately so that it is possible to modify F_i and have Δ_i vary from 0 to $+\infty$ independently of the values specified for q_i and γ_i.

We now move to characterizing the equilibrium pricing and search order.

2.2 Pricing

Consider a Firm $i < n$, and suppose it prices so that even a consumer who draws a match q_i with its product chooses not to search on. Because consumers follow an optimal search order in equilibrium, they compare the utility they currently hold with the highest reservation value among the remaining firms. In an equilibrium where consumers search in order from 1 to n, this highest reservation value should be that for Firm $i+1$. In other words, as per the Weitzman (1979) analysis, the optimal search rule is myopic and only considers the costs and benefits of searching Firm $i+1$ as if it were the only firm remaining. Because consumers expect utility $v_{i+1} - p_{i+1}$ with Firm $i+1$, the reservation utility, r_{i+1}, associated with Firm

$i+1$ is the unique solution to

$$\int_{r_{i+1}+p_{i+1}}^{q_{i+1}+B_{i+1}} (v - p_{i+1} - r_{i+1}) dF_{i+1}(v) = s.$$

It is immediate from comparing the above condition to (1) that we can write the reservation valuation as $r_{i+1} = q_{i+1} + \Delta_{i+1} - p_{i+1}$. From this observation, we conclude that the largest price, p_i, that Firm $i = 1, ..., n-1$ can charge such that a consumer with match q_i will decline to search Firm $i+1$ satisfies $q_i - p_i = r_{i+1} = q_{i+1} + \Delta_{i+1} - p_{i+1}$. This equality determines the candidate equilibrium pricing rule as

$$p_i = p_{i+1} + q_i - q_{i+1} - \Delta_{i+1}, \tag{2}$$

which therefore determines a first-order difference equation for prices.[10] We now need to find an initial condition, which is the price set by Firm n.

So consider Firm n's problem. It knows it is the last to be searched and that all consumers who get to it in equilibrium have zero valuation for all the other products. It is therefore in a monopoly position. As we do for the other firms, we seek an equilibrium price such that all consumers with valuations of at least q_n buy Firm n's product. The largest price n can charge which is consistent with all such consumers buying is $p_n = q_n$. By applying the recursive price relation (2) it follows by induction that equilibrium prices are

$$p_i = q_i - \sum_{j=i+1}^{n} \Delta_j, \quad i = 1, ..., n. \tag{3}$$

We establish below that this pricing sequence induces consumer search in the specified order. We now show (proof in Appendix) that the pricing behavior described above is indeed profit-maximizing as long as q_i is sufficiently large and under the mild condition that $f_i(q_i) > 0$ for all i.[11]

[10] Equation (2) highlights the role of competition between two neighboring firms: because $\Delta_{i+1} > 0$, Firm i must factor in search appeal on top of the quality difference in order to retain all interested consumers. Indeed, if we had $\Delta_i \leq 0$ for all i, then the equilibrium outcome would be in line with that of Diamond (1971) where each firm would charge its monopoly price q_i. The ensuing analysis of GSP auctions would then be analogous to previous work by Chen and He (2011) and Athey and Ellison (2011) if $q_i = q$ for all i, or Varian (2007) and Edelman et al. (2007) if $\gamma_i = \gamma$ for all i.

[11] The result also exploits the fact that B_i is finite.

Lemma 1 *If consumers search optimally from Firm 1 to Firm n expecting all the firms to price according to (3), then it is optimal for any Firm i to charge price p_i defined by (3) as long as q_i is sufficiently large, $i = 1, ..., n$.*

The intuition for the above result is the following. If all Firms $j > i$ charge p_j defined by (3), then Firm i faces a demand which is completely inelastic up to p_i defined by (3): for all prices up to this level, it serves all consumers with strictly positive valuation for its product and has demand $1 - \gamma_i$. If its price satisfies (3), then the marginal consumer has valuation q_i. Hence i's demand derivative for a price increase (the right derivative) is $f_i(q_i)$. If this is strictly positive, then with a large enough price (i.e. for a large enough q_i) the corresponding price elasticity is above one. A small price deviation is therefore unprofitable. Furthermore, the upper bound on the support of valuations $q_i + B_i$, implies an upper bound on a potentially profitable price increase and if the equilibrium price is large enough, then the relative price increment is too small to compensate for a resulting drop in demand which is not arbitrarily small (so that large price deviations cannot be profitable either).

Lemma 1 establishes that firms do not wish to deviate if all other firms price according to (3). If instead some Firm i expected the next firm to price much lower, i would not respond with a price so low that all those consumers with $v_i > 0$ want to buy. Instead, some of the consumers who would choose to search on will come back. Our pricing analysis couples with our demand model to enable us to avoid such returning consumers in equilibrium and thus avoid one of the main tractability problems with ordered search models. This device then allows us to incorporate multiple dimensions of demand heterogeneity.

To establish the equilibrium, we therefore merely need to verify that the specified search order is optimal for consumers, that is, $r_i \geq r_{i+1}$ for $i = 1, ..., n-1$. Recall that $r_i = q_i + \Delta_i - p_i$ which, from the pricing expression (3), implies $r_i = \sum_{j \geq i} \Delta_j$. So r_i indeed monotonically decreases as i increases. Hence qualities net of prices fall along the search order (so if all qualities are the same, prices rise). However, as we now argue, each firm's equilibrium profit is higher at an earlier position if the assumptions of Lemma 1 hold. That is, when the price equilibrium is ensured, an earlier slot is worth more to each firm (and this property is a key one for the auction analysis).

Formally, we show that Firm i's equilibrium profit (i.e. its profit when placed in slot i) is larger than what its profit would be in an equilibrium where it is searched in position $i+1$ while Firm $i+1$ is promoted to position i and all other firms stay put. First note that Firm i's equilibrium profit is at least as large as what it would earn if it deviated to charging the price it would charge in the equilibrium where it is in position $i+1$ (from (3) this would mean increasing its price by Δ_{i+1}). To establish the result, it thus suffices to show that this deviation profit is strictly larger than that of Firm i in the alternative equilibrium where it is in slot $i+1$.

The two situations to be compared have Firm i charging the same price, so establishing the result requires comparing the demands. In either case, Firm i only sells to consumers who have zero match with all products before $i-1$ and hence they search at least up to slot i. Among those, in the equilibrium where i is in slot $i+1$, i's demand comprises of all the consumers drawing zero with Firm $i+1$ (which they search before) and drawing at least q_i with Firm i. In the alternative situation where Firm i is in slot i but deviates up from equilibrium pricing, all consumers who then search Firm $i+1$ would still buy from Firm i if they get a 0 with product $i+1$: this is because the deviation price (which is also the price in the alternative equilibrium order), ensures that none of them wants to search on to slot $i+2$. It follows that all the consumers who buy in the alternative equilibrium order also buy in the original order at the deviated price. In addition, some of the consumers who choose to search on after the price deviation but find out their valuation with product $i+1$ is close to q_{i+1} will return to buy product i because their match with i is substantially above q_i. This is because the deviation price for product i net of quality q_i is equal to the price of product $i+1$ in that same equilibrium order net of quality q_{i+1}. Hence the demand for Firm i if it deviates up in price when it is in slot i is strictly larger than its equilibrium demand if it is moved down to slot $i+1$.

We summarize with the following Proposition.

Proposition 1 *Under the assumptions of Lemma 1, there exists an equilibrium where con-*

sumers search firms in the order of the firm index, $i = 1, ..., n$ and Firm i charges a price given by (3): $p_i = q_i - \sum_{j=i+1}^{n} \Delta_j$ with equilibrium demands $D_i = (1-\gamma_i) \prod_{j<i} \gamma_j$; $i = 1, ..., n$. Furthermore, each firm's profit is strictly higher at any equilibrium in which it is placed earlier.

The pricing sequence in (3) bears the hallmark property that firms that are searched early on extract less surplus from consumers than firms that are searched later. This property is needed for consumers to search in the stipulated order, and is consistent with the results in previous studies of products with symmetric match distributions (Armstrong *et al.*, 2009, Zhou, 2011 and Armstrong, 2017) that early firms charge lower prices.[12]

The equilibrium price has two components: a private value measured by q_i and an externality from the remaining firms in the search order measured by $\sum_{j>i} \Delta_j$. The private value is a surplus associated with the consumption of the firm's product, for which it captures any additional dollar through its price. The firm cannot however capture the entirety of this surplus because of the downward pressure on its price resulting from the option consumers have to search on to the firms down the line. The amount by which price is lowered can be interpreted as the total *search appeal* of the remaining products to be checked out by the consumer. Because of the "myopic" search rule used by the consumer, only the search appeal of the next product down, Δ_{i+1}, is directly relevant for Firm i's pricing. However, Firm i must also take into account the pricing behavior of Firm $i+1$, which depends on the search appeal of Firm $i+2$. This is why the total search appeal externality imparted on a firm is the cumulative search appeal of all the remaining firms.

In the benchmark case where firms all have identical product match distributions (so that $\Delta_i = \Delta$ for all i) the search appeal externality is merely $(n-i)\Delta$ for Firm i, which only depends on the *number* of firms following Firm i and prices step up by Δ from one firm to the next. In our setting where products are *ex ante* heterogenous, the externality also depends on the *identity* of the remaining firms. This property is key to the welfare analysis

[12] Armstrong (2017) shows this when the cdf of product valuations is log-concave. Other authors have this result in contexts with very specific asymmetries: merged and not merged products in Moraga-Gonzalez and Petrikaite (2012) or products with different degrees of match heterogeneity in Song (2013).

of the optimal ordering of firms for maximal total profit or consumer surplus. It also has important implications for the firms' willingness to pay to be searched earlier rather than later.

The search appeal externality reflects the competition that a firm faces from the following firms. But a firm does not directly compete in price with the preceding firms. This is because consumer search behavior factors in only the expected price at remaining firms: a firm has no way to steal customers from its predecessors by committing to a lower price than expected. However, preceding firms do affect Firm i's profit because they price so that all consumers who have a positive valuation with at least one earlier firm stop searching before reaching Firm i. As a result, only a fraction $\prod_{j<i} \gamma_j$ reach Firm i. This constitutes a *business stealing* externality which was previously analyzed by Chen and He (2011) and Athey and Ellison (2011) in models where prices are exogenous, or effectively so. If firms were *ex ante* symmetric, with $\gamma_i = \gamma$ for all i, market stealing for Firm i would depend only on the number of predecessors and the fraction of consumers reaching Firm i would merely be γ^{i-1}. Again, the identity of firms that are searched prior to Firm i becomes relevant once match distributions differ across products. Our analysis introduces novel insights for the interaction between business stealing and pricing, as shown in Section 3, where we show the tensions for consumers and firms among different search orders when there are position externalities. In Section 4 we endogenize the search order through a position auction.

We conclude this section with two points about the difference between equilibrium and socially optimal search. First, at this juncture (i.e., before we endogenize the search order via the position auction), any search order can constitute an equilibrium (Proposition 1). Yet the (first-best) optimum entails searching in decreasing order of the reservation values $q_i + \Delta_i$. To see this, first note that the social optimum entails pricing at marginal cost (here zero). The search problem for social welfare maximization is to achieve the best possible gross valuation net of search costs. The definition of Δ_i implies that the reservation value associated with searching Firm i is $q_i + \Delta_i$ - and hence the optimal order follows this statistic in decreasing order. The consumer should search i if she holds a lower value than $q_i + \Delta_i$.

Second, even for any given search order, *equilibrium search* (constrained by having to follow the stipulated search order) *is too low* compared to the social optimum. This result follows because equilibrium pricing chokes off further search as soon as the consumer gets a positive match value. Instead, a consumer should optimally search Firm i whenever her match is below $q_i + \Delta_i$. Then the benefit from searching Firm i alone is enough to justify incurring search cost s and, whenever the search order is suboptimal, there is also an option value from being able to search on beyond Firm i.[13] In equilibrium, the consumer searches Firm i if and only if her match is zero so that, if $q_{i-1} < q_i + \Delta_i$, then there is not enough search in equilibrium (regardless of whether the search order is optimal) because a consumer holding q_{i-1} at Firm $i-1$ should search but does not due to Firm $i-1$'s price discount.

3 Optimal rankings

The results of the previous Section indicate that ANY order of search can be sustained as an equilibrium to the game in which consumers follow their optimal search protocol and firms set their prices. Prices though differ across these equilibrium search orders when firms are asymmetric, and so the search order matters for various measures of market performance. Typically, the optimal order varies by market performance measure. We here determine the optimal orders, given equilibrium search and pricing, for total industry profit and consumer surplus. For short, call these TIP and CS respectively.

A priori, this is a complicated problem because position order affects all prices and search probabilities: with n active firms there are $n!$ configurations to compare. Nevertheless, the structure of our model delivers a simple and clean characterization for the optimal order under each criterion. The optimal order is described by ordering firms according to a simple summary statistic, which is different for each surplus criterion.

The idea is as follows. For any neighboring pair of firms, A and B, in slots i and $i+1$ respectively, (and for each surplus criterion), we can find a summary statistic Φ_k for Firm

[13] This option value is zero when the search order is optimal, as reflected in the myopia property of the optimal search rule.

k such that the maximand (CS or TIP) evaluated in these two slots is higher if $\Phi_A \geq \Phi_B$. Crucially, the summary statistics are derived solely from parameters of the match distribution of the corresponding product, F_A for Φ_A and F_B for Φ_B. Hence they do not depend on which two slots are flipped (e.g., first and second or fifteenth and sixteenth). The key property of our model is that such a flip affects the welfare objective only through the joint impact in the two consecutive slots: the welfare in all the other slots only depends on the joint externality that the two firms exert, either because they are in front (the business stealing externality) or because they come later in the search order (the search appeal externality that affects prices in those earlier slots). Thus, with $\Phi_A \geq \Phi_B$, A being in front of B (rather than the reverse) yields a higher welfare criterion computed over all the n slots. Clearly, a necessary condition for a maximum is that flipping the order of the two firms in each successive pair does not strictly increase the desired objective function. Because the flipping rule is independent of the positions i and $i+1$ to be flipped, this criterion induces an ordering of firms based on the indices Φ_k as claimed above. Put another way, any alternative order, with at least one pair of consecutive firms violating the pairwise flip condition, cannot be an optimum. Thus the ranking of firms by the size of their summary statistics is a necessary condition for optimality. It is also sufficient because, if there are no ties among firms in the sufficient statistics Φ_k, there is only one such order out of a finite set of possible configurations, and if there are ties, flipping two consecutive firms that are tied leaves the objective unchanged so that the multiple solutions obtained by ranking according to Φ_k are all optimal.

We now derive the summary statistics for the different criteria and give the intuition.

3.1 Total Industry Profit (TIP)

The profit for the firm in position i is

$$\pi_i = (q_i - \kappa_i)\,\lambda_i\,(1-\gamma_i)\,, \quad i = 1, ...n, \tag{4}$$

where we have defined $\kappa_i = \Sigma_{j>i}\Delta_j$ as the sum of all later price steps (where κ_n is taken to be zero), and $\lambda_i = \prod_{j<i}\gamma_j$ for $i > 1$ as the probability that a consumer has no interest in any of the previous products (and we let $\lambda_1 = 1$). The term in the first parenthesis in (4) is the equilibrium price (3) and it is multiplied by the probability that the consumer ends up searching Firm i, λ_i, and then buying product i, $1-\gamma_i$.

As explained above, to find the maximum TIP we just need to look at the change in profit from switching Firms A and B between slots i and $i+1$. So A precedes B as long as

$$\pi_A^i + \pi_B^{i+1} \geq \pi_B^i + \pi_A^{i+1}, \tag{5}$$

where π_k^i denotes the profit of Firm k when it is in slot i. Writing this out for our model,

$$(1-\gamma_A)(1-\gamma_B)(q_A - q_B) + (1-\gamma_B)\Delta_A - (1-\gamma_A)\Delta_B \geq 0. \tag{6}$$

To derive this, first notice that we can divide through by the total number of consumers who search up to slot i, i.e., λ_i, and then the terms in all prices after $i+1$ (i.e., κ_{i+1}) cancel out. Importantly, the condition is independent of the position in the overall order of the two slots that are switched.

Dividing through (6) by $(1-\gamma_A)(1-\gamma_B)$ delivers the TIP summary statistics such that A should precede B (in any consecutive pair and hence in the global maximum) as long as

$$\Phi_A^\pi \equiv q_A + \frac{1}{1-\gamma_A}\Delta_A \geq q_B + \frac{1}{1-\gamma_B}\Delta_B \equiv \Phi_B^\pi.$$

The TIP summary statistic is readily apparent from this inequality, and is given next:

Proposition 2 *An order of firms maximizes Total Industry Profit if and only if it follows the ranking of the summary statistic*

$$\Phi_k^\pi \equiv q_k + \frac{1}{1-\gamma_k}\Delta_k \tag{7}$$

and firms should follow a decreasing order of the Φ_k^π. Ceteris paribus, higher q_k, Δ_k, and γ_k should go earlier in the order.

This result and the discussion below require that the parameters γ_k and Δ_k can be moved independently. Claim 1 in the Appendix shows this can be done by modifying F_k appropriately.[14]

[14]The discussion following Proposition 3 describes why we cannot characterize the optimal order for consumer welfare via independent rankings of γ_k and Δ_k without further assumptions.

To understand this result, recall that, in equilibrium, firms that are early in the search order sell more but extract less consumer surplus (they have deeper quality discounts), whereas firms that come later sell less but extract more consumer surplus because their prices are closer to their qualities. TIP maximization is achieved by ensuring that firms that sell more extract as much surplus as possible and firms that extract the most surplus sell as much as possible. The first objective is achieved by having firms with a large quality q_k and a large search appeal Δ_k searched early. Having firms with least popular products (large γ_k) searched first serves the second goal.

One way to see these effects clearly in isolation is by looking at each as the sole source of heterogeneity (so the other parameters are set the same for all firms). A large quality ensures that there is much potential consumer surplus to be extracted by the sellers of such products, which should therefore have the most consumers sampling them. Notice that the quality effect is NOT an externality on the other firms.

High Δ firms cause low prices on all those which precede them. Switching a high-Δ firm with a low-Δ one that was initially earlier, raises the prices for all the firms in between the two slots, and so raises total profits. The idea of stacking up early all the high-Δ firms is to "clear-the-decks" of them to suppress their shadow on all prices that come earlier, which they would otherwise bring down. Put another way, having the firms that are most appealing to search early mitigates the search appeal externality they impart: they can keep their prices relatively high because consumers are not too eager to search the remaining firms.

Finally, it may seem surprising that firms with less popular products (large γ_k) should be presented first to consumers because these firms are less likely to make a sale. However, early slots have low prices, so the ranking uses up these slots on less likely prospects.[15] Firms that extract the most surplus from their customers have larger sales if the business stealing externality from earlier firms is limited. Both this feature and the search appeal externality already suggest that consumer surplus may run the opposite way from TIP, a property that

[15]For example, suppose there were two firms, and $\gamma_A = 0.1$ while $\gamma_B = 0.9$. Then the number of consumers who buy constitute 91 percentage points, regardless of the order of search. Having A first entails 89% buying at the high price, while B first means only 1% do.

is confirmed in broad-brush terms.

3.2 Consumer Surplus

We now consider the pairwise ranking condition for consumer surplus (given equilibrium firm pricing). Let p_k^i be Firm k's price when in slot i. Conditional on reaching this slot, the expected consumer surplus from Firm k in slot i is $-s+\int_{q_k}^{q_k+B_k} v-p_k^i dF_k(v)$ or, substituting s from (1), $\int_{q_k}^{q_k+\Delta_k} v-p_k^i dF_k(v)+(q_k+\Delta_k-p_k^i)(1-F_k(q_k+\Delta_k))$. Integrating by parts and then using $F_k(q_k)=\gamma_k$, this surplus can be written as

$$(1-\gamma_k)(q_k-p_k^i)+\int_{q_k}^{q_k+\Delta_k} 1-F_k(v)dv.$$

The pricing rule (3) gives $p_k^i=q_k-\Delta_{i+1}-\kappa_{i+1}$ and $p_k^{i+1}=q_k-\kappa_{i+1}$ where $\kappa_{i+1}=\Sigma_{j>i+1}\Delta_j$ denotes the sum of later price steps. Using this pricing equation to replace $q_k-p_k^i$ and $q_k-p_k^{i+1}$ respectively, allows us to write consumer surplus from Firm k in slot i as

$$(1-\gamma_k)\left(\Delta_{i+1}+\kappa_{i+1}\right)+\int_{q_k}^{q_k+\Delta_k} 1-F_k(v)dv \tag{8}$$

and consumer surplus from Firm k in slot $i+1$ as

$$(1-\gamma_k)\kappa_{i+1}+\int_{q_k}^{q_k+\Delta_k} 1-F_k(v)dv.$$

Figure 2 illustrates the components of surplus in (8) (firm surplus is $p_k^i(1-\gamma_k)$ in the figure). The term $\Delta_{i+1}+\kappa_{i+1}$ is the quality-price discount Firm k must offer to keep on-board all its consumers not drawing a zero match value, while ω_k corresponds to the integral in (8), which is the consumer surplus over the interval of valuations above q_k net of quality q_k and net of search cost s.

Hence, letting $\omega_k=\int_{q_k}^{q_k+\Delta_k} 1-F_k(v)dv$, consumer surplus from Firm k in slot i is $(1-\gamma_k)\left(\Delta_{i+1}+\kappa_{i+1}\right)+\omega_k$; it is $(1-\gamma_k)\kappa_{i+1}+\omega_k$ from Firm k in slot $i+1$. Consider now the consumer surplus from searching A then B (conditional on having reached A at some position i). Comparing it with the converse order using the analogous expression (switching subscripts) for the sequence BA, search order AB is preferable if and only if

$$(1-\gamma_A)\,\Delta_B+\omega_A+\gamma_A\omega_B\geq(1-\gamma_B)\,\Delta_A+\omega_B+\gamma_B\omega_A$$

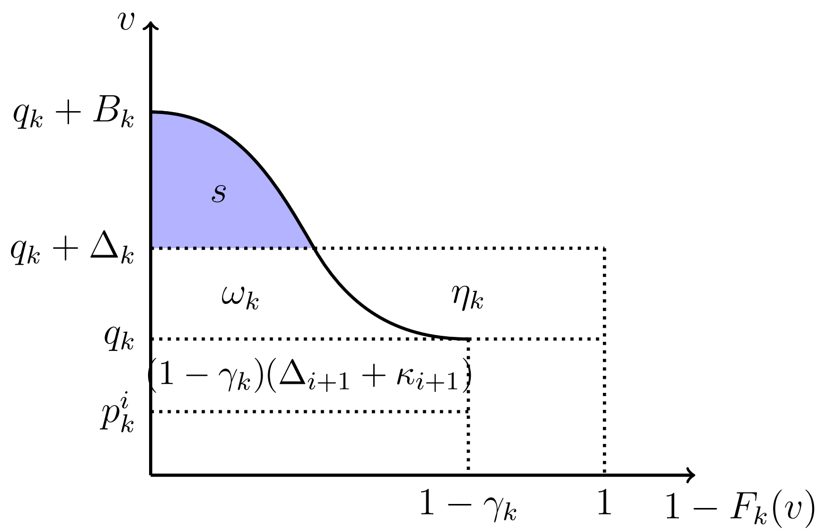

Figure 2: Consumer Surplus from Firm k in slot i and determination of η_k.

where the κ_{i+1} terms all cancel out because they are common to both firms' prices: hence the relevant calculus only depends on the identity of the two firms involved. Rearranging, A should precede B as long as

$$\frac{\omega_A - \Delta_A}{1-\gamma_A} \geq \frac{\omega_B - \Delta_B}{1-\gamma_B}$$

or

$$\Phi_A^{CS} \equiv \frac{-\eta_A}{1-\gamma_A} \geq \frac{-\eta_B}{1-\gamma_B} \equiv \Phi_B^{CS}$$

where we have defined $\eta_k \equiv \Delta_k - \omega_k$ (see Figure 2) and so

$$\eta_k = \int_{q_k}^{q_k+\Delta_k} F_k(v)dv \geq 0. \tag{9}$$

The implication for the summary statistic follows.

Proposition 3 *An order of firms maximizes Consumer Surplus if and only if it follows the ranking of the summary statistics*

$$\Phi_k^{CS} \equiv \frac{-\eta_k}{1-\gamma_k} < 0 \tag{10}$$

and firms should follow a decreasing order of the Φ_k^{CS}. The q_k value is irrelevant whereas higher η_k and γ_k should go later in the order, ceteris paribus.

The ranking of product qualities is irrelevant to consumer welfare because any quality advantage is entirely reflected in a higher price for the product. The relevant product dimensions for consumer welfare can be interpreted as follows. All else equal, consumers retain more surplus for products placed early in the search order because of pricing and also because they expend less search costs to get to them. Hence, the maximization of consumer surplus requires that the likelihood that the early products are bought is as large as possible, *ceteris paribus*, which is ensured by having low-γ products first. Second, the surplus enjoyed by consumers at those early slots conditional on buying should also be as large as possible, *ceteris paribus*, which means that products with a large Δ_k should come later so early products are cheaper, and/or the expected consumer valuation in excess of quality should be higher for early products. The term η_k encapsulates both these criteria.

The characterization in Proposition 3 might suggest that consumers prefer earlier products to have lower values for Δ_k (because a lower Δ_k means a lower η_k by (9)), and lower values for γ_k. Both effects are the reverse from what maximizes firms' total profit. However, while we know from Claim 1 that these two parameters can be adjusted independently from one another by modifying the distribution of consumer valuations appropriately, these modifications will also impact $\eta_k = \int_{q_k}^{q_k+\Delta_k} F_k(v)dv$ through the change in F_k. Hence, the two parameters cannot be moved independently *ceteris paribus* and we need to account for these changes in F_k to characterize the best product ranking for consumers solely in terms of Δ_k and γ_k so it can be readily compared to the TIP-maximizing ranking. First, we want to compare products with different market potentials, i.e. with different γ_k, but with the same search appeal, $\Delta_k = \Delta$ for all k. Second, we want to compare products with identical market potentials, $\gamma_k = \gamma$ for all k, and different search appeals, Δ_k. Delivering usable structures for both these criteria also yields valuable insights into some natural correlations between different dimensions of product demand.

To perform both comparisons we define $\bar{F}_k$ as the distribution function of $v_k - q_k$ conditional on $v_k \geq q_k$. As already pointed out in the proof of Claim 1 in Appendix A1, for any $v_k \geq q_k$, $F_k(v_k) = \gamma_k + (1-\gamma_k)\bar{F}_k(v_k - q_k)$. Then, because $\bar{F}_k < 1$ on $[0, B_k)$, an increase in

γ_k increases $F_k(v_k)$ for all $v_k \in [q_k, q_k + B_k)$. Furthermore, if $\Delta_k > 0$ it satisfies

$$(1-\gamma_k)\int_{\Delta_k}^{B_k} 1 - \bar{F}_k(\delta)d\delta = s, \tag{11}$$

so that, keeping $\bar{F}_k$ fixed, Δ_k is decreasing in γ_k. This reflects the intuition that a product for which it is less likely that the consumer's valuation is non-zero is less attractive to search.

Consider first products that are equally appealing to search, so $\Delta_k = \Delta$ for all k. From Claim 1 we know there is a way to modify $\bar{F}_k$ so as to keep $\Delta_k = \Delta$ while modifying the probability γ_k. As can be seen from equation (11) and following the argument in the proof of Claim 1, the change in $\bar{F}_k$ need not concern values of δ in $[0, \Delta]$. Now, for a given specification of $\bar{F}_k$ on $[0, \Delta]$, an increase in γ_k will induce an increase in F_k on $[q_k, q_k + \Delta]$ so that $\eta_k = \int_{q_k}^{q_k+\Delta_k} F_k(v)dv$ will be larger. It follows that a larger γ_k is associated with a lower index Φ_k^{CS} and we have the following result.

Proposition 4 *Assume $\Delta_k = \Delta$ and $\bar{F}_k(\delta)$ is the same for all k on $[0, \Delta]$, $k = 1, ..., n$. then consumer surplus is maximized by positioning products with higher market potentials earlier, i.e. γ_k should be smaller for earlier products.*

We now turn to a situation where products are equally popular, $\gamma_k = \gamma$, for all k. As already pointed out, if products with large search appeal are placed later, prices at earlier slots will be lower, which is desirable for consumers. This however is combined in the value of η_k with some other considerations pertaining to the expected consumer valuation between q_k and $q_k + \Delta_k$ and these values are higher in expectation for products with more search appeal: on this basis, keeping pricing fixed, consumers would prefer to access these products earlier in the search order. This is why there is no straightforward relation between search appeal and η_k and hence, between search appeal and the product index that determines the optimal ordering for consumers, ϕ_k^{CS}. However, we now draw on standard results on the impact of stochastic dominance on optimal search to establish a correlation between η_k and Δ_k that can easily be interpreted.

A standard situation where reservation utilities across search options are readily ranked is when utility distributions can be compared in terms of first order stochastic dominance

(FOSD). Here we exploit this idea while keeping a common γ for all products and applying a dominance relation to the inverse of $\bar{F}_k$.

Because F_k has a strictly positive density on $[q_k, q_k + B_k]$, $\bar{F}_k$ does as well for values on $[0, B_k]$, and $1 - \bar{F}_k$ is strictly decreasing and differentiable on $[0, B_k]$, so it admits an inverse V_k, which is strictly decreasing and differentiable on $[0, 1-\gamma]$. Note that $q_k + V_k$ can be viewed as an inverse demand for product k because for any probability $x \in [0, 1-\gamma]$, $q_k + V_k(x)$ is the highest price at which the product can be sold with a probability of at least x.

We now define two dominance orders across products.

Definition 1 *Assume $\gamma_i = \gamma_j = \gamma$. Then: (a) product i **dominates** product j in inverse demand if for all $x \in [0, 1-\gamma]$, $V_i(x) \geq V_j(x)$; (b) product i **monotonically dominates** product j in inverse demand if for all $x \in [0, 1-\gamma]$, $V_i(x) \geq V_j(x)$ and $V_i^{'}(x) < V_j^{'}(x)$.*

For identical qualities, $q_i = q_j$, inverse demand dominance means that product i's inverse demand is higher than product j's on $[0, 1-\gamma)$ (they are both q at $x = 1-\gamma$). It can be shown that $V_i > V_j$ implies $\bar{F}_i$ FOSD $\bar{F}_j$ which implies that $\Delta_i > \Delta_j$.[16] Monotonic inverse demand dominance means that V_i goes down faster so the two curves become closer as x increases. Combined with FOSD, it implies that $\eta_i > \eta_j$. While inverse demand dominance of V_i guarantees that Δ_i is larger thanks to FOSD, monotonicity ensures it does so without increasing consumer surplus too much, so the impact of search appeal on prices is the main driver of the ranking of the η's.[17] Combining this correlation with Proposition 3 we have the following result.

Proposition 5 *Assume $\gamma_k = \gamma$ for all $k = 1, ..., n$ and products can be ordered by monotonic inverse demand dominance. Then consumer surplus is maximized by positioning products with higher search appeal earlier.*

[16] The two properties are actually equivalent.

[17] A positive correlation between the Δ and η parameters also arises if products i and j are such that $\gamma_i = \gamma_j$, and $E(v_i - q_i|v_i) \geq E(v_j - q_j|v_j)$: then $\Delta_i > \Delta_j$ implies $\eta_i > \eta_j$.

3.3 Conflicting preferences between firms and consumers

By explicitly deriving endogenous prices and considering *ex ante* heterogeneous products, we highlight a conflict between the orders firms and consumers prefer, which is not present in previous literature. Here we characterize two dimensions of heterogeneity in product demands which naturally lead to opposite rankings for maximization of total industry profit and consumer welfare. Both exploit correlations between the demand parameters that are likely to arise along certain dimensions of product heterogeneity.

The first concerns products with different qualities for which it is natural that the match distributions can be compared in terms of FOSD. Here we assume that higher quality products monotonically dominate lower quality products in terms of inverse demand (see Definition 1). Proposition 5 shows that (for products with identical market potentials) this assumption will rank products by search appeal with the same ranking for the consumer welfare parameter η_k. The following assumption then allows us to correlate these key dimensions of optimal rankings with product quality.

Assumption 1 *Qualities are all different and higher quality products monotonically dominate lower quality products in inverse demand.*

It is immediate from our results earlier in this section that this assumption drives opposing ranking preferences when market potentials are the same. Proposition 5 says that, when products are ordered in terms of monotonic inverse demand dominance and regardless of the quality ranking, consumers prefer products with low search appeal early. These are the products with dominated inverse demands. Hence, under Assumption 1, they want low quality products to be positioned at the start of the search order. For firms, Proposition 2 indicates that their joint profit is largest when high quality products with high search appeals are earlier. From the proof of Proposition 5, Δ_k is larger for inverse demand dominant products. Hence, if Assumption 1 holds, high quality products have more search appeal and firms would prefer that they are positioned earlier. The following proposition summarizes the above conclusions.

Proposition 6 *Assume common market potentials. Under Assumption 1, total industry profit is maximized with a decreasing order of qualities, $q_i > q_{i+1}$, $i = 1, ..., n-1$ while consumer surplus is maximized with an increasing order of qualities, $q_i < q_{i+1}$, $i = 1, ..., n-1$.*

Next we consider products with different market potentials so γ's are different and we introduce a ranking that relies on the concept of a niche product for which demand is limited for low prices but a significant fraction of consumers are willing to pay high prices. Johnson and Myatt (2006) characterize such products by using a rotation of the inverse demand, where inverse demand is steeper for a product that is "more niche". In our setting, this means that demand is relatively small for prices in the neighborhood of q but the probability that valuations are large is high. Formally we have the following definition.

Definition 2 *Product i niche dominates product j if $q_i \geq q_j$, $\gamma_i > \gamma_j$, and $F_i(q_i + \delta) > F_j(q_j + \delta)$ for $\delta < \Delta_j$ and $F_i(q_i + \delta) < F_j(q_j + \delta)$ for $\delta > \Delta_j$.*

This is most clearly described for $q_i = q_j = q$ (which is the case we shall deploy in the auction analysis below) and is illustrated in Figure 3.[18] The demand for product i results from pivoting the demand for product j clockwise around the point with vertical coordinate $q + \Delta_j$.[19]

Niche dominance induces a crisp characterization of the correlation between the four parameters (γ, q, Δ, and η) which are relevant for optimal rankings. This is stated in the following Lemma.

Lemma 2 *If product i niche dominates product j, then $\Delta_i > \Delta_j$ and $\eta_i > \eta_j$.*

The intuition for the comparison of Δ_i and Δ_j can be seen from Figure 3. For the comparisons of η_i and η_j, recall that η takes into account both how the search appeal impacts

[18] The definition allows the quality of the niche dominant product to be weakly higher to allow for the possibility that it is sold at a higher price.

[19] The definition is restrictive in requiring that the pivot is at $q_i + \Delta_j$. Our results below still hold if the pivot is above that point provided a condition comparing expectations holds. A sufficient condition is that $(1 - \gamma_i)E\delta_i \geq (1 - \gamma_j)E\delta_j$.

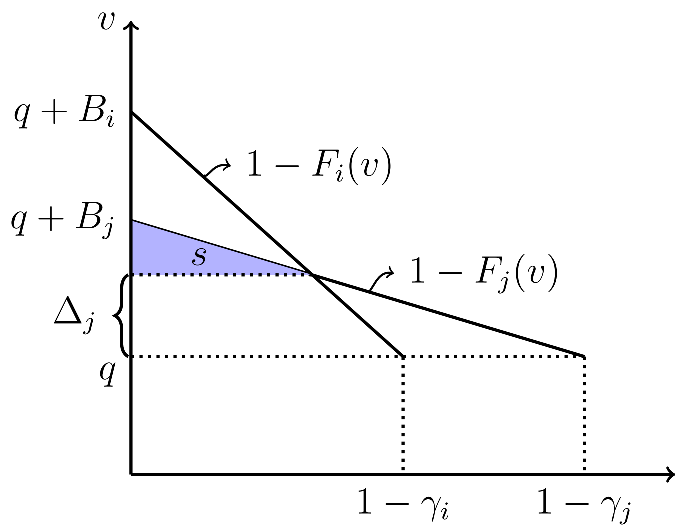

Figure 3: Product i niche dominates product j. The figure shows that $\int_{\Delta_j}^{B_i} 1-F_i(q+\delta)d\delta > s$ so that we must have $\Delta_i > \Delta_j$.

prices and the expected surplus of consumers in excess of q, measured by ω in the analysis leading to Proposition 3. Because $\Delta_i > \Delta_j$, the pricing effect increases η (so consumers want a more niche product i to come later). Distribution F_i being larger in the neighborhood of q, the probability that the match is close to q is increased. Hence the surplus measure is lower for product i, which also induces a lower η for product i. All other things equal, it should be placed later to optimize consumer welfare.

Now, because Definition 2, specifies that niche dominant products have a weakly higher q and a strictly higher γ, Lemma 2 along with Propositions 2 and 3 imply that firms prefer niche dominant products first while consumers prefer niche dominant products later.

Proposition 7 *Assume heterogeneous market potentials and products can be ordered according to niche dominance (Definition 2). Total industry profit is maximized with a decreasing order in niche dominance so* $\gamma_i > \gamma_{i+1}$, $i = 1, ..., n-1$ *while consumer surplus is maximized with an increasing order in niche dominance so* $\gamma_i < \gamma_{i+1}$, $i = 1, ..., n-1$.

These results, both for asymmetric qualities with inverse demand dominance and asymmetric market potentials with niche dominance, contrast with the previous literature. Our setting with different qualities can readily be compared with those of Varian (2007) and Edelman, Ostrovsky and Schwartz (2007). If we set $\Delta = 0$ for all products, we obtain a private value framework where Firm i charges price q_i, and so joint profit maximization requires that those with a high q_i are positioned at the top so as to sell more. This reflects the analysis in those papers but says nothing about which ranking consumers would want: we have seen that consumers do not care about the ranking of qualities in our model, and the two previous models provide no basis for thinking about what would be best for consumers. By exploring the properties of the match distribution above q and how they might be related to the product's quality through inverse demand dominance, we highlight some potential sources of disagreements between the two sides of the market.

The result on niche ordered products is also very different from those in the earlier work by Athey and Ellison (2011) and Chen and He (2011). The models in these papers can be construed as settings where products have different market potentials and are all sold at the same price: this would happen in our analysis if all qualities are the same and search appeals are zero. In Athey and Ellison (2011), the preferred order for both consumers and firms is that the most popular products (low γ) are first. The order matters in their model because they assume that search costs are heterogeneous and some consumers stop searching whereas they could have purchased a product that they like. This does not happen in our setting where the order matters because different products have different prices: these prices depend on the slot at which the product is on offer and consumer surplus depends both on the purchased product and the slot at which it is purchased. If we introduce heterogeneous search costs in our search environment, the optimal order for TIP maximization becomes ambiguous but the optimal order for consumer surplus would be unchanged.[20]

[20] In Chen and He (2011), all consumers have the same search cost and the order of products is irrelevant for TIP or CS maximization in the early slots for which they assume a low search cost. However, it is preferable that these early slots are occupied by the most popular products.

4 Auctions

In this section we explore whether an allocation rule that relies on the firms' private incentives can implement a "desirable" outcome. Our characterization of optimal rankings shows the search order that maximizes total industry profit is not necessarily the most attractive for consumers. Our goal here is to investigate how a particular type of auction might lead to joint profit maximization or rather to the maximization of consumer surplus.

The analysis in Section 2.2 shows that any search order could be an equilibrium order, where firms price optimally while expecting consumers to follow the search order and consumers search optimally in this order while correctly anticipating firms' prices. Hence, the determination of the equilibrium search order involves a coordination problem. The posting of ads with a certain priority order (top versus bottom, front page versus subpages for online ads) can be viewed as a coordination device determining that order. Specifically, we consider a generalized second price auction, in line with previous literature on auction mechanisms used for online search engines. We significantly extend the earlier literature by allowing the firms' position valuations to result from our full-fledged price competition model. Consequently, the auction involves two position externalities, business stealing and search appeal.

The auction game we consider is as follows. There are $n \geq 2$ firms competing for n slots. Each firm posts a bid. As in the pricing game of Section 2.2, all product demands are common knowledge among consumers and firms.[21] Let b^i denote the i^{th} highest bid, where two consecutive bids can be equal. The i^{th} slot, $i = 1, ..., n$, is allocated to the firm with the i^{th} highest bid, and that firm is charged b^{i+1}; if two firms have equal bids each is equally likely to be placed in front. Throughout the analysis below, Firm i, $i = 1, ..., n$ is the firm that is positioned in slot i in a candidate equilibrium. Firm n, which has the lowest bid, b^n, gets the last slot so it is searched last and pays nothing. Bids are lump sum amounts of money that are paid for being positioned in some slot i. We call this *per impression* bidding

[21] The analysis would be drastically different if prices were posted. Choi, Dai, and Kim (2018) and Haan, Moraga-Gonzalez, and Petrikaite (2018) consider pricing models in this vein with directed search and with heterogeneous prior consumer match information.

because all consumers are shown all ads.[22]

Recall from our earlier analysis (see (4)) that Firm i's gross equilibrium profit is

$$\pi_i = \left(q_i - \Sigma_{j>i}\Delta_j\right)\left(1-\gamma_i\right)\prod_{j<i}\gamma_j, \quad i = 1, ...n. \tag{12}$$

As shown in Proposition 1 gross profits for all firm types are higher when they are placed earlier under the maintained hypothesis that qualities are large enough.

We start with some general preliminary analysis that introduces the main concepts and explains how they can be used to analyze a position auction problem.

4.1 Preliminaries

We write the (generalized second-price) auction equilibrium in terms of firms' profits in different positions and the amounts they pay at each position. We consider some ordering of firms and analyze what happens to the payoff of the firm in the jth position (Firm $j = 1, ..., n$) when it moves to some other position i. In contemplating such a move, we hold fixed the *order* of the other firms. If j moves *up* to reach the ith position ($i < j$), the first $i - 1$ firms retain their positions, Firms i through $j - 1$ are demoted one slot down, and firms below j retain their positions. Conversely, if Firm j moves *down* to position $i > j$ then Firms 1 through $j - 1$ and Firms $i + 1$ through n retain their positions, while Firms $j + 1$ through i are promoted one slot up.

Let $\pi^i(j)$ denote the profit of Firm j when it moves to position i, so $\pi^j(j)$ is its profit at the status quo. For q_j large enough, Firm j earns more profit if placed earlier so that $\pi^i(j)$ is strictly decreasing in i (Proposition 1).

A *per impression position equilibrium* is a set of bids such that no firm wishes to switch its position if firms are placed in descending order of their bids with each paying the bid of the firm immediately below it. We split the no-switching conditions into there being no desire to jump down, and none to jump up. We deal with these in turn.

For Firm i (i.e., the incumbent type in position i) to not wish to jump down $k \geq 1$ slots

[22] Anderson and Renault (2021) analyze per click bidding with broadly similar results.

to position $i+k$, requires that

$$b^{i+1} - b^{i+k+1} \leq \pi^{i}(i) - \pi^{i+k}(i), \tag{13}$$

which says that the lost profit from jumping down exceeds the bid cost saving. Define

$$IV_{i+k}^{i}(j) = \pi^{i}(j) - \pi^{i+k}(j) \;\; j = 1, ..., n; \;\; i = 1, ..., n-1; \;\; k = 1, ..., n-i \tag{14}$$

as the *Incremental Value* to Firm j in position i over position $i+k$ ($k \geq 1$): note that this definition allows for Firm j moving from some position that differs from slot j (and this is useful for the analysis below). Incremental values are all strictly positive reflecting that firms always benefit from being moved earlier. The condition on bids for no jumping down (rewriting (13)) is therefore

$$b^{i+1} - b^{i+k+1} \leq IV_{i+k}^{i}(i), \qquad i = 1, ..., n-1; \;\; k = 1, ..., n-i. \tag{15}$$

We proceed analogously for jumps up. To do this, think of Firm $i+k$ jumping up to position i with $k \geq 1$, usurping the incumbent and paying its bid. In terms of the incremental cost-benefit, for Firm $i+k$ to prefer to stay put requires

$$b^{i} - b^{i+k+1} \geq \pi^{i}(i+k) - \pi^{i+k}(i+k), \tag{16}$$

or, using the definition of incremental values (14)

$$b^{i} - b^{i+k+1} \geq IV_{i+k}^{i}(i+k), \qquad i = 1, ..., n-1; \;\; k = 1, ..., n-i. \tag{17}$$

Finally, equilibrium requires that bids b^{i}, $i = 1, ..., n$, are decreasing in i so firms are ordered according to their decreasing bids. However this is implied by the no jump condition (17), for $k = 1$ and because incremental values are strictly positive. The *position equilibrium condition* on the vector of bids is therefore that (15) and (17) hold.

The next result provides conditions on incremental values (and hence on the underlying payoff structure) which ensure that the local equilibrium conditions for downward jumps, (15) for $k = 1$, is sufficient to rule out any downward deviation because larger deviations down are unprofitable too.

Lemma 3 *Assume firms are ordered such that* $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$ *for all* $i = 1, ...n-1$ *and* $k = 1, ..., n-i$. *Then if bids are such that one-step deviations down are not profitable, larger downward deviations are not either.*

The assumption on incremental values says that the drop in profit for the firm in slot $i+k$ from dropping down to slot $i+k+1$ is less than what a firm initially placed earlier would lose by making the same move after having been demoted to slot $i+k$.[23] The condition implies that firms in higher slots lose more from moving down by one slot than firms in lower slots. The proof then uses a transitivity argument whereby if a firm does not want to replace its downside neighbor and the latter does not want to replace its downside neighbor then the first firm does not want to jump down two slots.

Our equilibrium analysis below focuses on two dimensions of heterogeneity, asymmetric qualities and asymmetric market potentials respectively. In both cases, the assumption on incremental values in Lemma 3 holds, provided that products are in the appropriate order. When qualities differ, if products are ordered in the decreasing order of their qualities, then the firm with the ith highest quality would have a higher willingness to pay to stay in some slot $i+k$ rather than dropping down one slot than the incumbent firms in slot $i+k$, which has the $(i+k)$th highest quality. Similarly, when market potentials differ, but not qualities, if products are ordered from larger to smaller market potentials, the incremental value of the firm with the ith largest market potential for staying in some slot $i+k$ rather than moving down one slot is larger than that of the firm with the $(i+k)$th largest market potential. It is noteworthy that whether Lemma 3 holds does not depend on the ranking of the Δ's. This is because the two relevant incremental values only depend on the price in slot $i+k$ and the price in slot $i+k+1$, which prices in turn depend only on the search appeals of products $i+k+1$ and beyond. Hence the comparison of the two incremental values is unaffected by the ranking of Δ_i and Δ_{i+k}.

Our analysis below will stress that the ordering of products resulting from a GSP auction

[23] For instance, if $i = 5$, the condition compares the drop in profit for Firm 5 from moving to slot 6 with the drop in profit that some earlier firm, say Firm 3, would suffer moving from slot 5 to slot 6 after having first reached slot 5 (meaning that Firms 4 and 5 are now in front).

is quite different if product asymmetries pertain to qualities, q, or to market potentials, $1-\gamma$. In particular, we emphasize that, depending on the nature of product heterogeneity, the outcome can be more favorable to firms or consumers. As a first step, we consider the benchmark where all qualities and market potentials are identical.

4.2 Symmetric qualities and market potentials

Assuming that quality and market potential is the same for all products, we show that there exists an equilibrium for which bids are such that all firms would strictly lose from modifying their bids to reach a different position in the search order. Proving the existence of such an equilibrium is a useful stepping stone for deriving existence results when products have asymmetric demands in q or $1-\gamma$. The fact that the equilibrium is strict also yields as a corollary that if products have qualities and market potentials that are too similar, then any order can be sustained in equilibrium.

Writing out incremental values with identical qualities and market potentials is fairly straightforward. It provides a simple illustration of the general principles involved in constructing incremental values more generally. So assume that for all $i = 1, ..., n$, $q_i = q$ and $\gamma_i = \gamma$. Consider some Firm j positioned in the jth slot. Its profit in some position i would be $\pi^i(j) = (1-\gamma)\gamma^{i-1}(q - \sum_{\ell > i} \Delta_\ell)$ if $j \leq i$ and $\pi^i(j) = (1-\gamma)\gamma^{i-1}(q - \sum_{\ell \geq i; \ell \neq j} \Delta_\ell)$ if $j > i$. In the analysis below we will need to refer to the incremental value for some Firm j moving from some position $i+k$ to position i, $k \geq 1$, with $j \leq i$ or $j \geq i+k$. Using the profit expressions above and rearranging, it is given by

$$IV^i_{i+k}(j) = \pi^i(j) - \pi^{i+k}(j) = (1-\gamma)\gamma^{i-1}\left[(1-\gamma^k)(q - \sum_{\ell > i+k} \Delta_\ell) - \sum_{\ell = i+1}^{i+k} \Delta_\ell\right] \text{ if } j \leq i \quad (18)$$

$$IV^i_{i+k}(j) = \pi^i(j) - \pi^{i+k}(j) = (1-\gamma)\gamma^{i-1}\left[(1-\gamma^k)(q - \sum_{\ell \geq i+k; \ell \neq j} \Delta_\ell) - \sum_{\ell = i}^{i+k-1} \Delta_\ell\right] \text{ if } j \geq i+k. \quad (19)$$

The above expressions can readily be interpreted in terms of quantities (here probabilities of selling the product) and prices. The term in front of the bracket is the probability that

Firm j sells its product conditional on getting a click, $1-\gamma$, multiplied by the probability, γ^{i-1}, of getting a click if it is positioned in the earlier slot i. Inside the bracket we first have the price that Firm j charges if in slot $i+k$ times the increase in probability of getting a click, $1-\gamma^k$, from a consumer who reaches slot i if Firm j is in slot i rather than in slot $i+k$. The second term in the bracket is the decrease in Firm j's equilibrium price if it is moved from slot $i+k$ to slot i.

We have the following result.

Proposition 8 *Assume $q_i = q$ and $\gamma_i = \gamma$ for all i. For q sufficiently large, for any ordering of Δ_i, $i = 1, ..., n$, there exists an equilibrium with Firm i in slot i and bids satisfying*

$$b^i - b^{i+1} = \alpha IV_i^{i-1}(i-1),\ i = 2, ..., n-1,\ \textit{and } b^n = \alpha IV_n^{n-1}(n-1),\ \textit{with } \alpha \in [\gamma, 1) \quad (20)$$

where each firm strictly prefers its position to any deviation (i.e. equilibrium conditions (15) and (17) hold strictly).

Proposition 8 shows that, if products only differ in terms of their search appeals, then any order can be sustained in a strict equilibrium provided that quality is sufficiently high. If $\alpha = 1$ in the bid expression we still have an equilibrium but it is no longer strict. Firms in slots $i > n$ would be indifferent between remaining in their equilibrium positions and moving down to later slots.

The arguments used to prove the above result illustrate how to construct an equilibrium for the auction game using incremental values. The same logic applies to establish the existence of an equilibrium with asymmetric qualities or asymmetric market potentials below.

The bids ensure that a Firm $i-1$ which pays b^i to be in slot $i-1$ does not want to drop its bid to take slot i and pay b^{i+1}. This would result in a drop in profit of $IV_{i+1}^i(i-1)$ which exceeds the saving on fees $b^i - b^{i+1}$ because $\alpha < 1$. Hence, local downward deviations are unprofitable and, because Firm $i-1$ only needs to pay a fraction $\alpha < 1$ of its incremental value to be in slot $i-1$ rather than slot i, local downward deviations entail a strict loss. This generalizes to all downward deviations provided that Lemma 3 can be used. Taking some

Firm i and some other Firm $i+k$, $k \geq 1$, the relevant specification of incremental values for moving from slot $i+k+1$ to slot $i+k$ is (18) so that $IV_{i+k+1}^{i+k}(i+k) = IV_{i+k+1}^{i+k}(i)$ and hence, Lemma 3 applies.

Checking that upward jumps are not profitable either is a bit more involved. Short of imposing some restriction on the ordering of products in terms of search appeal (for instance assuming that Δ_i is decreasing in i)[24], it is not possible to use a recursive argument such as the one that underpins Lemma 3 for downward deviations. However we can show (as per the proof in the Appendix) that, for $\alpha > \gamma$, upward deviations are strictly dominated by the status quo. In a first step we show this for $\Delta_i = 0$, and the difference between the additional fees a firm has to pay in order to outbid those which are in front and the incremental value for such a move can be made arbitrarily large by increasing quality q. Then we use the property that incremental values are additively separable in quality and search appeals to show that it is possible to select a quality level such that an upward deviation is strictly unprofitable even if products have positive search appeal ($\Delta_i > 0$).

Proposition 8 establishes the existence of a strict equilibrium for firms with identical qualities and market potentials, no matter how they are ordered in terms of their product's search appeal. Then, if product qualities and market potentials do not differ too much, by continuity the strict equilibrium conditions should still hold. This is indeed the case as shown in the following result.

Proposition 9 *Assume q_1 is large enough. Then, for any ordering of Δ_i, $i = 1, ..., n$, there exists $\epsilon > 0$ such that if $|q_i - q_1| < \epsilon$ and $|\gamma_i - \gamma_1| < \epsilon$ for all $i = 1, ..., n$, there exists an equilibrium where Firm i is in slot i, $i = 1, ..., n$.*

This proposition suggests that it requires sufficient heterogeneity in qualities or market potentials to get sharp predictions about the ordering of products that can be sustained in equilibrium. Proposition 9 gives a general existence result for when heterogeneity is not too

[24] It is possible to use a recursive argument when $q_i + \Delta_i$ is decreasing in i and the γ_i are the same, as we do in Anderson and Renault (2021). Note that equilibrium conditions for upward and downward jumps are not symmetric. By deviating down by one slot, a firm ends up paying what the next firm down is paying. Deviating up, it must pay whatever the firm two slots ahead is paying.

large: the next two subsections treat significant heterogeneity in each dimension separately.[25] We first consider heterogeneous qualities.

4.3 Asymmetric qualities

In this section, all products are equally popular so they share the same market potential but qualities are allowed to differ significantly. Market potentials being identical, we have $\gamma_i = \gamma$ for all $i = 1, ..., n$. It follows that the sales of a firm in slot i are $\gamma^{i-1}(1-\gamma)$ independent of the product that is being sold. By contrast, from the analysis in Section 2 the price charged in some slot i depends on the quality of the product that is positioned in that slot as well as on the search appeal of the products placed afterwards: indeed, Firm i in slot i charges $p_i = q_i - \Sigma_{j>i}\Delta_j$. Because the increase in sales afforded by a better position is common to all firms, those firms which are better able to monetize this traffic advantage because they charge higher prices than others should be expected to be the highest bidders in the auction. This is reflected in the next proposition, which first shows that the GSP auction always has an equilibrium at which firms are ranked in decreasing order of product qualities.[26] We also invoke the following assumption to show that enough quality heterogeneity significantly restricts the equilibrium outcomes. Heterogeneity in qualities is characterized as follows.

Assumption 2 *For all pairs i, j with $i \neq j$, either $q_i > \frac{1+\gamma}{\gamma} q_j$ or $q_j > \frac{1+\gamma}{\gamma} q_i$.*

Section 3.3 above describes how different product qualities could result in an inverse demand dominance ordering of products as summarized in Assumption 1. We also show in Proposition 6 that consumers and firms then have opposite preferences over the ranking and hence we have the following result.

[25] Our constructive proofs for heterogeneity in either dimension alone are not readily extended to allow for significant heterogeneity in both.

[26] An alternative way to pin down the equilibrium outcome would be to analyze an asymmetric information version of the auction game, as in Edelman, Ostrovsky, and Schwartz (2007). In a setting that corresponds to ours with no search appeal and identical market potentials, they find that the unique equilibrium to a button auction has higher quality products ranked earlier. This might be generalized to allow for common search appeal.

Proposition 10 *Assume $\gamma_i = \gamma$ for all i, $q_i > q_{i+1}$ for $i = 1, ..., n-1$ and Assumption 1 holds.*

(i) Then for q_n sufficiently large there exists an equilibrium with Firm i in slot i. Furthermore, total industry profit is maximized at any such equilibrium.

(ii) If Assumption 2 holds and $n \geq 3$ then in any equilibrium the product with the ith highest quality, $i = 1, ..., n$ is placed at slot $i-1$, slot i, or slot $i+1$; and the preferred search order for consumers (with low-quality products placed earlier) ***cannot*** *be sustained as an equilibrium.*

Part (i) shows there is an equilibrium that sustains the ordering of products from higher to lower qualities (although Assumption 1 is not needed for this existence result[27]). As shown in the Appendix, it can be sustained with bids satisfying $b^i - b^{i+1} = IV_i^{i-1}(i-1)$, $i = 2, ..., n-1$, with $b^n = IV_n^{n-1}(n-1)$. These bids make Firms $i = 1, ..., n-1$ indifferent between staying in slot i and dropping down to lower slots. The same equilibrium order could be sustained with lower bids (cf. Proposition 8 where firms strictly prefer their positions over lower ones).

Assumption 1 guarantees that the top-down quality equilibrium maximizes TIP. The equilibrium has the protagonists at loggerheads. Indeed, Proposition 6 shows that consumer welfare is at its lowest if high quality products come first – although consumers are indifferent to quality per se (because it is priced out), they dislike products with monotonically dominant inverse demand to come early. This is because a large search appeal is positively correlated with a large value of η (via Proposition 5) and then the price-lowering power of search appeal is the consumers' main concern.

The last part of Proposition 10 is important because it shows that enough quality asymmetry as per Assumption 2 entails that any equilibrium ranking must be "close" to the

[27]Thus existence of the quality ranking equilibrium requires nothing about how search appeals are ordered. Assumption 1 provides the extra structure for the strong welfare results that follow.

TIP-maximizing decreasing-quality order. Perhaps more importantly still, it cannot maximize consumer surplus.[28]

4.4 Asymmetric market potentials

We now look at heterogeneity in market potentials, so that products have common q and different γ values. In contrast with quality heterogeneity above, the sales of a firm in a given slot depend on the firm's identity: all other things equal, a firm with a higher market potential sells more. Those sales also depend on the business stealing externality imparted by products positioned earlier which in turn is determined by those products' market potentials. As a result, the role of market potential in determining the firms' incentives to bid for a more favorable placement is more ambiguous than for quality, which is a purely private value component of a product's demand and only impacts the price charged for that product. Although firms selling more popular products have a stronger incentive to bid for better positions that generate more clicks, these incentives also depend on the attractiveness of the products that are jumped over: all other things equal, a firm's willingness to pay to reach a higher position is larger if the firms that are being demoted have higher market potential because the move induces more additional clicks. In short, although larger firms have a stronger incentive to jump over other sellers, it is also more profitable for other sellers to jump over them than over smaller firms. Nonetheless, the main results in this section show that a ranking of products in decreasing order of their market potentials is a robust equilibrium outcome of the GSP auction, whereas the reverse order is not.

As for the analysis with asymmetric qualities above, our approach to robustness is to allow the differences in market potentials to become large. This is formally reflected in the following assumption.

Assumption 3 *For each pair $i, j = 1, ..., n$ with $i \neq j$, either $\frac{1-\gamma_i}{\gamma_i} < (1-\gamma_j)^2$ or $\frac{1-\gamma_j}{\gamma_j} < (1-\gamma_i)^2$*

[28] Recall that by contrast (as was earlier shown in Proposition 9) the quality-ordered ranking is not the unique ranking when qualities are too similar, in which case any search order can be sustained via bidding.

This means that market potentials are sufficiently heterogeneous. We will also assume that products can be ordered in terms of niche dominance so that a lower market potential is associated with a more niche product (though here we assume that all qualities are the same). Recall from Proposition 7 in Section 3.3 that if there is a niche ranking then firms and consumers have opposite preferences over orders.

Proposition 11 *Assume $q_i = q$ for all i, $\gamma_i < \gamma_{i+1}$ for $i = 1, ..., n-1$, products can be ordered in terms of niche dominance and q large enough:*

(i) there exists an equilibrium with Firm i in slot i (maximum consumer surplus order);

(ii) under Assumption 3 with $n \geq 3$, there is no equilibrium with $\gamma_i > \gamma_{i+1}$, $i = 1, ..., n-1$, (maximum joint profit order).

Proposition 11 reflects the propensity of the equilibrium bidding to gravitate towards higher market potentials early.[29] This property presages problems for firm total profit as each firm steals a march on the others, but also suggests good news for consumers. The last part of the proposition confirms that the TIP order cannot sustain under sufficient heterogeneity in market potentials (Assumption 3).

The equilibrium described in Proposition 11 is reminiscent of the analysis in Athey and Ellison (2011)[30] and Chen and He (2011) in that it places more popular products earlier, which is the ranking that prevails in equilibrium in those papers. However, the implications for joint profit and consumer welfare are opposite. As already discussed in Section 3.3, under niche dominance, this order is the worse for firms. Yet it does ensure an optimal consumer welfare as is also the case in these previous works.

[29] Proposition 11(i) can be satisfied when bids satisfy

$$b^i - b^{i+1} = IV_i^{i-1}(i-1),\ i = 2, ..., n-1,\ \text{with } b^n = IV_n^{n-1}(n-1).$$

This means that bid differences are at incremental values: see the analogous statement after Proposition 8.

[30] Their auction game assumes asymmetric information across bidders. It might be possible to follow their analysis by introducing a sufficient common search appeal to get an equilibrium with the result of Proposition 11i.

We now describe some robust conclusions as well as some possible tensions when there are fewer slots than firms ($k < n$). For concreteness, following Athey and Ellison (2011) we suppose that firms which do not get a slot are excluded from the market (and we do not treat the platform's choice of how many slots to offer optimally). We here address the results in Proposition 11 and we indicate the analogous arguments that can be made for Proposition 10.

First, the equilibrium in Proposition 11i is still an equilibrium, with the first k firms bidding following the same principles as before and simply replacing n by k in the pricing rule (3), though now Firm k bids its own incremental value on top of that of its predecessor. The firms active in the market will be those with highest market potentials (those which are niche dominated). The excluded Firms $k+1$ through n earn 0 and bid Firm k's incremental value for being in slot k over being out. None would want to bid high enough to enter the market because the excluded firms would need to pay more than Firm k's incremental value for being in the market, which in turn exceeds its own because a lower market potential means a lower profit in the final slot k. Analogously, with quality heterogeneity as assumed in Section 4.3, there is an equilibrium where qualities for participating firms are ranked from top to bottom as in Proposition 10i and firms with the lower qualities stay out.

In the equilibrium under asymmetric market potentials described above, the ordering of included firms from the largest to the smallest maximizes consumer surplus and minimizes joint profit, as in Proposition 11i. Furthermore, the selection of active firms corresponds to what consumers prefer, provided that the heterogeneity in η is small relative to that of market potentials: consumers prefer that the products available in the market are those they are more likely to buy. However, the set of firms selected by the auction is also the most favorable for joint profit because it maximizes sales by selecting products with high market potentials and minimizes the search appeal externality at all slots by selecting niche dominated products which have a low search appeal. In the case of heterogeneous qualities as per Section 4.3, the equilibrium with high quality firms on the platform and low quality firms out ranks active firms in accordance with joint profit maximization and opposite to what consumers would prefer as in Proposition 10. The selection of products is also favorable to firms, provided that heterogeneity in search appeal is small compared to differences in quality. Nothing definite can be said about the the impact on consumer welfare of equilibrium selection of products because there is no clear argument for selecting high η or low η products[31] and consumers are indifferent about the quality of the products they buy as a higher quality is entirely capitalized into a higher price.

5 Conclusions

Ordered search characterizes the lion's share of the modern online economy, which is only growing in importance. Yet research so far has been stymied for lack of a tractable set-up, even in the symmetric case, let alone dealing with the full set of product distinguishers as we do here. One main accomplishment of the paper is to deliver a clean analysis for ordered search under asymmetry. Our specification of product demands allows for characterizing key dimensions of product heterogeneity that result in different pricing behavior and different sales among the competing firms. We can also provide a crisp characterization of the two position externalities, *business stealing* and *search appeal* that depend on the identity of firms positioned ahead or after a given slot but not on the order of those firms. This key property enables us to determine summary statistics for firms, which are firm specific and independent of position and which are used to determine optimal rankings of firms that maximize either total profits or consumer surplus.

We apply our analysis of optimal product rankings to two situations where products can be compared via *inverse demand dominance* and *niche dominance*. The first product ordering corresponds to a form of first order stochastic dominance between the different match distributions. It is particularly relevant when products differ by quality, and quality heterogeneity translates into price heterogeneity. The second order is characterized by a rotation of demand between products and applies when products have different market potentials

[31] Recall that $\eta = \Delta - \omega$ and consumers prefer products with a large Δ to be active, which brings down prices, but they also prefer products with a large ω to be in because they generate more consumer surplus when consumed.

so firm heterogeneity translates into different levels of sales. These two dominance relations across products highlight the potential conflict between firms and consumers: their preferred rankings of the search order of products conflict in both cases, with firms preferring inverse demand dominant or niche dominant products to come first.

We find that higher quality firms are willing to pay more (ceteris paribus) to obtain a better position. Under inverse demand dominance, the auction outcome therefore favors sellers. If instead market potentials are heterogeneous, firms with higher ones bid more. Then, under niche dominance, the ranking that results from an auction is optimal for consumers rather than firms.

Like (nearly all of) the extant ordered search literature, all consumers follow the same search order in our model. This is an important step because most papers on search and firm pricing involve random search settings. Still this may seem somewhat at odds with actual online search. In practice, different consumers may follow different search orders. Theoretically, this could be interpreted as resulting from some mixing behavior by consumers facing different search options among which they are indifferent as is the case in symmetric random search settings.[32] Perhaps more importantly, there are many obvious reasons why different consumers do not face *ex ante* symmetric circumstances for their search opportunities. In practice, they might be targeted by different advertising campaigns or, relatedly, they might be treated differently by a platform's product steering policy. They might also have different priors about their match with the various available products so they will input different search queries on search platforms. The latter possibility is captured in the setting studied by Choi, Dai, and Kim (2018) and Haan, Moraga-González, and Petrikaite (2018) where each consumer receives an informative signal about their matches so different consumers follow different search orders. Actual consumer online search behavior is less idiosyncratic than is assumed in these models. There is substantial correlation in search orders and the allocation of advertising slots plays a key role for a firm's online prominence. By abstracting from consumer *ex ante* heterogeneity, our analysis provides a rich environment capturing multi-

[32] In the symmetric version of our setting, we expect that random search would lead to a symmetric price equilibrium analogous to that characterized in Watson (2006).

ple dimensions of product heterogeneity and how they interact with firms' priority in the search order. Although future research should definitely investigate how the insights from our setting might be nuanced by the various sources of pre-search consumer information that impact individual search behavior, we hope that our findings retain substantial validity and relevance in such complex environments.

Section 4 highlights the ranking of sponsored search engine results and its impact on pricing, profit, and consumer surplus. Although consumers still rely on search engines, they now have other options for searching purchase opportunities. The advance of vertical platforms devoted to product categories such as hotels, flights, and real estate often provide effective shopping alternatives. Search rankings on such platforms are typically not based on bidding. Still, our analysis in Section 2 provides insights how those rankings might effect pricing, while Section 3 explores their impact on joint profit and consumer welfare (modulo the caveat that we abstract from heterogeneous search behavior).

Our analysis of advertising slot allocation through GSP cannot account for the many ways in which firms can pay to make themselves more prominent. We believe however that our approach in terms of incremental values is useful beyond the restrictive setting of an auction. One would expect that firms that advertise more are more likely to be searched earlier. Advertising slots are still often awarded based on some auction-like mechanism, even though advertisers sometimes now just determine an overall budget with some criteria for how it should be allocated, leaving it up to the platform to decide when and how exactly to attribute the money. Firms' willingness to pay for advertising are likely related to the incremental values we characterize.

Future research might exploit the analysis here and extend the setting to address new questions and incorporate dimensions which are currently left out, while retaining the tractability of the pricing characterization. One obvious issue is the platform's optimal design problem. This includes the choice of the number of slots, building on the setting sketched out at the end of Section 4.4, and auction design like the use of a reservation price or more general mechanism design.[33] Our characterizations of optimal rankings in Section 3 and the potential tension between joint profit and consumer welfare maximization are also important to the platform's two-sided market problem. As already discussed, we have ruled out the possibility of consumers observing any pre-search information, other than what they infer from the ranking outcome of the auction. This means that we do not deliver any heterogeneity in search behavior by consumers and search is solely directed by the positions of products on the platform. In particular, we do not account for price-directed search which is often quite relevant in practice. A useful extension would allow consumers to observe prices while maintaining tractable pricing behavior by firms.

Appendix

A1 Results from Section 2

Claim 1 *For any $\gamma_i \in (0,1)$, $q_i > 0$ and $\Delta_i > 0$ there exists a distribution function F_i, whose support has a maximum $q_i + B_i < \infty$, which satisfies (1).*

Proof. It is useful to define $\delta_i = v_i - q_i$ for all $v_i \geq q_i$: then the support of δ_i should be a subset of $[0, B_i]$. Let $\bar{F}_i$ be the distribution function for δ_i. We can then rewrite (1) using $F_i(v) = \gamma_i + (1-\gamma_i)\bar{F}_i(v - q_i)$ for $v \geq q_i$ as

$$(1-\gamma_i)\int_{\Delta_i}^{B_i}\left(1-\bar{F}_i(\delta)\right)d\delta = s. \tag{21}$$

The left-hand side can be made arbitrarily close to $(1-\gamma_i)(B_i - \Delta_i)$ by moving all the weight of the distribution of δ_i in the neighborhood of B_i (so that $\bar{F}_i$ is nearly 0 on $[0, B_i]$, except in a small neighborhood of B_i) and, equal to 0, by shifting all the weight of δ_i below Δ_i (so $\bar{F}_i(\delta) = 1$ for $\delta > \Delta_i$). Furthermore, $B_i - \Delta_i$ can be made as large as necessary by increasing B_i. Hence, for any $\Delta_i > 0$, $\gamma_i \in (0,1)$, $q_i > 0$ and $s > 0$, it is possible to find some specification of $\bar{F}_i$ such that (21) holds. ■

[33] Note that here we do not have a private value setting because the two position externalities depend on the firms' "types". As a result, VCG is not applicable, except in the case with different qualities and identical market potentials and search appeal.

Lemma 1 *If consumers search optimally from Firm 1 to Firm n expecting all the firms to price according to (3), then it is optimal for any Firm i to charge price p_i defined by (3) as long as q_i is sufficiently large, $i = 1, ..., n$.*

Proof. If Firm $i < n$ charges its candidate equilibrium price p_i, it earns per click profit $(1 - \gamma_i)p_i$. At this price, it sells to all consumers with strictly positive willingness to pay for its product who have reached it. Hence, it cannot gain additional profit by charging a lower price. Assume therefore that it charges a price that is $\Delta p > 0$ in excess of p_i. Its corresponding profit is then at most

$$(p_i + \Delta p)\left(1 - \gamma_i - (1 - F_{i+1}(q_{i+1} + \Delta_{i+1}))(F_i(q_i + \Delta p) - \gamma_i)\right).$$

This upper bound on deviation profit is obtained as follows. First, if Firm i deviates to $p_i + \Delta p$, then all consumers with valuations less than Δp in excess of q_i search Firm $i + 1$ (recall that at price p_i consumers holding match q_i with Firm i are just indifferent between buying product i and searching on). Hence, the probability that a consumer who does not search in equilibrium chooses to search Firm $i+1$ is $F_i(q_i+\Delta p)-\gamma_i$. Among those searching consumers, those with valuations $v_{i+1} > q_{i+1} + \Delta_{i+1}$ with Firm $i + 1$, strictly prefer buying product $i + 1$ and never return to Firm i (indeed, since the equilibrium price difference is Δ_{i+1}, those consumers would prefer buying product $i + 1$ even if Firm i had not increased its price from its equilibrium level and they had chosen to search nonetheless). Hence, Firm i's demand at price $p_i + \Delta p$ is at most $1 - \gamma_i - (1 - F_{i+1}(q_{i+1} + \Delta_{i+1}))(F_i(q_i + \Delta p) - \gamma_i)$ and the profit gain from the price increase is bounded above by

$$\Delta p(1 - \gamma_i) - (p_i + \Delta p)(1 - F_{i+1}(q_{i+1} + \Delta_{i+1}))(F_i(q_i + \Delta p) - \gamma_i).$$

First consider a small deviation with Δp close to zero. Because $f_i(q_i) = F'(q_i)$, $\frac{F_i(q_i+\Delta p)-F_i(q_i)}{\Delta p}$ tends to $f_i(q_i)$ as Δp tends to 0. Hence there exists $\bar{\delta}$ such that if $\Delta p < \bar{\delta}$, then $F_i(q_i+\Delta p) - F_i(q_i) > \frac{f_i(q_i)}{2}\Delta p$. Then, because $F_i(q_i) = \gamma_i$, the benefit from deviating is bounded above by

$$\Delta p\left((1 - \gamma_i) - (p_i + \Delta p)(1 - F_{i+1}(q_{i+1} + \Delta_{i+1}))\frac{f_i(q_i)}{2}\right),$$

which is negative if p_i is large enough, since $f_i(q_i) = a_i > 0$.

Now take a large deviation, $\Delta p > \bar{\delta}$. If follows that $F_i(q_i + \Delta p) \geq F_i(q_i + \bar{\delta}) > \gamma_i$. Since a price above $q_i + B_i$ would yield zero profit, an upper bound for the deviation gain is

$$(q_i + B_i - p_i)(1 - \gamma_i) - (p_i + \Delta p)(1 - F_{i+1}(q_{i+1} + \Delta_{i+1}))(F_i(q_i + \bar{\delta}) - \gamma_i).$$

From the pricing expression (3), price p_i is increasing in q_i and $q_i - p_i$ does not depend on q_i. Hence, for q_i large enough, the above upper bound on the profit change from a large price increase is negative, so that such a deviation is not profitable.

The above arguments go through for $i = n$, where $F_{i+1}(q_{i+1} + \Delta_{i+1})$ is replaced by 1 (all the consumers who give up buying product n at $q_n + \Delta p$ select not to buy any product so none of them return to Firm n). ■

A2 Results from Section 3

A2.1 Results from Section 3.2

Proposition 5 *Assume $\gamma_k = \gamma$ for all $k = 1, ..., n$ and products can be ordered by monotonic inverse demand dominance. Then consumer surplus is maximized by positioning products with higher search appeal earlier.*

Proof. From Proposition 3 we have that products should be placed in order of decreasing order of η_k to maximize consumer surplus if γ_k is constant (see also (10)). Hence to show that the consumer-optimal order follows the order of decreasing Δ_k, it suffices to show that if product i monotonically dominates product j in inverse demand and γ_k is constant, then $\Delta_i > \Delta_j$ and $\eta_i > \eta_j$. The remainder of the proof establishes this last statement.

First we rewrite equation (11) in terms of the net valuation function, V_k, which is the inverse of $1 - \bar{F}_k$ so that if $V_k(x) = \delta$, then $1 - \bar{F}_k(\delta) = x$ (where $x \in [0, 1 - \gamma]$ is the probability that $v_k - q_k$ exceeds δ for distribution $\bar{F}_k$). Then (11) can be written as $\frac{s}{1-\gamma} = \int_{\Delta_k}^{B_k} [1 - \bar{F}_k(\delta)] d\delta = \int_{\hat{x}_k}^{0} x V_k'(x)\, dx$ where $\hat{x}_k = [1 - \bar{F}_k(\Delta_k)]$, so that $V_k(\hat{x}_k) = \Delta_k$ and we then use the change of variables with $V_k'(x)\, dx = d\delta$. Integrating by parts gives

$$\frac{s}{1-\gamma} = \int_0^{\hat{x}_k} V_k(x)\, dx - \hat{x}_k V_k(\hat{x}_k), \tag{22}$$

Taking the derivative of the RHS with respect to $\hat{x}_k$ yields $-\hat{x}_k V_k'(\hat{x}_k)$ which is positive because $V_k' < 0$. Hence the RHS of (22) is increasing in $\hat{x}_k$. Further note that it is zero for $\hat{x}_k = 0$.

We next suppose that i monotonically dominates j in terms of inverse demand.

Since $V_i'(x) < V_j'(x)$ we must have $\hat{x}_i < \hat{x}_j$ because the RHS of (22) increases faster for product i. In turn, $\hat{x}_i < \hat{x}_j$ and $V_i > V_j$ imply $\Delta_i = V_i(\hat{x}_i) > V_i(\hat{x}_j) > V_j(\hat{x}_j) = \Delta_j$, which establishes the first claim.[34]

We now show that $\eta_i > \eta_j$. By definition of η_k, using $F_k(v) = \gamma + (1-\gamma)\bar{F}_k(\delta)$, for $v \geq q_k$ and $\delta = v - q_k$, and then again using the change of variable $d\delta = V_k'(x)dx$ for $x = 1 - \bar{F}_k(\delta)$ and then integrating by parts, we have

$$\begin{aligned}
\eta_k &= \int_{q_k}^{q_k+\Delta_k} F_k(v)dv = \gamma\Delta_k + (1-\gamma)\int_0^{\Delta_k} \bar{F}_k(\delta)d\delta \\
&= \gamma\Delta_k + (1-\gamma)\int_1^{\hat{x}_k} (1-x)V_k'(x)dx \\
&= \gamma\Delta_k + (1-\gamma)\left[(1-\hat{x}_k)\Delta_k - \int_{\hat{x}_k}^1 V_k(x)dx\right] \\
&= \gamma\Delta_k + (1-\gamma)\int_{\hat{x}_k}^1 (\Delta_k - V_k(x))\, dx.
\end{aligned}$$

Now compare η_i and η_j. Because $\Delta_i > \Delta_j$, the first term is larger for Firm i. Because $\Delta_k - V_k(x) > 0$ for $x > \hat{x}_k$ and $\hat{x}_i < \hat{x}_j$, the second term is also guaranteed larger for i if

$$\int_{\hat{x}_j}^1 (\Delta_i - V_i(x))\, dx > \int_{\hat{x}_j}^1 (\Delta_j - V_j(x))\, dx,$$

or

$$\int_{\hat{x}_j}^1 (\Delta_i - \Delta_j)\, dx > \int_{\hat{x}_j}^1 (V_i(x) - V_j(x))\, dx,$$

where the last line is true because V_i decreasing and $\hat{x}_i < \hat{x}_j$ imply $\Delta_i - \Delta_j = V_i(\hat{x}_i) - V_j(\hat{x}_j) > V_i(\hat{x}_j) - V_j(\hat{x}_j)$ and $V_i - V_j$ is decreasing. ■

A2.2 Results from Section 3.3

Lemma 2 *If product i niche dominates product j, then $\Delta_i > \Delta_j$ and $\eta_i > \eta_j$.*

[34] This does not require that $V_i' < V_j'$. As noted in the text, $V_i > V_j$ is equivalent to $\bar{F}_i$ FOSD $\bar{F}_j$ which implies $\Delta_i > \Delta_j$.

Proof. Niche dominance of i over j implies that $B_i > B_j$. We now show that $\int_{\Delta_j}^{B_i} 1 - F_i(q_i+\delta)d\delta > \int_{\Delta_j}^{B_j} 1 - F_j(q_j+\delta)d\delta = s$, which in turn implies $\Delta_i > \Delta_j$. We have

$$\int_{\Delta_j}^{B_i} 1 - F_i(q_i+\delta)d\delta = \int_{\Delta_j}^{B_i} 1 - F_j(q_j+\delta)d\delta + \int_{\Delta_j}^{B_i} F_j(q_j+\delta) - F_i(q_i+\delta)d\delta$$

The second term on the RHS is positive because $F_j(q_j+\delta) > F_i(q_i+\delta)$ for $\delta > \Delta_j$, so the desired strict inequality holds. Now, since $\Delta_i > \Delta_j$, $\eta_j = \int_0^{\Delta_j} F_j(q_j+\delta)d\delta < \int_0^{\Delta_j} F_i(q_i+\delta)d\delta < \int_0^{\Delta_i} F_i(q_i+\delta)d\delta = \eta_i$, which completes the proof. ■

A3 Results from Section 4

A3.1 Preliminaries (Section 4.1)

Lemma 3 *Assume firms are ordered such that $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$ for all $i = 1, ...n-1$ and $k = 1, ..., n-i$. Then if bids are such that one-step deviations down are not profitable, larger downward deviations are not either.*

Proof. Assume $n \geq 3$. We wish to show that, for all $i = 1, \ldots, n-1$, if (15) holds for $k = 1$, then (15) holds for $k = 1, ...n-i$. This is clearly true for $k = 1$.

Now we just need to show that, for $i = 1, ..., n-2$, if this is true for some $k = 1, ..., n-i-1$, then it is true for $k+1$. If (15) holds for k then

$$b^{i+1} - b^{i+k+1} \leq IV_{i+k}^{i}(i). \tag{23}$$

Because one step deviations down are unprofitable, Firm $i+k$ does not want to deviate to slot $i+k+1$ so $b^{i+k+1} - b^{i+k+2} \leq IV_{i+k+1}^{i+k}(i+k)$. The Lemma also assumes that $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$, so that

$$b^{i+k+1} - b^{i+k+2} \leq IV_{i+k+1}^{i+k}(i). \tag{24}$$

Adding (24) to (23) yields

$$b^{i+1} - b^{i+k+2} \leq IV_{i+k}^{i}(i) + IV_{i+k+1}^{i+k}(i) = \pi^{i}(i) - \pi^{i+k+1}(i) = IV_{i+k+1}^{i}(i),$$

so Firm i does not want to deviate to slot $i+k+1$. ■

A3.2 Symmetric qualities and market potentials (Section 4.2)

Proposition 8 *Assume $q_i = q$ and $\gamma_i = \gamma$ for all i. For q sufficiently large, for any ordering of Δ_i, $i = 1, ..., n$, there exists an equilibrium with Firm i in slot i and bids satisfying*

$$b^i - b^{i+1} = \alpha IV_i^{i-1}(i-1),\ i = 2, ..., n-1,\ \text{and } b^n = \alpha IV_n^{n-1}(n-1),\ \text{with } \alpha \in [\gamma, 1) \quad (25)$$

where each firm strictly prefers its position to any deviation (i.e. equilibrium conditions (15) and (17) hold strictly).

Proof. Assume bids are given by

$$b^{i+1} = b^{i+2} + \alpha IV_{i+1}^i(i),\ i = 1, ..., n-1, \quad (26)$$

where $b^{n+1} = 0$ and $\gamma < \alpha < 1$.

First, q large enough ensures that all incremental values are strictly positive so bids are strictly decreasing in i. Second, these bids ensure that each Firm i is strictly better off at slot i than moving down to slot $i+1$. Third, since $q_i = q$ and $\gamma_i = \gamma$ for all i, $IV_{i+k+1}^{i+k}(i+k) = IV_{i+k+1}^{i+k}(i) = \gamma^{i+k-1}(1-\gamma)\left((1-\gamma)(q - \Sigma_{j>i+k+1}\Delta_j) - \Delta_{i+k+1}\right)$, $i = 1, ..., n-1$, $k = 1, ..., n-i$, so Lemma 3 applies. Hence, no downward deviation is profitable and a firm strictly prefers staying in its equilibrium slot.[35]

To analyze upward deviations, first assume that $\Delta_i = 0$ for all i. Then price is q for all i and since $q_i = q$ and $\gamma_i = \gamma$ for all i, we have an anonymous setting where incremental values depend only on the number of slots that a firm jumps over and on the slot it starts at: i.e. $IV_{i+k}^i(j) = (1-\gamma)(\gamma^{i-1} - \gamma^{i+k-1})q$ for any firm j. Then bids are $b^i = \alpha\Sigma_{k=i-2}^{n-2}\gamma^k(1-\gamma)^2 q = \alpha(\gamma^{i-2} - \gamma^{n-1})(1-\gamma)q$. Then Firms $i+k$ would be strictly worse off deviating upwards if and only if

$$b^i - b^{i+k+1} = \alpha(\gamma^{i-2} - \gamma^{i+k-1})(1-\gamma)q > IV_{i+k}^i(i+k) = (\gamma^{i-1} - \gamma^{i+k-1})(1-\gamma)q,\ \ i = 1, ..., n,\ k = 1, ..., n-i. \quad (27)$$

which holds if $\alpha > \gamma\frac{1-\gamma^k}{1-\gamma^{k+1}}$. A sufficient condition is $\alpha > \gamma$.

[35] Here we are applying a strict version of Lemma 3, which clearly holds as can be seen from the details of the proof: if one-step downward jumps are strictly dominated then so are larger downward jumps.

The difference between the left-hand side and the right-hand side of inequality (27) can be made arbitrarily large by increasing q. Now allowing Δ_i to be non-zero for all i, define $D^i_{i+k}(j) = IV^i_{i+k}(j) - (\gamma^{i-1} - \gamma^{i+k-1})(1-\gamma)q$, and the latter term cancels out any term in q from the IV expression so that $D^i_{i+k}(j)$ is independent of q.

Then the left-hand side of (27) becomes $b^i - b^{i+k+1} = \alpha\left[(\gamma^{i-2} - \gamma^{i-k-1})(1-\gamma)q + \Sigma_{\ell=i}^{i+k-1} D^{\ell}_{\ell+1}(\ell)\right]$ and the right-hand side is $IV^i_{i+k}(i+k) = (\gamma^{i-1} - \gamma^{i+k-1})(1-\gamma)q + D^i_{i+k}(i+k)$. So for q large enough, the inequality always holds and all firms would strictly lose from deviating up. ■

Proposition 9 *Assume q_1 is large enough. Then, for any ordering of Δ_i, $i = 1, ..., n$, there exists $\epsilon > 0$ such that if $|q_i - q_1| < \epsilon$ and $|\gamma_i - \gamma_1| < \epsilon$ for all $i = 1, ..., n$, there exists an equilibrium where Firm i is in slot i, $i = 1, ..., n$.*

Proof. From Proposition 8, if $q_i = q_1$ and $\gamma_i = \gamma_1$ for all i, then there is a strict equilibrium where Firm i is in slot i for all i and there are bids $b^1, ..., b^n$ such that

$$b^{i+1} - b^{i+k+1} < IV^i_{i+k}(i), \quad i = 1, ..., n-1, \; k = 1, ..., n-i, \tag{28}$$

and

$$b^{i} - b^{i+k+1} > IV^i_{i+k}(i+k), \quad i = 1, ..., n-1, \; k = 1, ..., n-i. \tag{29}$$

Keeping q_1, γ_1 and $\Delta_1, ..., \Delta_n$ fixed, $IV^i_{i+k}(i)$ and $IV^i_{i+k}(i+k)$ are both continuous functions of $(q_2, ..., q_n, \gamma_2, ...\gamma_n)$. Because (28) and (29) involve a finite number of inequalities, there exists $\epsilon > 0$ such that, if $|q_i - q_1| < \epsilon$ and $|\gamma_i - \gamma_1| < \epsilon$ for all $i = 2, ..., n$, then these inequalities remain satisfied for the same bid values as in (28) and (29). ■

A3.3 Asymmetric qualities

Before stating the main results, we construct the incremental value for some Firm j for being in some slot i rather than in some slot $i+k$ following the logic described at the beginning of Section 4.2. The relevant cases for the equilibrium analysis below are when j is positioned weakly earlier than i (which applies for downward deviations by j) and when j is positioned weakly farther down than $i+k$ (which applies for upward deviations by j). We have the following expressions. For coming down from an earlier slot:[36]

$$IV^i_{i+k}(j) = (1-\gamma)\gamma^{i=1}\left((1-\gamma^k)(q_j - \sum_{\ell>i+k}\Delta_\ell) - \sum_{\ell=i+1}^{i+k}\Delta_\ell\right) \text{ if } j \leq i, \tag{30}$$

(because if some Firm $j < i$ ends up in slot i, Firm i is promoted to slot $i-1$) and for coming up from a later slot:

$$IV^i_{i+k}(j) = (1-\gamma)\gamma^{i-1}\left((1-\gamma^k)(q_j - \sum_{\ell\geq i+k;\ell\neq j}\Delta_\ell) - \sum_{\ell=i}^{i+k-1}\Delta_\ell\right), \text{ if } j \geq i+k.$$

(because if some Firm $j > i+k$ ends up in slot $i+k$, Firm $i+k$ is demoted down to slot $i+k+1$ and contributes to the search appeal externality in slot $i+k$).

Proposition 10 *Assume $\gamma_i = \gamma$ for all i, $q_i > q_{i+1}$ for $i = 1, ..., n-1$ and Assumption 1 holds.*

(i) *Then for q_n sufficiently large there exists an equilibrium with Firm i in slot i. Furthermore, total industry profit is maximized at any such equilibrium.*

(ii) *If Assumption 2 holds and $n \geq 3$ then in any equilibrium the product with the ith highest quality, $i = 1, ..., n$ is placed at slot $i-1$, slot i, or slot $i+1$; and the preferred search order for consumers (with low-quality products placed lower)* ***cannot*** *be sustained as an equilibrium.*

Proof. The implications for welfare are straightforward from our discussion in the text so here we merely prove existence in (i) and the restriction on the equilibrium order in (ii).

To show existence in Part (i) we specify the following bid sequence:

$$b^i - \gamma b^{i+1} = IV^{i-1}_i(i-1),\ i = 2, ..., n-1, \text{ with } b^n = IV^{n-1}_n(n-1). \tag{31}$$

First, from the quality ranking, $q_i \geq q_n$ for all i and hence, if q_n is large enough, all incremental values are strictly positive. This ensures that bids are strictly decreasing in i.

[36]For completeness, note that for $i < j < i+k$ (a case that is not used in the analysis), we have $IV^i_{i+k}(j) = (1-\gamma)\left((1-\gamma^k)(q_j - \sum_{\ell>i+k}\Delta_\ell) - \sum_{i\leq\ell\leq i+k;\ell\neq j}\Delta_\ell\right)$.

Let us now turn to showing that firms do not want to deviate to lower slots. The specification of bids in (31) implies that Firm $i-1$ is indifferent between staying in slot $i-1$ and moving down to slot i, for all $i = 2, ..., n$. Furthermore, because $q_i \geq q_{i+k}$ for all $i = 1, ..., n-1$, $k = 1, .., n-k$, $IV^{i+k}_{i+k+1}(i) \geq IV^{i+k}_{i+k+1}(i+k)$ (notice that the two incremental value expressions depends neither on Δ_i nor on Δ_{i+k} so only the quality differences are relevant) and Lemma 3 applies, so Firm $i-1$ does not want to deviate to any lower slot.

To show that upward deviations are not profitable, first assume that $\Delta_i = 0$ for all i. The proof of Proposition 8 establishes that, for any $i = 1, ..., n-1$, $k = 1, ..., n-i$, if $q_\ell = q_{i+k}$ for $i-1 \leq \ell \leq i+k-1$ then $b^i - b^{i+k+1} > IV^i_{i+k}(i+k)$: this is obtained by taking $\alpha = 1$ and $q = q_{i+k}$ when computing the bids. Besides, the bid difference is increasing in the qualities of products between $i-1$ and $i+k-1$ so that the equilibrium condition remains satisfied with $q_\ell \geq q_{i+k}$, which is the case with qualities ranked in a decreasing order. Now the difference between the bid difference and the incremental value can be made arbitrarily large by increasing qualities and hence by increasing the lowest quality q_n. Then we can apply the same argument as in the proof of Proposition 8 to show that even if $\Delta_i > 0$ for some i, an upward deviation is not profitable for Firm $i+k$.

For part (ii), consider some $i = 2, ..., n-1$ and $r = 1, ..., n-i$. Equilibrium condition (15) evaluated at $k = r+1$ yields $b^i - b^{i+r+1} \leq IV^{i-1}_{i+r}(i-1)$ and equilibrium condition (17) evaluated at $k = r$ yields $b^i - b^{i+r+1} \geq IV^i_{i+r}(i+r)$ so we must have $IV^{i-1}_{i+r}(i-1) \geq IV^i_{i+r}(i+r)$ which requires that

$$(1-\gamma^{r+1})(q_{i-1} - \sum_{\ell > i+r} \Delta_\ell) - \sum_{\ell=i}^{i+r} \Delta_\ell \geq (\gamma - \gamma^{r+1})(q_{i+r} - \sum_{\ell > i+r} \Delta_\ell) - \sum_{\ell=i}^{i+r-1} \Delta_\ell$$

Rearranging, we must have

$$(1-\gamma^{r+1})q_{i-1} - (\gamma - \gamma^{r+1})q_{i+r} \geq (1-\gamma) \sum_{\ell > i+r} \Delta_\ell + \Delta_{i+r}$$

The RHS is positive so the LHS should be positive as well, and we must have

$$\frac{1-\gamma^{r+1}}{\gamma - \gamma^{r+1}} q_{i-1} \geq q_{i+r}. \text{ Now, } \frac{1-\gamma^{r+1}}{\gamma - \gamma^{r+1}} = 1 + \frac{1-\gamma}{\gamma - \gamma^{r+1}},$$

which decreases in r. Hence it is bounded above by $\frac{\gamma+1}{\gamma}$. So we must have $q_{i-1} > \frac{\gamma}{1=\gamma} q_{i+r}$. Under the assumptions in the Proposition this requires that $q_{i-1} > q_{i+r}$. This must hold for all $i = 2, ..., n-1$ and $r = 1, ..., n-i$. So the result holds. ■

A3.4 Asymmetric market potentials

As for heterogeneous qualities, we start by writing the incremental value for some Firm j for being in some slot i rather than in slot $i+k$. The incremental value is[37]

$$IV_{i+k}^{i}(j) = (1-\gamma_j)\Pi_{\ell \leq i; \ell \neq j}\gamma_\ell \left((1 - \Pi_{\ell=i+1}^{i+k}\gamma_\ell)(q - \Sigma_{\ell > i+k}\Delta_\ell) - \Sigma_{\ell=i+1}^{i+k} \right), \text{ if } j \leq i, \tag{32}$$

for a firm that would have initially moved down to slot i, while

$$IV_{i+k}^{i}(j) = (1-\gamma_j)\Pi_{\ell=1}^{i-1}\gamma_\ell \left((1 - \Pi_{\ell=i}^{i+k-1}\gamma_\ell)(q - \sum_{\ell \geq i+k; \ell \neq j} \Delta_\ell) - \Sigma_{\ell=i}^{i+k-1}\Delta_\ell \right), \text{ if } j \geq i+k, \tag{33}$$

for a firm that would have initially moved up to slot $i+k$.

Proposition 11 *Assume $q_i = q$ for all i, $\gamma_i < \gamma_{i+1}$ for $i = 1, ..., n-1$, products can be ordered in terms of niche dominance and q large enough:*

(i) *there exists an equilibrium with Firm i in slot i (maximum consumer surplus order);*

(ii) *under Assumption 3 with $n \geq 3$, there is no equilibrium with $\gamma_i > \gamma_{i+1}$, $i = 1, ..., n-1$, (maximum joint profit order).*

Proof. Welfare implications are straightforward in the text so here we focus on existence of an equilibrium in (i) and non-existence of an equilibrium in (ii).

To prove existence in part (i) we specify the following bids:

$$b^i - \gamma_i b^{i+1} = IV_i^{i-1}(i-1), \; i = 2, ..., n-1, \text{ with } b^n = IV_n^{n-1}(n-1). \tag{34}$$

First, for q sufficiently large, all incremental values are strictly positive so that bids as specified in the proposition are strictly decreasing in i.

[37] Again, the case $i < j < i+k$ is not relevant to the analysis: we would then have $IV_{i+k}^{i}(j) = (1-\gamma_j)\left((1 - \Pi_{i \leq \ell \leq i+k, \ell \neq j}\gamma_\ell)(q - (n-i-k)\Delta) - k\Delta\right)$.

We now turn to showing that firms do not want to deviate to lower slots. The specification of bids in (34) implies that Firm $i-1$ is indifferent between staying in slot $i-1$ and moving down to slot i, for all $i = 2, ..., n$. Furthermore, because $1-\gamma_i \geq 1-\gamma_{i+k}$ for all $i = 1, ..., n-1$, $k = 1, .., n-i$, $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$ because Firm $i+k$ sells with a lower probability conditional on a click and because its number of clicks in slot $i+k$ when positioned after Firm i is smaller than what Firm i would obtain in that same slot when positioned after Firm $i+k$. Hence Lemma 3 applies and Firm $i-1$ does not want to deviate to any lower slot.

To deal with upward jumps consider a deviation upwards by some Firm $i+k$, $i = 1, ..., n-1$, $k = 1, ..., n-i$ to slot i. Assume first $\Delta_i = 0$ for all i. Then we have $IV_j^{j-1}(j-1) = IV_j^{j-1}(j) = \Pi_{\ell=1}^{j=1}\gamma_\ell(1-\gamma_{j-1})(1-\gamma_j)q$ for all $j = 1, ..., n$. Now, the gain for Firm $i+k$ from moving up to slot i gross of the change in fees can be written as $IV_{i+k}^{i}(i+k) = \pi^i(i+k) - \pi^{i+k}(i+k) = \Sigma_{j=i+1}^{i+k} IV_j^{j-1}(i+k)$. The corresponding increase in fees is $b^i - b^{i+k+1} = \Sigma_{j=i}^{i+k} IV_j^{j-1}(j-1)$. Because market potentials for all firms $j < i+k$ are larger than that of product $i+k$, all the terms for $j = i+1, ..., i+k$ are larger for $b^i - b^{i+k+1}$ than for $IV_{i+k}^{i}(i+k)$ and, since $IV_i^{i-1}(i-1) > 0$ we clearly have

$$b^i - b^{i+k+1} > IV_{i+k}^{i}(i+k).$$

Both terms in the above inequality being proportional to q, the difference can be made arbitrarily large by increasing q so the case where $\Delta_i > 0$ for some i can be dealt with using the same argument as in the proof of Proposition 8. This proves (i).

Consider some $i = 2, ..., n-1$. Equilibrium conditions (15) and (17) imply

$$IV_{i+1}^{i-1}(i-1) \geq b^i - b^{i+2} \geq IV_{i+1}^{i}(i+1).$$

Hence we must have

$$\begin{aligned} & \left((1-\gamma_i\gamma_{i+1})(q - \Sigma_{j>i+1}\Delta_j) - \Delta_i - \Delta_{i+1}\right)(1-\gamma_{i-1}) \\ \geq\ & \left(\gamma_{i-1}(1-\gamma_i)(q - \Sigma_{j>i+1}\Delta_j) - \Delta_{i+1}\right)(1-\gamma_i), \end{aligned}$$

or

$$\frac{(1-\gamma_i\gamma_{i+1})(q-\Sigma_{j>i+1}\Delta_j)-\Delta_i-\Delta_{i+1}}{\gamma_{i-1}(1-\gamma_i)(q-\Sigma_{j>i+1}\Delta_j)-\Delta_i}(1-\gamma_{i-1})\geq(1-\gamma_{i+1}).$$

As q tends to infinity the LHS tends to $\frac{(1-\gamma_i\gamma_{i+1})}{\gamma_{i-1}(1-\gamma_i)}(1-\gamma_{i-1})$. Now if $1-\gamma_{i-1}\leq 1-\gamma_i$ then the assumption in the proposition implies that $1-\gamma_{i-1}<\gamma_{i-1}(1-\gamma_i)^2$, so for q large enough equilibrium would require that $(1-\gamma_i\gamma_{i+1})(1-\gamma_i)>(1-\gamma_{i+1})$.

Because $1-\gamma_i\gamma_{i+1}<1$ the above inequality cannot hold if $1-\gamma_i\leq 1-\gamma_{i+1}$. ■

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