Pseudo Lindahl Equilibrium as a Collective Choice Rule†

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Abstract

A collective choice problem specifies a finite set of alternatives from which a group of expected utility maximizers must choose. We associate a collective pseudo market with every collective choice problem and establish the existence and efficiency of *pseudo Lindahl equilibrium (PLE)* allocations. We also associate a cooperative bargaining problem with every collective choice problem and define a set-valued solution concept, the *ω-weighted Nash bargaining set* where ω is a vector of welfare weights. We provide axioms that characterize the ω-weighted Nash bargaining set. Our main result shows that ω-weighted Nash bargaining set payoffs are also the PLE payoffs of the corresponding collective pseudo market with the same utility functions and incomes ω. We define a *pseudo core* for collective pseudo markets and show that pseudo Lindahl equilibria are in the pseudo core. We characterize the set of PLE outcomes of discrete allocation problems and show that they contain the set of pseudo Walrasian equilibrium outcomes.

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1. Introduction

An organization must choose among several alternatives that affect the welfare of its members. The choice must be equitable and efficient. Monetary transfers among its members are not feasible. Examples of this type of situation include a community's decision on how to allocate infrastructure investments among neighborhoods, the allocation of office space among groups within an organization, or the assignment of college roommates.

In this paper, we analyze a pseudo-market solution to this problem. Each group member is given a budget of fiat money, that is, money that has no value beyond the confines of this specific market, and confronts a price for each alternative. As in standard consumer theory, members choose an alternative that maximizes their utility subject to the budget constraint. The organization acts as an auctioneer and implements an alternative that maximizes revenue. We allow choices to be stochastic; that is, agents choose lotteries over social outcomes.

In a standard public goods setting (Foley, 1969), individuals contribute tangible resources to the public good and, by varying these contributions, can transfer resources among each other. In our setting, agents cannot be asked to contribute or transfer tangible resources. Hylland and Zeckhauser (1979) introduced pseudo markets to study efficient allocations of indivisible private goods when real money transfers are not feasible. We apply their approach to collective choice problems and, therefore, refer to our mechanism as a *collective pseudo market*. A collective pseudo market treats each social outcome as a collection of personalized goods, one for each agent, and agents behave like consumers in a competitive economy. Thus, collective pseudo market equilibria resemble the Lindahl equilibria (Foley, 1969) of the public goods literature. Therefore, we refer to them as pseudo Lindahl equilibria (PLE).

As an example, consider an organization that must allocate office space to its members. A member's utility depends not only on her own office but also on who occupies nearby offices. Thus, utilities depend on the office *allocation*. The organization must choose among several plans. In the collective pseudo market, every member, i, has ω_i units of fiat money[1] and faces price p_i^j for plan j; she must choose an optimal lottery over plans

[1] We normalize the fiat money endowments so that $\sum_i \omega_i = 1$.

subject to her budget constraint. In equilibrium, the organization chooses a lottery that maximizes revenue, and that choice must coincide with each member's optimal choice.

The organization's decision problem can also be described as an n-person bargaining problem. A bargaining problem takes as primitive the set of attainable utility profiles. Thus, we map the organization's decision problem into a bargaining problem by identifying each lottery over outcomes with the corresponding vector of members' utilities. We define and characterize a set-valued and weighted version of the Nash bargaining solution for n-person bargaining problems called the weighted-Nash bargaining set (wNBSet).[2]

The simplest bargaining problem is one of pure conflict where a single prize must be allocated via a lottery, and the utility of each member is equal to the probability with which she receives the prize. In this situation, the wNBSet contains a unique payoff vector that assigns the prize to agent i with a probability equal to i's bargaining weight ω_i. We interpret the pure-conflict case as a benchmark that reveals the organization's attitude to equity among its members. If every agent has the same bargaining weight, then the organization treats each agent equally; other weights reflect the priority of some members over others. For example, if the agents represent neighborhoods of a city, the bargaining weight might be proportional to the neighborhood's population. For bargaining games that are not of the pure-conflict form, the wNBSet specifies all payoff vectors that can be reconciled with the organization's normative judgment derived from the benchmark case.

Our main result (Theorem 1) relates pseudo Lindahl equilibria to the wNBSet. It shows that the set of equilibrium payoff vectors of a collective pseudo market with fiat money endowments $(\omega_i)_{i=1}^n$ is the same as the wNBSet of the corresponding bargaining problem with weights $(\omega_i)_{i=1}^n$. By relating pseudo Lindahl equilibria to wNBSet payoff vectors, Theorem 1 clarifies the normative judgment implied by the fiat money endowments: each person's share of the budget reflects the probability that they would be awarded the prize in a pure conflict situation.

We provide axiomatic foundations for the wNBSet in Theorem 3. It is similar to the characterization of the Nash bargaining solution, except that we allow the solution to be set-valued. As in Nash's theorem (Nash, 1950), the substantive axioms are efficiency and

[2] The wNBSet is a multivalued solution concept even for a fixed set of weights. We normalize the bargaining weights ω so that ω_i is between zero and one and $\sum_i \omega_i = 1$.

a version of independence of irrelevant alternatives (IIA). Our notion of IIA, which we call Consistency, requires that x is a solution to the bargaining problem B if it is a solution to a bargaining problem A that dominates B in the weak set order. The bargaining problem A dominates B in the weak set order if for every utility profile $y \in B$ there is a utility profile $x \in A$ such that $y \leq x$, and conversely, for every utility profile $x \in A$ there is a utility profile $y \in B$ with $y \leq x$.

1.1 A Three Person Example

Consider an organization with three members who have equal bargaining weights, given by the vector $\sigma = (1/3, 1/3, 1/3)$. The symmetric Nash bargaining set — i.e., the weighted Nash bargaining set with weights σ — for a bargaining problem $B \subset \mathbb{R}^3$ consists of all utility profiles $x \in B$ that are the Nash bargaining solution of some set A that dominates B in the weak set order.[3]

Suppose a single prize can be awarded to one of the members, and that each member's utility is 1 when receiving the prize and 0 otherwise. The unit simplex

$$\Delta := \{x \in \mathbb{R}^3_+ \,|\, x_1 + x_2 + x_3 \leq 1\}$$

describes this situation. The Nash bargaining solution of Δ is σ, which is also the unique element of the symmetric Nash Bargaining set. This uniqueness holds because if a set B dominates Δ in the weak set order, then the Nash bargaining solution of B is in Δ if and only if $B = \Delta$.

For any $a = (a_1, a_2, a_3)$ and $d = (d_1, d_2, d_3)$ such that $a_i > 0$, $d_i \geq 0$ for all i, we let $a \odot \Delta + d$ denote the simplex B such that

$$B = \{y \,|\, \text{for some } x \in \Delta, y_i = a_i x_i + d_i \text{ for } i = 1, 2, 3\}$$

Because our bargaining solution is invariant under affine transformations of utilities, the unique solution of $a \odot \Delta + d$ is $a/3 + d$.

[3] Recall that for bargaining problem B with disagreement point d, the Nash bargaining solution is the unique maximizer of $f(x) = \sum_{i=1}^{3} \log(x_i - d_i)$ over $x \in B$.

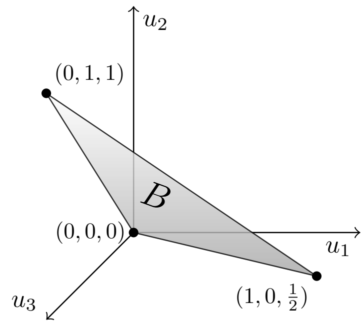

Figure 1

Next, suppose that the organization must choose between two new locations for its office space. The status quo yields the utility profile $u^s = (0,0,0)$; location r yields $u^r = (1,0,1/2)$; and location l yields $u^l = (0,1,1)$. The corresponding bargaining problem, B, is the convex hull of u^s, u^r and u^l, as indicated in Figure 1.

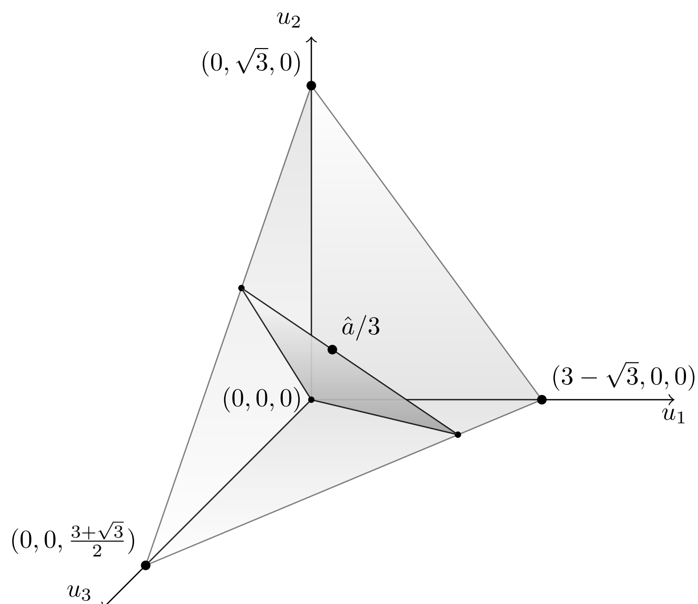

Figure 2

Figure 2 depicts the simplex $\hat{A}$ that dominates the bargaining problem B. The simplex $\hat{A}$ has the disagreement point $(0,0,0)$ and the vertices $(3-\sqrt{3},0,0), (0,\sqrt{3},0)$, and $(0,0,\frac{3+\sqrt{3}}{2})$. As Figure 2 illustrates, every payoff vector in B is dominated by some payoff vector in $\hat{A}$, and every payoff vector in $\hat{A}$ dominates some payoff vector in B. Therefore, $\hat{A}$ dominates B in the weak set order.

The bargaining problem $\hat{A}$ has $\hat{a}/3$ as the unique element of the symmetric Nash bargaining set, where $\hat{a} = (3-\sqrt{3}, \sqrt{3}, \frac{3+\sqrt{3}}{2})$ and $\hat{A} = \hat{a} \odot \Delta$. Since $\hat{a}/3$ is an element of B, it is an element of the symmetric Nash bargaining set of B. Note that the payoff vector $\hat{a}/3$ is the Nash bargaining solution of B and corresponds to the allocation $q^r = 1 - \frac{\sqrt{3}}{3}, q^l = \frac{\sqrt{3}}{3}$; that is, the organization chooses location r with probability $1 - \frac{\sqrt{3}}{3}$ and location l with probability $\frac{\sqrt{3}}{3}$.

The symmetric Nash bargaining set of B contains more than one solution. To identify a second solution, let $a = (1, 2, 1)$ and $d = (0, 0, 1/2)$. Then, let $A = a \odot \Delta + d$ be the simplex with the extreme points $(0, 0, 1/2), (1, 0, 1/2), (0, 2, 1/2)$ and $(0, 0, 3/2)$. Figure 3, below, depicts this bargaining problem and its intersection with the original bargaining problem B. As Figure 3 shows, A also dominates B in the weak set order.

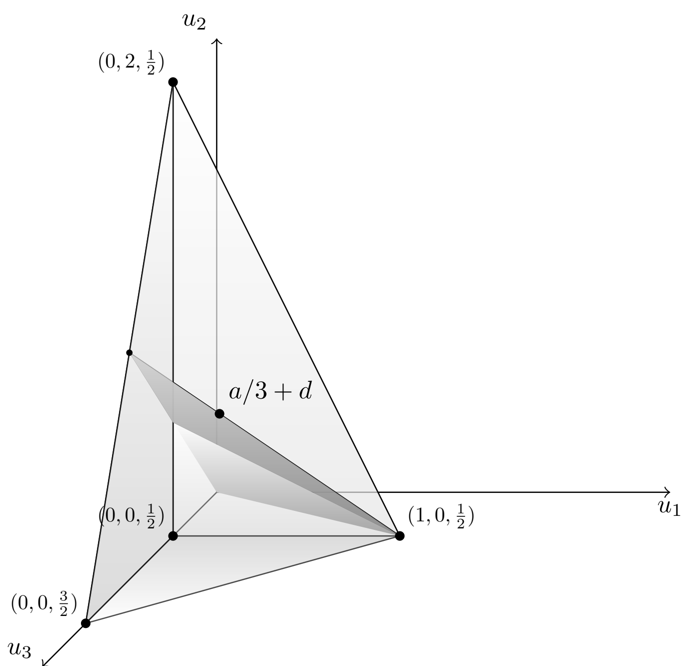

Figure 3

Note that player 3's minimal payoff in A is $1/2$ while it is 0 in B, and, therefore, A raises the disagreement utility of player 3. The Nash bargaining solution of A is $a/3 + d$ and, since $a/3 + d$ is an element of B, it is also an element of the symmetric Nash bargaining set of B. This solution corresponds to the outcome $q^r = 1/3, q^l = 2/3$.

The symmetric Nash bargaining set consists of all convex combinations of the solutions in Figures 2 and 3. The solution in Figure 2, $\hat{a}/3$, yields the lowest payoff for player 3 while the solution in Figure 3, $a/3 + d$, yields the highest. For every other solution, x, there is a simplex $\tilde{A} = \tilde{a} \odot \Delta + \tilde{d}$ such that $B \leq \tilde{A}$, $x = \tilde{a}/3 + \tilde{d}$ and $\tilde{d} = (0, 0, c)$ with $0 \leq c \leq 1/2$.[4]

[4] If $c > 1/2$ then, for any $\tilde{a}$, the simplex $\tilde{a} \odot \Delta + (0, 0, c)$ either does not dominate B or its Nash bargaining solution is not in B. Thus, the solution in Figure 3 is an extreme point of the symmetric Nash bargaining set.

The organization can implement every element of the symmetric Nash bargaining set through a collective pseudo market with equal budgets. In such a market, each of the three agents is endowed with 1/3-units of fiat money and confronts the personalized prices $p_i = (p_i^r, p_i^l)$. Agents treat the collective goods r and l as if they were private goods, available for purchase at the indicated prices. Agents may also choose a lottery over the two goods and, if they do, they must pay the lottery's expected cost. The organization acts as an auctioneer and picks the lottery that maximizes revenue.

Suppose that $q = (q^r, q^l) = (1/3, 2/3)$, the solution in Figure 3, is the desired payoff vector. The dominating simplex in Figure 3 is $A = a \odot \Delta + d$ such that $a = (1, 2, 1)$ and $d = (0, 0, 1/2)$. Using the parameters of A, we define p_i as follows:

$$p_i = \left(p_i^r, p_i^l\right) = \left(\frac{u_i^r - d_i}{a_i}, \frac{u_i^l - d_i}{a_i}\right)$$

which yields $p_1 = (1, 0), p_2 = (0, 1/2)$ and $p_3 = (0, 1/2)$. Note that $p_i \cdot q = \omega_i = 1/3$ for $i = 1, 2, 3$, and therefore all three agents can afford q. Furthermore, q maximizes the utility of each agent at those prices. To verify that (p, q) is a PLE, it remains to check auctioneer optimality. The auctioneer's revenue from selling r is $\sum_i p_i^r$ while the revenue from selling l is $\sum_i p_i^l$. Since

$$\sum_i p_i^r = 1 = \sum_i p_i^l,$$

the lottery q maximizes revenue. Thus, (p, q) is a PLE.

Using the simplex $\hat{A}$, the same algorithm yields the equilibrium prices for the solution $\hat{a}/3$, depicted in Figure 2. For the solutions between $\hat{a}/3$ and $a/3 + d$, the prices can be found with an analogous construction. Thus, every element of the symmetric Nash bargaining set corresponds to an equilibrium with equal budgets. Theorem 1, our main result, shows that the reverse implication holds as well: every PLE utility profile is in the symmetric Nash bargaining set. Hence, convex combinations of $\hat{a}/3$ and $a/3 + d$ are the only PLE utilities with equal budgets.

1.2 Properties and Applications

Foley (1970) shows that in standard public goods economies every Lindahl equilibrium is in the core. In his framework, agents hold physical resources, and a coalition can block an allocation by reallocating these resources to achieve higher utility for its members than what they would enjoy with the proposed allocation. In our model, however, agents are endowed only with fiat money, which neither generates utility nor produces public goods. Therefore, the usual notion of a blocking coalition does not apply.

To define the pseudo core, we reinterpret each agent's fiat money endowment as stochastic property rights over allocations. A coalition, I, with collective fiat money endowment, ω_I, can block an allocation if it can find a feasible alternative that assigns probability $1 - \omega_I$ to the disagreement point *and* makes all its members better off. The pseudo core is the set of all allocations that no coalition can block.

Theorem 2 shows that every PLE (and thus every element of the weighted Nash bargaining set) belongs to the pseudo core. Moreover, when there are two agents, the pseudo core coincides with the set of pseudo Lindahl equilibria.[5]

Pseudo Lindahl equilibria (PLEs) are generally not unique. However, in Proposition 1, we show that they are unique if utilities are *binary*, that is, if every agent views each outcome as either ideal or no better than the disagreement point.

In the paper's final section, we explore the connection between PLE allocations and pseudo Walrasian equilibrium allocations in discrete allocation problems. Here, a finite set of indivisible goods must be allocated among the agents. A *private pseudo market* transforms the original allocation problem into an exchange economy by endowing agents with fiat money and introducing an auctioneer who maximizes (fiat money) revenue. We show that every pseudo Walrasian equilibrium in such an economy is a PLE of the corresponding collective choice market. Our main theorem then implies that all pseudo Walrasian equilibrium payoffs of a private pseudo market with endowments ω are in the $\omega-$weighted Nash bargaining set. However, while a PLE always exists, a pseudo Walrasian equilibrium may not. In particular, pseudo Walrasian equilibria exist if the goods are gross substitutes[6], but they may fail to exist under more general preferences that allow for complementarities.

We also show that pseudo Walrasian equilibrium payoffs are sensitive to how the traded goods are specified (i.e., how property rights are defined), whereas PLE payoffs are not.

[5] Our definition of the pseudo core is related to the definition of the core in Fain, Goel and Munagala (2016) who also analyze an economy without individual endowments. In their definition, each coalition can allocate a share of the common resources proportional to the coalition size. Their model has a unique Lindahl equilibrium and they show it is in the core.

[6] See Gul, Pesendorfer and Zhang (2024)

As Sertel and Yildiz (2003) demonstrate in a different context, two exchange economies can yield the same bargaining set (i.e., the same set of feasible utility vectors) and have the same money endowments, yet still produce different pseudo Walrasian equilibrium payoffs. In contrast, as our main result shows, PLE payoffs depend only on the set of attainable utilities and money endowments.

Finally, we provide a condition under which pseudo Walrasian and PLE payoffs coincide. An example satisfying this condition is a matching market where individuals must form pairs. Such markets include the one-sided roommate problem or the classic two-sided matching problem[7]. Our results imply that pseudo Walrasian equilibrium payoffs in a matching market with non-transferable utility and endowments, $(\omega_i)_{i=1}^n$, coincide with the payoffs in the weighted Nash bargaining set, with weights $(\omega_i)_{i=1}^n$.

1.3 Related Literature

Our paper is related to the extensive literature on axiomatic bargaining theory (see Thomson (1994) for a survey). Theorem 3 is related to Nash (1950) and Harsanyi (1959) and we discuss this relationship in detail below. For 2-person bargaining, the symmetric NBSet includes the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) and the Perles-Maschler solution (Perles and Maschler (1981)). Kaneko (1980) provides a set-valued extension of the Nash bargaining solution to incorporate non-convex bargaining sets. His solution, when restricted to convex sets – i.e., in our setting – becomes the (single-valued) Nash bargaining solution.

Hylland and Zeckhauser (1979) were the first to propose pseudo Walrasian equilibria with randomization as solutions to allocation problems in situations with indivisibilities and without transfers. Miralles and Pycia (2021), Gul, Pesendorfer, and Zhang (2024), and Nguyen and Vohra (2024) extend Hylland and Zeckhauser from unit demand preferences to more general preferences. The collective pseudo markets studied in this paper allow for arbitrary preferences, public goods, and externalities and hence provide a further generalization of the environment considered in these papers.

Eisenberg and Gale (1959) show that Walrasian equilibria of a Fisher market[8] with equal budgets yield the Nash bargaining solution. Eisenberg (1961) extends this result

[7] Gale and Shapley (1962)

[8] A Fisher market is an exchange economy with fiat money but no indivisibilities.

to non-equal budgets and the weighted Nash bargaining solution. As is well known, this connection breaks down in the presence of indivisibilities. With indivisibilities, pseudo Walrasian equilibria of Fisher markets may fail to exist and, if they exist, they generally do not coincide with the Nash bargaining solution. By contrast, PLEs in our model always exist and contain the set of pseudo Walrasian equilibrium outcomes of the associated discrete allocation problem. Moreover, the weighted Nash bargaining set contains the weighted Nash bargaining solution. Thus, using more permissive solution concepts, we extend the classic results of Eisenberg and Gale (1959) and Eisenberg (1961) to include indivisibilities and externalities.

Fain, Goel and Munagala (2016) consider an economy in which the planner must allocate a fixed budget among a finite collection of public goods. Agents' utilities for the public goods are a linear function of the funding levels. They show that the Nash bargaining solution yields a PLE allocation of an economy in which each agent controls an equal share of the budget. We do not assume their linear structure[9] and, as a result, there are typically multiple pseudo Lindahl equilibria. Nonetheless, we show that pseudo Lindahl equilibria implement a suitable set-valued generalization of the Nash bargaining solution. Brandl, Brandt, Peters, Stricker and Suksompong (2020) examine the incentive properties of the Nash bargaining solution and show that it satisfies a weak form of incentive compatibility.

Foley (1967), Schmeidler and Vind (1972) and Varian (1974) associate fairness with envy-freeness. Pseudo Walrasian equilibria with equal budgets are envy free and, thus, these authors establish a connection between Walrasian outcomes and fairness. In a public goods setting, it is difficult to identify fairness with envy-freeness. Efficiency may require two agents to contribute different amounts to the same public good and, therefore, the notion of envy-freeness is difficult to apply. Several authors have developed notions of fairness for public goods settings with transfers. Sato (1987) adapts the notion of envy-freeness to Lindahl equilibria by assuming that agent i converts j's actual consumption of the public good into a virtual quantity based on j's utility of that good. Buchholz and Peters (2007) characterize Lindahl equilibria with axioms on agents' marginal rates

[9] In our model, randomization creates some linearity. However, allowing lotteries is fundamentally different from having a linear production technology since probabilities are constrained to be in the unit simplex. This "probability constraint" is the source of multiplicity in our model.

of substitution between private and public goods. Silvestre (2003) develops a connection between Lindahl's normative ideas and the core of an economy.

Chen and Zeckhauser (2018) analyze a public goods provision game between countries. They discuss the Nash bargaining solution and a version of Lindahl equilibrium. In a two country application, they find that their version of Lindahl equilibrium is close (but not identical) to the Nash bargaining solution. Our model differs from theirs in two key aspects. First, agents in Chen and Zeckhauser's model contribute tangible resources to the public good and, second, the disagreement outcome in their setting is not exogenous; it is the Nash equilibrium of a non-cooperative contribution game.

Finally, our paper is related to the analysis of random assignment problems and fairness (Abdulkadiroglu and Sonmez (1998), Bogomolnaia and Moulin (2001), Budish (2011), Basteck (2018)). That literature takes the agents' ordinal preferences over the relevant alternatives as a primitive while the pseudo-market approach, followed here, takes the agents' lottery preferences as a primitive and, therefore, imposes stronger efficiency requirements.

2. PLE and ω-NBSet

In this section, we describe two different solutions to a social choice problem. Our main result establishes their equivalence and the subsequent sections take advantage of this equivalence to study properties of *the* solution.

There are n agents, $i \in \{1, \dots, n\}$, who must decide on one of k social outcomes, $j \in K = \{1, \dots, k\}$ or settle for the disagreement outcome (outcome 0). A *random outcome* is a probability distribution over social outcomes. We let

$$Q := \{q \in [0,1]^k \mid \textstyle\sum_{j \in K} q^j \leq 1\}$$

be the set of random outcomes with the understanding that $1 - \sum_{j \in K} q^j$ is the probability of disagreement.

Agents are expected utility maximizers; i's utility from outcome $j \in K$ is $u_i^j \geq 0$ and $u_i = (u_i^1, \dots, u_i^k)$ denotes i's von Neumann-Morgenstern utility index. The disagreement outcome, outcome 0, yields the utility profile $u^0 = o := (0, \dots, 0)$. We dismiss all agents who have no stake in the collective decision; that is, we assume that for every utility u_i, there is some j such that $u_i^j > 0$.

A social choice rule, then, is a mapping that associates a (feasible) set of utility vectors $S(u) \subset \mathbb{R}_+^n$ with every profile of utilities u. Feasibility requires that each $x \in S(u)$ can be achieved by some probability distribution, q, over random outcomes; that is,

$$S(u) \subset \{(u_1 \cdot q, \ldots, u_n \cdot q) \,|\, q \in Q\}$$

Below, we consider two classes of social choice rules: The first is an application of PLE to our setting while the second is a multi-valued version of the Nash bargaining solution, which we call the weighted-Nash bargaining set (wNBSet).

2.1 Collective Pseudo Markets and Pseudo Lindahl Equilibrium

Hylland and Zeckhauser (1979) introduce pseudo markets as a way of allocating indivisible private goods in settings where transfers are not possible. The designer (or auctioneer) of a pseudo market issues fiat money, that is, money without value beyond the market's confines. Participants use this money to purchase probabilistic shares of the indivisible goods; the auctioneer earns revenue by selling these shares. A collective pseudo market applies this idea to a setting with collective goods. As in a standard exchange economy, all agents, including the auctioneer, are price takers. Market clearing requires every agent, including the auctioneer, to (optimally) choose the same outcome.

Agent i is endowed with $\omega_i > 0$ units of fiat money and confronts the personalized prices $p_i = (p_i^1, \ldots, p_i^k) \geq 0$ for the k social outcomes. We normalize the aggregate money endowments to 1, that is,

$$\sum_i \omega_i = 1$$

Agents in a collective pseudo market choose random social outcomes as if they were private goods. Let

$$U_i(p, \omega_i) = \max_{q \in Q} u_i \cdot q \text{ subject to } p_i \cdot q \leq \omega_i \tag{1}$$

be the maximal utility agent i can attain at prices p_i and money endowment ω_i. The random outcome q is a solution to the agent's problem if $u_i \cdot q = U_i(p, \omega_i)$. It is a *least-cost* *solution* if, in addition, $p_i \cdot \hat{q} \geq p_i \cdot q$ for every $\hat{q}$ such that $u_i \cdot \hat{q} = U_i(p, \omega_i)$. Throughout our analysis we will assume that agents choose least-cost solutions. This restriction ensures that the first welfare theorem holds and is standard in the literature on pseudo markets with private goods (see Mas-Colell, 1992). Without it, there may be additional, possibly inefficient, equilibria.[10]

The social planner (or auctioneer) takes prices as given and chooses the random outcome that maximizes revenue. Specifically, the auctioneer chooses q to solve

$$R(p) = \max_{q \in Q} \sum_{i=1}^{n} p_i \cdot q \tag{2}$$

Market clearing requires that all agents and the auctioneer choose the same random outcome.

Definition: *The pair (p, q) is a pseudo Lindahl equilibrium (PLE) for the collective pseudo market (u, ω) if, at prices p, q is a least-cost solution to every consumer's maximization problem and solves the auctioneer's maximization problem.*

The collective pseudo-market approach yields the following class of social choice rules, indexed by the fiat-money endowments ω:

$$L_\omega(u) = \{(u_1 \cdot q, \ldots, u_n \cdot q) \,|\, q \text{ is a PLE outcome for } (u, \omega)\}$$

The social choice rule $L_\omega(u)$ identifies all utility profiles that correspond to pseudo Lindahl equilibria of the collective pseudo market (u, ω).

2.2 The Bargaining Problem and the weighted-Nash Bargaining Set

Next, we define our set-valued generalization of the Nash bargaining solution, the weighted-Nash Bargaining set (wNBSet). As in the standard Nash bargaining solution, the weights are welfare weights.

[10] Inefficiency arises because, with finitely many social alternatives, utilities cannot satisfy local non-satiation. For example, consider the following 2-person, 2-alternative problem. Let $\omega = (1/2, 1/2)$; that is, the two agents have equal endowments of fiat money, and $u_1 = (1, 1); u_2 = (1, 0)$. In this example, $p_1 = (1, 0), p_2 = (0, 1)$, and $q = (1/2, 1/2)$ is an inefficient PLE. Notice that q is not a least-cost solution for agent 2 since $\hat{q} = (1, 0)$ costs agent 2 zero and yields the same utility as q.

For any $x, y \in I\!R^n$, we write $x \leq y$ to mean $x_i \leq y_i$ for all i. For any bounded set X, $d(X)$ is the greatest lower bound of X and $b(X)$ is the least upper bound of X. We refer to $d(X)$ as the *disagreement point* and to $b(X)$ as the *bliss point*. A polytope is the convex hull of finitely many points in $I\!R^n$. A bargaining problem is a polytope that contains its disagreement point and some payoff vector that strictly dominates the disagreement point. Let $\mathcal{B}$ be the set of polytopes B such that $o \leq d(B) \in B$ and, for some $x \in B$, $d_i(B) < x_i$ for all i. Thus, $\mathcal{B}$ is the set of bargaining problems.

We map the utility u of the social choice problem to the bargaining problem

$$B_u := \text{conv}\{(u_1^j, \ldots, u_n^j) \,|\, j \in K \cup \{0\}\}$$

where conv$X$ is the convex hull of $X \subset I\!R^n$ and $(u_1^0, \ldots, u_n^0) = o$. Note that $B_u \in \mathcal{B}$ with $d(B_u) = o$. Thus, the set of social choice problems consists of all those elements of $\mathcal{B}$ that have o as their disagreement point. The larger class of bargaining problems with non-negative disagreement points is needed to define our bargaining solution and to provide an axiomatic foundation for it.

We write $B \leq A$ if A dominates B in the *weak set order*, that is, if for every utility vector in B, there is a corresponding utility vector in A that dominates it and, conversely, for every utility vector in A, there is a corresponding utility vector in B that is dominated by it.

Definition: *For $A, B \in \mathcal{B}$, $B \leq A$ if for every $x \in B, y \in A$, there exist $x' \in B$, $y' \in A$ such that $x' \leq y$ and $x \leq y'$.*

The Nash bargaining solution (Nash (1950), Harsanyi (1959)) is the unique element of B that maximizes $f(B, \cdot)$ where

$$f(B, x) := \sum_i \log(x_i - d_i(B))$$

Let $\omega = (\omega_1, \ldots, \omega_n)$ be a set of weights; that is $\omega_i > 0$ for all i and $\sum_i \omega_i = 1$. Then, let

$$f_\omega(B, x) := \sum_i \omega_i \log(x_i - d_i(B))$$

The unique element of B that maximizes $f_\omega(B,\cdot)$ is the ω-weighted Nash bargaining solution, denoted $\eta_\omega(B)$. The ω-weighted Nash bargaining set of B, denoted $N_\omega(B)$, consists of all those $x \in B$ that are the ω-weighted Nash bargaining solutions of bargaining problems that dominate B:

Definition: $N_\omega(B) = \{\eta_\omega(A) \in B \,|\, B \leq A\}$

The set $N_\omega(B)$ is non-empty since $\eta_\omega(B) \in N_\omega(B)$. In our applications and examples, we often choose symmetric weights, that is, $\omega = \sigma := (1/n, \cdots, 1/n)$. We refer to $N_\sigma(B)$ as the *symmetric* Nash bargaining set. We interpret the weights as an exogenous parameter reflecting the normative judgment of the social planner. However, it is instructive to consider the range of payoff vectors that can be obtained by varying these weights. For any Pareto efficient $x \in B$ with $x_i > d(B)_i$ for all i, there are weights, ω, such that $x = \eta_\omega(B)$.[11] The converse is also true: if x is a weighted Nash bargaining solution for some weights, then it is Pareto efficient, and every x_i is strictly greater than agent i's disagreement utility. The same is true for the weighted Nash bargaining set: $x \in N_\omega(B)$ for some weights ω if and only if $x \in B$, x is Pareto efficient, and $x_i > d(B)_i$ for every i.

Let $e^i = (0,\ldots,0,1,0,\ldots,0)$ denote the n-dimensional vector with i coordinate 1 and all other coordinates zero. Let $e = (1,\ldots,1)$, and let $\Delta = \operatorname{conv}\{o, e^1, \ldots, e^n\}$ be the unit simplex. As we noted in the introduction, Δ represents a situation of pure conflict in which a single prize must be awarded to one of the n agents. For any $a, x \in I\!R^n$, let $a \odot x = (a_1 \cdot x_1, \ldots, a_n \cdot x_n)$, $a \odot B = \{a \odot x \,|\, x \in B\}$, and $B + z = \{x + z \,|\, x \in B\}$. The bargaining set, A, is a *simplex* if, for some $a > 0$ and $d \geq 0$, $A = a \odot \Delta + d$. Every simplex represents the same situation of pure conflict as the unit simplex with utilities undergoing the affine transformation $x_i \mapsto a_i x_i + d_i$. It is well known (and easy to verify) that $\eta_\omega(\Delta) = \omega$ and

$$\eta_\omega(a \odot \Delta + d) = \omega \odot a + d \tag{3}$$

Lemma 1 shows that for simplices the weighted Nash bargaining set contains no other solutions.

[11] We provide a proof of this assertion in Lemma A3(iv). Recall that welfare weights must be strictly positive for every agent. The converse – that is, $\eta_\omega(B)$ is Pareto efficient and $\eta_\omega(B)_i > d(B)_i$ – follows from the fact that all weights are strictly positive, that B contains x such that $x_i > d(B)_i$ for all i, and that the weighted Nash bargaining solution is Pareto efficient.

Lemma 1: *(i) For any simplex A, $N_\omega(A) = \{\eta_\omega(A)\}$; (ii) For any $B \in \mathcal{B}$, there is a simplex A such that $B \leq A$ and $\eta_\omega(B) = \eta_\omega(A)$.*

Proof: See proof of Lemma A3 (ii) and (iii) in the appendix.

Part (ii) of Lemma 1 shows that in our definition of N_ω, above, we can assume, without loss of generality, that A is a simplex. Thus, $x \in B$ is an element of the $\omega-$weighted Nash bargaining set for the bargaining problem B if and only if there is a simplex $A = a \odot \Delta + d$ such that $B \leq A$ and $x = a \odot \omega + d$.

2.3 The Main Result

Our main result, Theorem 1 below, shows that the weighted Nash Bargaining set with weights ω coincides with the set of PLE payoffs with budgets ω.

Theorem 1: *The set of PLE payoffs of (u, ω) is the same as the wNBSet of the corresponding bargaining problem with weights ω: $L_\omega(u) = N_\omega(B_u)$ for all u, ω.*

Theorem 1 enables us to use the term Lindahl-Nash solution and write LN_ω for both the bargaining solution and social choice rule. Thus, $LN_\omega(u) = L_\omega(u) = N_\omega(B_u)$.

Below, we provide intuition for Theorem 1 by outlining the argument for the two-person case. Consider the two-person bargaining problem B_u, depicted in Figure 4a below, and let x be an element of its ω-weighted Nash bargaining set.

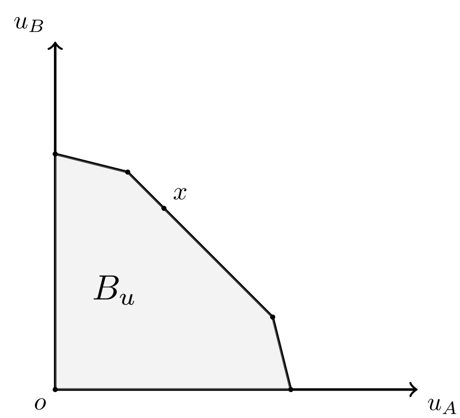

Figure 4a

We will show that x is the payoff vector of a PLE with endowments ω. By Lemma 1, x is the ω-weighted Nash bargaining solution of some simplex $A = a \odot \Delta + d$, depicted in Figure 4b, below:

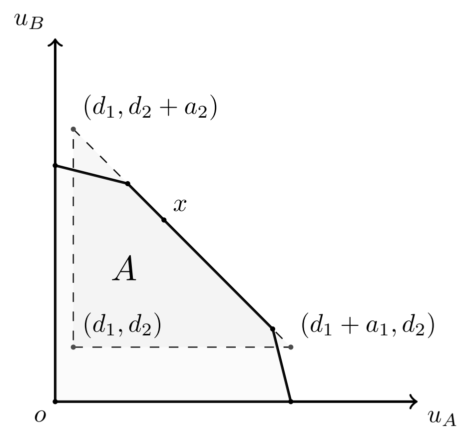

Figure 4b

Since $x = \eta_\omega(A)$, we have $x_i = a_i\omega_i + d_i$ for $i = 1, 2$. Since x is undominated, there is a distribution q over Pareto efficient outcomes that delivers the utility vector x. We will show that q is a PLE allocation. To do so, we must find the corresponding PLE prices. Let j be any outcome that is undominated in A, so that (u_1^j, u_2^j) is a convex combination of the extreme points $(a_1, 0)$ and $(0, a_2)$. Then, there are $z_1^j, z_2^j \geq 0$ such that $z_1^j + z_2^j = 1$ and $(a_1 z_1^j + d_1, a_2 z_2^j + d_2) = (u_1^j, u_2^j)$. Let $p_i^j = z_i^j$ and note that $p_1^j + p_2^j = 1$ for those outcomes. Since A dominates B_u in the weak set order, any other j is dominated by some vector in A. Therefore, there are $z_1^j, z_2^j \geq 0$ such that $z_1^j + z_2^j \leq 1$ and $(a_1 z_1^j + d_1, a_2 z_2^j + d_2) \geq (u_1^j, u_2^j)$. Again, let $p_i^j = z_i^j$ for those outcomes.

We have constructed prices that yield revenue 1 to the auctioneer for every outcome in the support of q and revenue no greater than 1 for all outcomes. Hence, q maximizes the auctioneer's revenue. It remains to show that q also maximizes the utility of every agent. By construction, $u_i^j \leq a_i p_i^j + d_i$ for all outcomes j. Therefore, for any alternative allocation $\hat{q}$, we have

$$\sum_j u_i^j \cdot \hat{q}^j \leq \sum_j (a_i p_i^j + d_i)\hat{q}^j = a_i \sum_j p_i^j \hat{q}^j + d_i \sum_j \hat{q}^j \tag{4}$$

The budget constraint implies that $\sum p_i^j \hat{q}^j \leq \omega_i$, and the fact that $\hat{q}$ is a probability (the probability constraint) implies that $\sum_j \hat{q}^j \leq 1$. Therefore,

$$\sum_j u_i^j \cdot \hat{q}^j \leq a_i\omega_i + d_i = \sum_j u_i^j \cdot q^j = x_i \tag{5}$$

and, thus, q maximizes utility.

As (4) and (5) reveal, we have constructed the prices p such that the parameters (a, d) of the supporting simplex A are the shadow prices of the agents' maximization problems. In particular, a_i is the shadow price of i's budget constraint, and d_i is the shadow price of i's probability constraint. This observation enables us to reverse the argument: for a given pseudo Lindahl equilibrium, (p, q), we can find the supporting simplex $A = a \odot \Delta + d$ by identifying the shadow prices, a_i, d_i, in agent i's utility maximization problem. To prove that the pseudo Lindahl equilibrium is in $N_\omega(B_u)$, we must also show that $B_u \leq A$. To do this, we use the fact that q also solves the auctioneer's revenue maximization problem. For the general case of $n \geq 2$, our proof relies on a linear programming duality argument to establish this relationship between the shadow prices a, d and the dominating simplex $A = a \odot \Delta + d$.

3. Properties of the Lindahl-Nash Solution

In this section, we use Theorem 1 to show that the Lindahl-Nash payoffs are in the *pseudo core* and to identify problems that have a unique Lindahl-Nash payoff.

3.1 The Pseudo Core

For any coalition of agents $I \subset \{1, \ldots, n\}$, define $\omega_I = \sum_{i \in I} \omega_i$ and

$$Q_I := \left\{ q \in Q \,\middle|\, \textstyle\sum_j q^j \leq \omega_I \right\}$$

Thus, random outcomes in Q_I allocate probability less than or equal to the share of coalition I's collective money endowment to the k social outcomes.

Definition: *The coalition $I \subset \{1, \ldots, n\}$ blocks the payoff vector x if there is $q \in Q_I$ such that $u_i \cdot q \geq x_i$ for all $i \in I$ with at least one inequality strict. The payoff $x \in B_u$ is in the pseudo core, $C_\omega(u)$, if no coalition blocks it.*

The definition of the pseudo core, above, translates fiat money endowments into probabilistic property rights over social outcomes. Every coalition of agents controls a probabilistic share of the social outcome corresponding to its share of the total supply of fiat money. Theorem 2, below, shows that Lindahl-Nash payoffs must respect those stochastic property rights. With two agents, the pseudo core coincides with the set of PLE payoffs; with more than two agents the pseudo core typically includes payoffs that are not Lindahl-Nash.

Theorem 2: *If x is a Lindahl-Nash payoff, then it is in the pseudo core; the converse also holds for $n = 2$. That is, $LN_\omega(u) \subset C_\omega(u)$ for all n; for $n = 2$, $C_\omega(u) = LN_\omega(u)$.*

Foley (1970) introduces Lindahl equilibria in a standard competitive model and shows that Lindahl equilibria are in the core of that economy. In Foley's model, as in the standard definition of the core, a coalition's resources are the sum of the physical endowments of its members. A coalition can block a proposed allocation if it can deploy its resources in a way that improves the utility of the coalition members over the utility of the proposed allocation. In a collective pseudo market, agents are endowed with fiat money that has no intrinsic value and, therefore, the standard definition of the core would render coalitions powerless. By converting the endowments of fiat money into stochastic property rights over social outcomes, we restore the coalitions' blocking power and obtain the appropriate definition of the core for collective pseudo markets.

3.2 Uniqueness

The Lindahl-Nash social choice rule is typically not single valued. Proposition 1 below identifies a class of cases in which it is. Suppose each agent considers every outcome either ideal or no better than the disagreement outcome. In such situations, we can assume, without loss of generality, that each agent's utility takes on the values 0 or 1, that is, $u_i \in \{0,1\}^K$.

Proposition 1: *If $u_i^j \in \{0,1\}$ for all i, j, then $LN_\omega(u) = \{\eta_\omega(B_u)\}$.*

Note that the Lindahl-Nash social choice rule always includes the $\omega-$weighted Nash bargaining solution and, therefore, uniqueness implies that $LN_\omega(u) = \{\eta_\omega(B_u)\}$. Conversely, since the $\omega-$weighted Nash bargaining solution is unique, establishing that every element of $LN_\omega(u)$ is the $\omega-$weighted Nash bargaining solution is enough to establish uniqueness.

The proof of Proposition 1 in the appendix uses a characterization of PLE prices together with the fact that the equilibrium allocation must maximize the auctioneer's revenue. To provide an alternative argument, consider the case in which no agent can receive their bliss point (that is, utility 1) in any weighted Lindahl-Nash payoff vector $x^* \in LN_\omega(u)$. This happens, for example, if no agent shares an ideal outcome with all other agents. We will argue that x^* must be the weighted Nash bargaining solution $\eta_\omega(B_u)$.

Let q be a PLE allocation that yields the utility profile x, that is, $u_i \cdot q = x_i$. Because agent i's utility is less than 1, there must be an outcome j such that $q^j > 0$ and $u_i^j = 0$. As a result, the probability constraint, $\sum_j q^j \leq 1$, must be slack and, therefore, this constraint must have a shadow price $c_i = 0$. Since $x \in LN_\omega(u)$, there is a dominating simplex A that has x as its solution. As we note in the discussion following Theorem 1, in this dominating simplex an agent's disagreement point equals the shadow price of that agent's probability constraint. In our case, this means that $d(A) = 0 = d(B_u)$. We conclude that $d(A) = d(B_u)$ and $x \in B_u \subset A$. Since x maximizes the $\omega-$weighted Nash product in A, it must also maximize the $\omega-$weighted Nash product in B_u, that is, $x = \eta_\omega(B_u)$.

The uniqueness result does not extend to cases with more general utilities. For example, if $u_i^j \in \{r, s\}$ with $0 < r < s$, then equilibria are typically not unique. Proposition 1 allows only two distinct utility values, one of which must be the outside option. Therefore, it does not cover this case.

4. A Characterization of the weighted Nash Bargaining Set

This section provides an axiomatic foundation of the weighted Nash bargaining set. The axioms are closely related to Nash's (1950) axioms. The main difference is that we abandon single-valuedness. A set-valued bargaining solution is a mapping $S : \mathcal{B} \to 2^{I\!R^n} \backslash \emptyset$ such that $S(B) \subset B$.

The first axiom, scale-invariance, is shared by most bargaining solutions including the Nash bargaining solution and the Kalai-Smorodinsky solution. It asserts that positive affine transformations of utilities lead to a corresponding positive affine transformation of the solution set.

Scale Invariance: $S(a \odot B + z) = a \odot S(B) + z$ *whenever* $a_i > 0$ *for all* i.

The second axiom, efficiency, applies only to the bargaining problem Δ. It ensures that a unique x is chosen from Δ, that this x is undominated, and that it yields positive utility to all agents.

Efficiency: *$S(\Delta) = \{x\}$ for some x such that $x \cdot e = 1$ and $x_i > 0$ for all i.*

The simplex Δ represents a situation in which a single prize must be awarded to one of the n agents and agents' utilities are equal to the probabilities with which they receive the prize. The efficiency axiom says that there is a single undominated x that results in this situation of pure conflict. This x encapsulates the planner's fairness judgment and determines the bargaining weights.

The third axiom is related to independence of irrelevant alternatives (IIA) of the Nash bargaining solution. The set valued version of IIA, due to Harsanyi (1959) and Kaneko (1980), considers two bargaining problems A and B with identical disagreement points. If $B \subset A$, then IIA requires that any solution to the bargaining problem A that is available in B must also be a solution to B, that is, $S(A) \cap B \subset S(B)$. The consistency axiom below, modifies this assumption by replacing $B \subset A$ with $B \leq A$ and omitting the requirement that A and B must share a common disagreement point.

Consistency: *$B \leq A$ implies $S(A) \cap B \subset S(B)$.*

Note that $B \leq A$ implies $d(B) \leq d(A)$; that is, Consistency requires that the disagreement point in A (weakly) dominates the disagreement point in B. The logic behind Consistency is as follows. If $x \in S(A)$, then x is judged a reasonable choice from A; that is, no agent can raise a compelling objection to x. Now suppose $B \leq A$ and $x \in B$. Since B does not offer an alternative that improves upon those in A, and since each agent's disagreement utility in B is lower (or no higher) than in A, there remains no compelling objection to x in B. Hence, Consistency requires $x \in S(B)$. Note that Consistency is "stronger" than the set-valued version of IIA because, whenever $B \subset A$ and $d(A) = d(B)$, we have $B \leq A$. As a result, in any situation where the set-valued version of IIA dictates that $S(A) \cap B \subset S(B)$, Consistency imposes the same requirement.

The fourth axiom, completeness, requires that no outcomes other than those necessitated by the preceding three axioms are included in the set of solutions. This axiom is due to Harsanyi (1959).

Completeness: *If $\hat{S}$ satisfies the three axioms above and $\hat{S}(B) \subset S(B)$ for all B, then $\hat{S}(B) = S(B)$ for all B.*

Theorem 3, below, establishes that the four axioms above characterize the weighted Nash bargaining set:

Theorem 3: *A bargaining solution, S, satisfies the four axioms above if and only if there are weights, ω, such that $S(B) = N_\omega(B)$ for all B.*

The Nash bargaining set is a set-valued solution concept with parameter ω. This parameter is the planner's vector of welfare weights and is pinned down by a single decision of the planner: the probability that each agent gets the prize in the pure conflict situation Δ. If we replace Efficiency with *Symmetric Efficiency*, below, then the bargaining weights become equal. Let $\sigma = \frac{1}{n}e$.

Symmetric Efficiency: $S(\Delta) = \{\sigma\}$.

To see how our axioms relate to the axioms for the Nash bargaining solution (Nash 1950), assume that $S(B)$ is a singleton for all B. Then, replace efficiency with symmetric efficiency. Then, restricting Consistency to cases in which $d(A) = d(B)$ yields the Nash bargaining solution. Alternatively, we can replace Consistency with the set-valued version of Nash's IIA to again obtain the Nash bargaining solution. Indeed, this is the axiomatization of the n-person Nash bargaining solution in Harsanyi (1959) who shows that symmetric efficiency, IIA and completeness imply the n-person Nash bargaining solution. Strengthening IIA to Consistency makes $S(B)$ larger at every B and, therefore, we obtain a more permissive solution concept.

Thomson (1994) provides a comprehensive survey of existing, single-valued, bargaining solutions. In the symmetric two-agent case, the ω-weighted Nash bargaining set includes all scale-invariant solutions discussed in Thomson's survey. These are the Nash bargaining solution, the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975), the Perles-Maschler solution (Perles and Maschler, 1981), and the Raiffa solution.[12] The following example illustrates how the symmetric Nash bargaining set for the two person case relates to other bargaining solutions.

4.1 Two Cakes

Ann and Bob would like to divide two cakes, a peanut butter cake and a chocolate cake, in a reasonable manner. Utilities are linear but Bob is allergic to peanuts while Ann

[12] It excludes solutions, such as the egalitarian solution, that violate Scale Invariance.

likes the peanut butter cake just as much as the chocolate cake. Ann and Bob have equal bargaining weights. Figure 5, below, illustrates this bargaining problem.

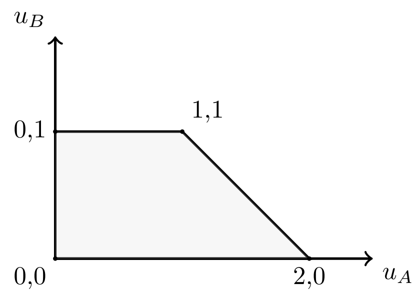

Figure 5

As we show in Lemma 1 above, x is in the symmetric bargaining set if and only if there exists a dominating simplex that has this x as its (unique) solution. The bargaining problem in Figure 6 is one such simplex. This bargaining problem has the unique solution $(1, 1)$. Therefore, $(1, 1)$ is an element of the symmetric Nash bargaining set of the original problem. Note that this solution is achieved by giving the peanut butter cake to Ann and the chocolate cake to Bob and corresponds to the Nash bargaining solution.

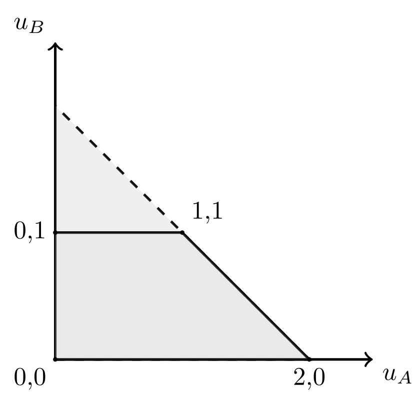

Figure 6

The utility profile $(1.5, 0.5)$ is a second element of the symmetric Nash bargaining set. In this case, the dark-shaded bargaining problem in Figure 7, below, is the dominating simplex. This simplex has disagreement point $(1, 0)$ and the unique solution $(1.5, .5)$. This payoff vector can be achieved if we divide the chocolate cake equally between the two players and give the peanut butter cake to Ann. Note that $(1.5, .5)$ corresponds to the Perles-Maschler solution (Perles and Maschler (1981)) of the original bargaining problem.

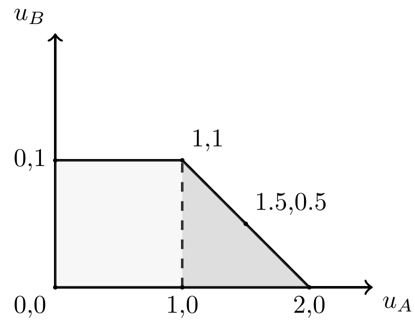

Figure 7

The two solutions, $(1,1)$ and $(1.5,.5)$, are the extreme points of the symmetric Nash bargaining set which consists of all utility profiles of the form $\lambda(1,1)+(1-\lambda)(1.5,.5)$ for $\lambda \in [0,1]$. These payoff vectors correspond to allocations in which Ann receives the peanut butter cake and at most 1/2 of the chocolate cake. This set contains all standard bargaining solutions that satisfy scale invariance (see Thomson (1994) for a discussion of bargaining solutions). For example, the Kalai-Smorodinsky solution would give 1/3 of the chocolate cake to Ann.

As the example shows, the choice of a dominating simplex can favor one or the other agent. Our solution concept takes no position on which of those payoff vectors is normatively best or fairest; in effect, it concedes that both the Perles-Maschler axioms and Nash's axioms make valid normative arguments.

5. Discrete Allocation Problems

A finite set of indivisible goods must be allocated, without transfers, to a group of agents. More precisely, a discrete allocation problem $\mathcal{A}$ is a triple $(H, \mathcal{F}, v)$ where H is the (finite) set of goods, $\mathcal{F}$, the production set, is a collection of non-empty subsets of H that represent feasible supply choices,[13] and $v = (v_1, \ldots, v_n)$ is a vector of utility functions.

Throughout this section, we assume every agent, i, has a utility function $v_i : 2^H \to I\!R$ such that $v_i(\emptyset) = 0$ and $v_i(M) \leq v_i(\hat{M})$ whenever $M \subset \hat{M} \subset H$. We write $v_i(h)$ instead of $v_i(\{h\})$. Without loss of generality, we assume $\bigcup_{F \in \mathcal{F}} F = H$. That is, we ignore any good that is never available. Finally, we assume that for every i there exists some $F \in \mathcal{F}$ such that $v_i(F) > 0$.

[13] In pseudo markets, often the constraints that define $\mathcal{F}$ are logical rather than technological. For example, if agent i gets the unique good j, then l cannot get j or if i matches with l then l must match with i. Nevertheless, the term production set seems appropriate.

An allocation assigns each agent a bundle of goods. The allocation is feasible if the assigned bundles are non-overlapping and their union is a feasible supply. Thus, a *(feasible) allocation* is a vector $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ such that $\mathbf{a}_i \subset H$ for all i, $\mathbf{a}_i \cap \mathbf{a}_l = \emptyset$ whenever $i \neq l$, and $\bigcup_i \mathbf{a}_i \in \mathcal{F}$. Henceforth, allocation will mean feasible allocation. The set of social outcomes, K, is the set of allocations, $\{\mathbf{a}^1, \ldots, \mathbf{a}^k\}$. As in previous section, we will identify K with $\{1, \ldots, k\}$ and let Q denote the set of random allocations. To map discrete allocation problems to collective choice problems, let $u_i^j = v_i(\mathbf{a}_i^j)$. The assumptions on v and $\mathcal{F}$ ensure that, for all i, $u_i^j \geq 0$ for all $j \in K$ and $u_i^j > 0$ for some $j \in K$. Therefore, the utilities satisfy the assumptions in section 2 and the weighted Lindahl-Nash solution is well defined. We write $LN_\omega^{\mathcal{A}}$ to denote the Lindahl-Nash solutions of the allocation problem $\mathcal{A}$ with weights ω.

Hylland and Zeckhauser (1979) introduce private pseudo markets as an efficient allocation mechanism for a class of discrete allocation problems.[14] A pseudo market transforms the discrete allocation problem into an exchange economy with random private consumption. Next, we describe this approach and relate it to the Lindahl-Nash solutions. As in the collective pseudo market, each agent is endowed with ω_i units of fiat money. Agents choose *consumption lotteries*, while a price taking firm chooses the (random) supply. Hence, each agent chooses a (personal) consumption bundle not an entire allocation as in the collective pseudo market. Formally, let D be the set of all probability distributions over 2^H and let $\mathbf{P} = I\!R_+^{|H|}$ be the set of all prices. We write $\mathbf{p}(M)$ to mean $\sum_{h \in M} \mathbf{p}(h)$. Then, given any price $\mathbf{p}$ and endowment ω_i, agent i's budget set is

$$\mathbf{B}(\mathbf{p}, \omega_i) := \left\{ d \in D \,\middle|\, \sum_M \mathbf{p}(M) d(M) \leq \omega_i \right\}$$

and the agent's utility maximization problem is:

$$V_i(\mathbf{p}, \omega_i) = \max_{d \in \mathbf{B}(\mathbf{p}, \omega_i)} \sum_M v_i(M) d(M)$$

A least-cost solution to this problem is an optimal random consumption such that no other optimal random consumption costs less.[15]

[14] Hylland and Zeckhauser consider unit demand preferences. Miralles and Pycia (2021), Gul, Pesendorfer and Zhang (2024) and Nguyen and Vohra (2022) extend the analysis to multi-unit demand.

[15] The restriction to least-cost solutions is a standard assumption in pseudo markets to ensure Pareto efficiency of the resulting equilibrium allocations (see Mas-Colell (1992) in the discussion in Section 2.1).

On the supply side, the firm must choose a probability distribution over $\mathcal{F}$. Let

$$\mathbf{B}_{\mathcal{F}} := \Big\{ d \in D \,\Big|\, \sum_{F \in \mathcal{F}} d(F) = 1 \Big\}$$

be the feasible random supplies. Then, the firm's revenue maximization problem is

$$R(\mathbf{p}, \mathcal{F}) = \max_{d \in \mathbf{B}_{\mathcal{F}}} \sum_{F \in \mathcal{F}} \mathbf{p}(F) d(F)$$

Recall that elements of Q are probability distributions over the set of feasible allocations K. For $q \in Q$, let $d_i^q \in D$ be i's random consumption; that is,

$$d_i^q(M) = \sum_{\{j \in K \,|\, \mathbf{a}_i^j = M\}} q^j$$

Let d_T^q be the random aggregate consumption, that is,

$$d_T^q(F) = \sum_{\{j \in K \,|\, \cup_i \mathbf{a}_i^j = F\}} q^j$$

The pair $(\mathbf{p}, q)$ is a pseudo Walrasian equilibrium if, for all i, d_i^q is a least-cost solution to consumer i's utility maximization problem and d_T^q solves the firm's revenue maximization problem. Let $W_\omega^{\mathcal{A}}$ be the set of Walrasian equilibrium utility vectors in this pseudo market. The following theorem shows that Walrasian equilibrium payoffs are Lindahl-Nash payoffs:

Theorem 4: $W_\omega^{\mathcal{A}} \subset LN_\omega^{\mathcal{A}}$.

The construction of the PLE prices (p_i^j) from the pseudo-Walrasian prices $\mathbf{p} \in \mathbf{P}$ is straightforward: we set $p_i^j = \mathbf{p}(\mathbf{a}_i^j)$. At those prices, every consumption plan that is affordable in the collective pseudo market is affordable in the private pseudo market. That is, agent i can afford q in the collective pseudo market only if she can afford d_i^q in the private pseudo market. Thus, q is least-cost optimal in the collective pseudo market if the random consumption d_i^q is least-cost optimal in the private pseudo market. Next, consider the auctioneer. For any allocation $j \in K$, we have

$$\sum_i p_i^j = \sum_i \mathbf{p}(\mathbf{a}_i^j) = \mathbf{p}(\cup_i \mathbf{a}_i^j)$$

The left hand side is the auctioneer's revenue from allocation j in the collective pseudo market while the right hand side is the revenue from the corresponding supply, $F = \cup_i \mathbf{a}_i^j$, in the private pseudo market. It follows that the auctioneer receives the same revenue from d_T^q in the private pseudo market as he does from allocation q in the collective pseudo market and, since d_T^q maximizes revenue in the private pseudo market, q must maximize revenue in the collective pseudo market.

In general, it is not possible to go in the reverse direction. In a PLE, (p, q), it may be that $q^j, q^k > 0, \mathbf{a}_1^k = \mathbf{a}_2^j$ and $p_1^j \neq p_2^k$; that is, different agents may be charged different prices for the same bundle. The results in sections 2-4 establish limits on these asymmetries: the asymmetries must be consistent with the normative criteria established in Theorem 3 and they must not conflict with the pseudo core. The next sections explore the relationship between pseudo Walrasian and pseudo Lindahl equilibria in more detail.

5.1 Examples

While Lindahl equilibria of a collective pseudo market always exist, pseudo Walrasian equilibria of a private pseudo market may not. The following example[16] illustrates the difference between the two mechanisms.

5.1.1 Example of Non-Existence of Pseudo Walrasian Equilibrium

There are two goods, $H = \{a, b\}$, and two agents. Agent 1 receives utility 1 if she consumes both goods and zero otherwise; thus, the two goods are complements for agent 1. Agent 2 receives utility 0 if he consumes nothing and 1 otherwise. The allocation $(H, \emptyset)$ yields the utility profile $(v_1, v_2) = (1, 0)$ and all other allocations either yield the utility profile $(v_1, v_2) = (0, 1)$ or the disagreement point $(0, 0)$. Therefore, the corresponding bargaining problem is a simplex, $B_v = \Delta$. It follows that there is a unique symmetric PLE payoff vector, $\sigma = (1/2, 1/2)$.

There is no pseudo Walrasian equilibrium in corresponding private pseudo market. To see this, note that by Theorem 4, any candidate for a pseudo Walrasian equilibrium must yield the unique symmetric Nash-Lindahl payoff vector. In a private pseudo market with endowments $\sigma = (1/2, 1/2)$, $V_1(\mathbf{p}, \omega_1) = 1/2$ if and only if the pseudo Walrasian price

[16] This example is used to illustrate non-existence of pseudo-market equilibria in Gul, Pesendorfer and Zhang (2024)

$\mathbf{p}$ satisfies $\mathbf{p}(a) + \mathbf{p}(b) = 1$. Then, $\min\{\mathbf{p}(a), \mathbf{p}(b)\} \leq 1/2$ and therefore, $V_2(\mathbf{p}, \omega_2) = 1$, a contradiction. The presence of complementarities creates well-known difficulties with existence of pseudo Walrasian equilibria in private pseudo markets.[17] By contrast, collective pseudo markets have no difficulties incorporating complementarities.

5.1.2 Sensitivity to the specification of the commodity space

Even when pseudo Walrasian equilibria exist, payoffs may be sensitive to how the traded goods are defined. Pseudo Lindahl equilibria of collective pseudo markets, however, depend only on the set of feasible payoffs. We say that the allocation problems $\mathcal{A}$ and $\mathcal{A}^*$ are utility-equivalent if they yield the same bargaining problem. That is, the convex hull of the feasible utility profiles is the same for $\mathcal{A}$ and $\mathcal{A}^*$. Theorem 1, above, shows that the weighted Lindahl-Nash payoff vectors of the two allocation problems must coincide. This is not true for pseudo Walrasian equilibrium payoff vectors as the two economies below demonstrate.

Economy 1: There are two goods, $H = \{a, b\}$ and two consumers with utilities v_1 and v_2 such that, $v_i(\emptyset) = 0, v_1(b) = 2$, and $v_1(M) = 3$ if $a \in M$; $v_2(b) = 1, v_2(M) = 2$ if $a \in M$. The bargaining set associated with this economy is the convex hull of the set $\{o, (3, 1), (2, 2)\}$. Let $\omega = \sigma = (1/2, 1/2)$ so that the two consumers have identical budget sets. The two efficient allocations are $\mathbf{a} = \{\{a\}, \{b\}\}$ and $\mathbf{a}' = \{\{b\}, \{a\}\}$. Since consumers have identical budgets and both prefer a over b, the two efficient allocations $\mathbf{a}$ and $\mathbf{a}'$ must be chosen with equal probabilities in equilibrium. The associated equilibrium payoff vector is $(5/2, 3/2)$.

Economy 2: There are two consumers and four goods, $H = \{a, b, c, d\}$. The four goods are perfect substitutes. Both consumers have additive utility functions but consumer 1 can derive utility from at most three goods while consumer 2 can derive utility from at most two goods. Specifically, $v_1(M) = \min\{|M|, 3\}$ and $v_2(M) = \min\{|M|, 2\}$ where $|M|$ is the cardinality of the set M. This economy yields the same bargaining set as the one above: the convex hull of the set $\{o, (3, 1), (2, 2)\}$. Again, let $\omega = \sigma$. We claim that the unique pseudo Walrasian equilibrium payoff is $(2, 2)$. First, note that the efficient allocations are

[17] See Gul, Pesendorfer and Zhang (2024), for a more detailed discussion

of the form (M_1, M_2) such that $|M_1| + |M_2| = 4, 2 \leq |M_1| \leq 3$. Since both consumers can afford the same consumption plans, equilibrium allocations (M_1, M_2) must be of the form $|M_1| = |M_2| = 2$. Any of these allocations yields the payoffs $(2, 2)$.

Theorem 4 and Theorem 1 imply that both $(5/2, 3/2)$ and $(2, 2)$ are symmetric Lindahl-Nash payoff vectors for either specification of this economy. The observation above is related to Sertel and Yildiz (2003) who provide a general statement and proof of the existence of distinct standard n-person exchange economies that yield the same set of feasible payoffs but have disjoint sets of pseudo Walrasian equilibrium payoffs. The example above shows that the same is true in a pseudo-market setting.

Theorem 1 and Theorem 4 above reveal an advantage of pseudo Lindahl equilibria over pseudo Walrasian equilibria: the set of PLE payoffs depends only on the implied bargaining problem whereas the set of pseudo Walrasian equilibria depends on how commodities are defined. Moreover, pseudo Lindahl equilibria exist even in the presence of complementarities. On the other hand, pseudo Walrasian equilibria are simpler than pseudo Lindahl equilibria because the former often involve fewer prices. This is so because the number of allocations typically exceeds the number of goods and because Lindahl prices are personal while pseudo Walrasian prices are not. Nonetheless, there are cases in which pseudo Lindahl and pseudo Walrasian equilibrium payoffs coincide. The following section provides a sufficient condition for this to occur.

5.2 Single-minded Consumers and Matching

Consumer i has unit demand if she never benefits from consuming more than one good. That is, for all M, $v_i(M) = \max_{h \in M} v_i(h)$. The unit demand consumer i is *single-minded* at $F \in \mathcal{F}$ if there is at most one $h \in F$ such that $v_i(h) > 0$, that is, i has demand for at most one of the goods in F. We say that consumers are single-minded if every consumer has unit demand and is single minded at every $F \in \mathcal{F}$.[18] Theorem 5 shows that PLE and pseudo Walrasian payoffs coincide if consumers are single-minded:

Theorem 5: *Let $\mathcal{A} = (H, \mathcal{F}, v)$ be a discrete allocation problem. If consumers are single-minded, then $W_\omega^{\mathcal{A}} = LN_\omega^{\mathcal{A}}$.*

[18] We are grateful to an anonymous referee for suggesting this condition.

As an example of single-mindedness, let $\mathcal{F} = \{\{h\} \,|\, h \in H\}$, that is, every feasible supply choice offers a single good. In this case, the PLE assigns each consumer their most preferred good with probability ω_i.[19] The same outcome is also a pseudo Walrasian equilibrium. To see this, set the price of every good equal to one so that consumer i can afford an ω_i-probability of her favorite good.[20]

A more interesting application of Theorem 5 are matching problems. Suppose a group of agents must choose roommates or partners. A matching is a bijection ι from the set of all agents to itself such that $\iota(\iota(i)) = i$ for all $i \in N$. If $\iota(i) = i$, then i is said to be unmatched. Let K denote the set of feasible matchings. We write ι^j for the feasible matching $j \in K$.

In this case, goods are ordered pairs (i, l) denoting i's right to match with l. Let $H = \{(i, l)|i, l \in N\}$ denote the commodity space. Let $v_i : H \to I\!R$ be the utility of agent i; that is, $v_i(i, l)$ is agent i's utility of matching with l. We normalize agents' utilities so that being unmatched yields 0 utility for every agent, i.e., $v_i(i, i) = 0$, and assume that $v_i(i, \iota^j(i)) > 0$ for some $j \in K$. We eliminate all matchings that are not individually rational and, therefore, we assume $v_i(i, \iota^j(i)) \geq 0$ for all $j \in K$. We set $v_i(i', \iota^j(i')) = 0$ if $i' \neq i$. Thus, every agent i can purchase $(i', \iota^j(i'))$ but only agent i' can benefit from this purchase.

The firm supplies the goods (matches). Since it is impossible to match agents 1 and 2 and, at the same time, match agents 1 and 3, not all goods can be supplied at the same time. Specifically, feasible F must correspond to some feasible matching ι^j. Thus, let $F^j = \{(i, l)|l = \iota^j(i)\}$. Then, $\mathcal{F} = \{F^j \,|\, j \in K\}$ captures the supply constraint for the matching problem. In this economy, consumers are single-minded because the only good in F^j that may yield strictly positive utility for agent i is $(i, \iota^j(i))$. Since each agent can match with at most one person, consumers have unit demands.

Therefore, Theorems 1 and 5 establish that in matching markets, pseudo Walrasian equilibrium exists and that the ω-weighted Nash bargaining solution is a pseudo Walrasian

[19] The most preferred good of consumer i may not be unique. In that case, the equilibrium allocation is not unique. However, the equilibrium utility profile is unique.

[20] To see why the unit demand assumption is needed, consider the following example: $\mathcal{F} = \{\{a\}, \{b\}\}$ and there is a single consumer with utility $v(a) = v(b) = 1$ and $v(\{a, b\}) = 4$. The auctioneer can either supply a or b but not both. This allocation problem has a PLE but no pseudo Walrasian equilibrium.

equilibrium payoff of the matching market with endowment ω. In particular, the Nash bargaining solution is a pseudo Walrasian equilibrium payoff of the matching market with equal endowments.

The matching market features personalized commodities, that is, good (i,l) yields zero utility for all agents other than i,l. We can use Theorem 5 and the idea of personalized goods to show that for each allocation problem there exists a utility-equivalent alternative in which pseudo Walrasian and pseudo Walrasian payoffs coincide. As in the previous section, we say that the allocation $\mathcal{A}$ and $\mathcal{A}^*$ are utility equivalent if they correspond to the same bargaining problem. For $\mathcal{A} = (H, \mathcal{F}, v)$, construct the following utility-equivalent allocation problem $\mathcal{A}^* = (H^*, \mathcal{F}^*, v^*)$. The goods are personalized commodity bundles:

$$H^* = \{h^* | h^* = (i, M) \in N \times 2^H\}$$

The feasible supply choices $\mathcal{F}^*$ correspond to the feasible allocations:

$$\mathcal{F}^* = \{\{(i, \mathbf{a}_i^j) \,|\, i \in N\}_{j \in K}\}$$

The utilities are unit demand and respect personalization:

$$v_i^*(l, M) = \begin{cases} v_i(M) & \text{if } i = l \\ 0 & \text{otherwise.} \end{cases}$$

We have constructed $\mathcal{A}^*$ to be utility-equivalent with $\mathcal{A}$ and, therefore, Theorem 1 implies that $LN_\omega^{\mathcal{A}} = LN_\omega^{\mathcal{A}^*}$; furthermore, consumers in $\mathcal{A}^*$ are single-minded and, therefore, Theorem 5 implies that $LN_\omega^{\mathcal{A}^*} = W_\omega^{\mathcal{A}^*}$. Thus, we have the following:

Corollary: *For every $\mathcal{A}$, there exists a utility-equivalent $\mathcal{A}^*$ such that $LN_\omega^{\mathcal{A}^*} = W_\omega^{\mathcal{A}^*}$.*

The number of goods in $\mathcal{A}^*$ can grow exponentially as the number of goods in $\mathcal{A}$ grows. Therefore, the pseudo Walrasian setting loses much of its simplicity advantage over the collective pseudo market setting once goods are personalized. Hence, applying pseudo Walrasian methods in discrete allocation problems entails a trade-off: either fairly restrictive assumptions must be imposed on preferences in the original market to ensure existence, or the complexity of personalization has to be confronted. In many situations, neither of these alternatives will be plausible and pseudo Lindahl equilibrium might be a reasonable alternative.

6. Conclusion

When there are indivisibilities, randomization may be needed to achieve compromise and avoid unfair outcomes. In practice, this randomization is often employed ex ante to assign priorities to the agents while the mechanism remains deterministic. For example, when allocating offices, an organization may randomly determine a priority order and then ask members to choose their preferred office sequentially. As Hylland and Zeckhauser (1979) point out, such mechanisms lead to ex ante inefficiency. As an alternative, they propose a market mechanism in which agents are given a budget of fiat money and choose lotteries over the available offices. The pseudo Walrasian mechanism proposed by Hylland and Zeckhauser is efficient but limited in its applicability to unit demand preferences. Gul, Pesendorfer, and Zhang (2024) extend Hylland and Zeckhauser's approach from unit demand to multi-unit demand with gross substitutes utilities. As is shown in that paper, demand complementarities create existence problems for the standard market mechanism. Moreover, externalities and public goods render pseudo Walrasian equilibria inefficient. In contrast, the collective pseudo markets proposed in the current paper are broadly applicable to all discrete allocation problems and are always efficient.

In a collective pseudo market, each agent expresses her demand for social alternatives rather than private outcomes. This formulation allows us to deal with a much broader range of applications. However, the number of social alternatives can be large and, therefore, the collective pseudo market may be too unwieldy to implement in practice. For matching markets, we have shown that pseudo Lindahl equilibria coincide with standard Walrasian equilibria and, therefore, each agent need only consider the set of possible partners and not the set of possible matchings (allocations) when formulating her demand. An important direction for future research is to examine other circumstances in which a smaller set of markets (and prices) suffices to implement pseudo Lindahl equilibria.

7. Appendix

7.1 Preliminaries

Let $p_i \in I\!R_+^K$ be a price. We say that q is a solution to (u_i, p_i, ω_i) if q solves

$$U_i(p_i, \omega_i) = \max_q u_i \cdot q \text{ subject to } p_i \cdot q \leq \omega_i, e \cdot q \leq 1, 0 \leq q \tag{P}$$

The constraint $0 \leq q$ never binds since $0 \leq u_i^j, \forall j$ and, therefore, we drop it in the remainder of the analysis. We say that q is a least-cost solution to (u_i, p_i, ω_i) if $p_i \cdot q' \geq p_i \cdot q$ for all solutions q' to (u_i, p_i, ω_i). The dual of the maximization problem (P) is

$$\min_{\mu^0, \mu^1 \geq 0} \mu^0 + \mu^1 \omega_i \text{ subject to } \mu^0 e + \mu^1 p_i \geq u_i \tag{D}$$

The vector (q, μ^0, μ^1) is feasible for (u_i, p_i, ω_i) if q satisfies the constraints of (P) and (μ^0, μ^1) satisfies the constraints of (D). A feasible vector (q, μ^0, μ^1) is optimal (that is, q solves (P) and μ^0, μ^1 solves (D)) if and only if

$$\begin{aligned} \mu^0(e \cdot q - 1) &= 0 \\ \mu^1(p_i \cdot q - \omega_i) &= 0 \\ \text{for all } j, \quad q^j(\mu^0 + \mu^1 p_i^j - u_i^j) &= 0 \end{aligned} \tag{CS}$$

For any utility u_i and $c_i \geq 0$, let $\bar{u}_i^j(c_i) = \max\{0, u_i^j - c_i\}$ and let $\bar{u}_i(c_i) = (\bar{u}_i^j(c_i))_{j=1}^K$. If $c_i < \max_j u_i^j$ for some j, then $\bar{u}_i(c_i)$ is a utility; that is, $\bar{u}_i^j(c_i) \geq 0$ for all j and $\bar{u}_i^j(c_i) > 0$ for some j.

The vector (q, c_i, α_i) is an *ideal solution* to (u_i, p_i, ω_i) if (1) q is a least-cost solution to (u_i, p_i, ω_i) and (2) $\alpha_i p_i^j \geq u_i^j - c_i$ for all j and $\alpha_i p_i^j = u_i^j - c_i$ for j such that $q^j > 0$.

Lemma A1: *If $q \in Q$ is a least-cost solution to (u_i, p_i, ω_i) and $p_i \cdot q = \omega_i$, then there are $c_i \geq 0$ and $\alpha_i > 0$ such that*

(i) (q, c_i, α_i) is an ideal solution to (u_i, p_i, ω_i)

(ii) $\bar{u}_i(c_i)$ is a utility and q is a least-cost solution to $(\bar{u}_i(c_i), p_i, \omega_i)$.

Proof: Let

$$J(q) = \begin{cases} \{j \mid q^j > 0\} & \text{if } \sum_{j \in K} q^j = 1 \\ \{j \mid q^j > 0\} \cup \{0\} & \text{otherwise} \end{cases}$$

and set $u_i^0 = 0 = p_i^0$. Let μ^0, μ^1 be the associated solution of the dual (D). First, consider the case in which $u_i^j \neq u_i^m$ for some $j, m \in J(q)$. Then, (CS) implies $\mu^1 > 0$. Set $c_i = \mu^0, \alpha_i = \mu^1$. Feasibility and (CS) imply $\alpha_i p_i^l \geq u_i^l - c_i$ with equality if $q^l > 0$, proving that (q, c_i, α_i) is an ideal solution.

Assume, without loss of generality, that $u_i^j > u_i^m$. Since $\alpha_i p_i^m = u_i^m - c_i$ and $p_i^m \geq 0$, we must have $u_i^j - c_i > u_i^m - c_i \geq 0$. Hence, $\bar{u}_i(c_i)$ is a utility. Next, we show that q is a solution to $(\bar{u}_i(c_i), p_i, \omega_i)$. Since (q, c_i, α_i) is an ideal solution to (u_i, p_i, ω_i), we have $\alpha_i p_i^l \geq u_i^l - c_i \geq 0$; that is, $\alpha_i p_i^l \geq \bar{u}_i^l(c_i)$ for all l. Since $u_i^l - c_i = \bar{u}_i^l(c_i)$ for all $l \in J(q)$ we conclude that $q^l(\alpha_i p_i^l - \bar{u}_i(c_i)) = 0$ for all l. Hence, $(q, 0, \alpha_i)$ is feasible and satisfies (CS) for $(\bar{u}_i(c_i), p_i, \omega_i)$; that is, q is a solution to $(\bar{u}_i(c_i), p_i, \omega_i)$.

Second, consider the case in which there is $\beta = u_i^j$ for all $j \in J(q)$ and $\mu^1 > 0$. Set $\alpha_i = \mu^1$ and $c_i = \mu^0$. Then, arguing as above, we conclude that feasibility and (CS) imply $\alpha_i p_i^l \geq u_i^l - c_i$ with equality if $q^l > 0$, and, therefore, (q, c_i, α_i) is an ideal solution. Since q is a least-cost solution to (u_i, p_i, ω_i) and $p_i \cdot q = \omega_i$, we must have $p_i^j = \omega_i > 0$ for all $j \in J(q)$. Then, since (q, c_i, α_i) is an ideal solution, we have $u_i^j - c_i > 0$ for all $j \in J(q)$. We conclude that $\bar{u}_i(c_i)$ is a utility and that $\alpha_i p_i^j = \bar{u}_i^j(c_i)$ for all $j \in J(q)$. To see that q solves $(\bar{u}_i(c_i), p_i, \omega_i)$, note that $(q, 0, \alpha_i)$ is feasible and satisfies (CS).

Finally, consider the case in which there is $\beta = u_i^j$ for all $j \in J(q)$ and $\mu^1 = 0$. Then, by (CS), $\beta = \mu^0 > 0$ and $0 \notin J(q)$. Hence, $\sum_{j \in K} q^j = 1$. Since q is least-cost optimal, p_i^j must be constant for all $j \in J(q)$. Let π be this constant. Note that $\pi > 0$ since $p_i \cdot q = \omega_i$. Note, also, that $\mu^0 = \beta \geq u_i^j$ for all j since (μ^0, μ^1) is feasible for (D) and $\mu^1 = 0$. Set

$$c_i = \begin{cases} \max\{u_i^j : u_i^j < \beta\} & \text{if } \{u_i^j : u_i^j < \beta\} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$
$$\alpha_i = \frac{\beta - c_i}{\pi}$$

Clearly $\alpha_i > 0$, $\alpha_i p_i^j = \alpha_i \pi = \beta - c_i = u_i^j - c_i$ for all $j \in J(q)$. For $j \notin J(q)$, if $u_i^j = \beta$, the least-cost optimality of q implies $p_i^j \geq \pi$ and therefore $\alpha_i p_i^j \geq \beta - c_i = u_i^j - c_i$; if $u_i^j < \beta$, then $u_i^j - c_i \leq 0$ and hence, $\alpha_i p_i^j \geq u_i^j - c_i$ in this case as well. This proves that (q, c_i, α_i) is an ideal solution. The function $\bar{u}_i(c_i)$ is a utility since $\beta - c_i > 0$. To see that q is a solution to $(\bar{u}_i(c_i), p_i, \omega_i)$, note that $\bar{u}_i^j(c_i) \leq \beta - c_i = \bar{u}_i(c_i) \cdot q$ for all j.

In all cases, we have established that q solves $(\bar{u}_i(c_i), p_i, \omega_i)$ and that $\bar{u}_i(c_i) \cdot q > 0$ since $\bar{u}_i(c_i)$ is a utility. It remains to show that q is a least-cost solution to $(\bar{u}_i(c_i), p_i, \omega_i)$. Let $\hat{q}$ be any other solution to $(\bar{u}_i(c_i), p_i, \omega_i)$. If $\hat{q}$ solves (u_i, p_i, ω_i), then $p_i \cdot \hat{q} \geq p_i \cdot q$ since q is a least-cost solution to (u_i, p_i, ω_i). If $\hat{q}$ does not solve (u_i, p_i, ω_i), then $\bar{u}_i^j(c_i) + c_i > u_i^j$ for some j such that $\hat{q}^j > 0$. In that case, $\bar{u}_i^j(c_i) = 0$. Since $\hat{q}$ solves $(\bar{u}_i(c_i), p_i, \omega_i)$ there is m in the support of $\hat{q}$ such that $\bar{u}_i^m(c_i) > 0$. Together with $\hat{q}^j > 0$, this implies that the budget constraint cannot be slack, that is, $p_i \cdot \hat{q} = \omega_i$ and, therefore, $p_i \cdot \hat{q} \geq p_i \cdot q = \omega_i$ also in this case. $\square$

Lemma A2: *Let $o \leq \lambda$ and $v_i = u_i + (\lambda_i, \ldots, \lambda_i)$ for all i. Then, (p, q) is a PLE of (u, ω) implies (p, q) is a PLE of (v, ω).*

Proof: Suppose $o \leq \lambda$ and let (p, q) be a PLE of (u, ω). Then, for all i and $\hat{q}$ such that $e \cdot \hat{q} \leq 1$, $u_i \cdot \hat{q} \leq u_i \cdot q = v_i \cdot q - \lambda_i$ and $v_i \cdot \hat{q} - \lambda_i \leq u_i \cdot \hat{q}$. It follows that q is a least-cost solution to consumer i's maximization problem in v. Clearly, q is a solution to the firm's maximization problem in the collective pseudo market v and hence (p, q) is a PLE. $\square$

Recall that $A \in \mathcal{B}$ is a *simplex* if $A = a \odot \Delta + b$ for some a such that $a_i > 0$ for all i and b such that $b_i \geq 0$ for all i. We say that the simplex A supports B at x with (exterior normal) θ if $x \in B \subset A$, $d(B) = d(A)$ and $\theta \cdot y \leq \theta \cdot x$ for all $y \in A$. Let $\nabla f_\omega(B, x)$ denote the gradient of $f_\omega(B, \cdot)$ at x. Define $A_{\omega x} = \text{conv}\,\{o, (x_1/\omega_1)e^1, \ldots, (x_n/\omega_n)e^n\}$ to be the simplex with disagreement point o and bliss point x_i/ω_i for agent i.

Lemma A3: *Let $B \in \mathcal{B}$ and let A be a simplex.*

(i) $\eta_\omega(B) = \eta_\omega(B - d(B)) + d(B)$, $\eta_\omega(A) = \omega \odot (b(A) - d(A)) + d(A)$ and $N_\omega(B) = N_\omega(B - d(B)) + d(B)$.

(ii) Let $x = \eta_\omega(B) - d(B)$. Then, $A_{\omega x} + d(B)$ supports B at $\eta_\omega(B)$ with $\nabla f_\omega(B, x)$.

(iii) $N_\omega(A) = \{\eta_\omega(A)\}$.

(iv) If $x \in B$ is undominated and $x_i > d(B)_i$ for all i, then $x = \eta_\omega(B)$ for some ω.

Proof: The proof of (i) is straightforward and, therefore, omitted. To prove (ii), first assume $d(B) = 0$. Let $x = \eta_\omega(B)$ and $\theta = \nabla f_\omega(B, \eta_\omega(B)) = (\omega_1/x_1, \ldots, \omega_n/x_n)$. Then, $\theta \cdot y \leq \theta \cdot x$ for all $y \in B$. Note that $\theta \cdot x = 1$. Hence, the simplex $A_{\omega x} =$

$\text{conv}\,\{o, (x_1/\omega_1)e^1, \ldots, (x_n/\omega_n)e^n\}$ supports B at $\eta_\omega(B)$ with $\nabla f_\omega(B, \eta_\omega(B))$. If $d(B) \neq 0$, let $C = B - d(B)$ so that $d(C) = o$. By the previous argument, $A_{\omega\eta_\omega(C)}$ supports C with $\theta = \nabla f_\omega(C, x)$. By (i), $\eta_\omega(B) = \eta_\omega(C) + d(B)$. Since $\nabla f_\omega(B, \eta_\omega(B)) = \nabla f_\omega(C, \eta_\omega(C))$, it follows that $A_{\omega\eta_\omega(C)} + d(B)$ supports B at $\eta_\omega(B)$ with $\nabla f_\omega(B, \eta_\omega(B))$.

To prove part (iii), first note that $\eta_\omega(A) \in N_\omega(A)$ since $A \leq A$. It remains to show that $N_\omega(A)$ is a singleton set. Assume $x \in N_\omega(A)$. Then, there is B such that $A \leq B$ and $\eta_\omega(B) = x$. By part (ii) there is a simplex A' such that $B \subset A'$, $d(B) = d(A')$ and $x = \eta_\omega(B) = \eta_\omega(A')$. Hence $B \leq A'$ and therefore $A \leq A'$. Since both A and A' are simplices either they are equal or $\eta_\omega(A')$ strictly dominates $\eta_\omega(A)$. Since $\eta_\omega(A') = x$, we conclude $x = \eta_\omega(A') = \eta_\omega(A)$.

To prove (iv), it is enough to consider only B such that $d(B) = o$ and appeal to (i) above. Let x be undominated with $x_i > 0$ for all i. Since B is a polytope (and hence finitely generated), Theorem 1 in Arrow, Barankin and Blackwell (1953) shows that there exists v with $v_i > 0$ for all i such that $v \cdot x \geq v \cdot y$ for all $y \in B$. Let $\lambda_i = v_i x_i$, $\lambda = \sum \lambda_i$ and let $\omega_i = \lambda_i/\lambda$. Then, x maximizes $f_\omega(B, \cdot)$ since $\lambda \nabla f_\omega(B, x) = v$. $\square$

7.2 Proof of Theorem 1

Lemma A4(i), below, proves one direction of Theorem 1: $x \in N_\omega(B_u)$ implies $x \in L_\omega(u)$.

Lemma A4: *Let $x \in N_\omega(B_u)$. Then, there is (p, q, c) such that*

(i) (p, q) is a PLE of (u, ω) and $(u_1 \cdot q, \ldots, u_n \cdot q) = x$.

(ii) $c \in I\!R^n_+$ such that for all i, $c_i < x_i$ and $\frac{u_i^j - c_i}{p_i^j} = \frac{x_i - c_i}{\omega_i}$ if $p_i^j > 0$,

(iii) $u_i^j \leq c_i$ if and only if $p_i^j = 0$,

(iv) $u_i^j \geq c_i$ if $q^j > 0$

Proof: Let $x \in N_\omega(B_u)$. Then, $x = \eta_\omega(B)$ for some B such that $B_u \leq B$. By Lemma A3, $B \subset A_{\omega\hat{x}} + d(B)$ such that $\hat{x} = x - d(B)$ and $x = \eta_\omega(A_{\omega\hat{x}} + d(B))$. For any $j \in K$, let $y^j = (u_1^j, \ldots, u_n^j)$. Since $x \in B_u$ there is a $q \in Q$ such that $x = \sum y^j q^j$. Since x is on the efficient frontier of $A_{\omega\hat{x}} + d(B)$, $q^j > 0$ implies y^j is also on the efficient frontier of $A_{\omega\hat{x}} + d(B)$.

First, consider the case where $d(B) = o$. Then, $x = \hat{x}$ and $B_u \subset A_{\omega x}$. Let $y^j = (u^j_1, \ldots, u^j_n)$. Since $y^j \in A_{\omega x}$, it is a convex combination of the extreme points of $A_{\omega x}$. Let z^j_i be the weight of $(x_i/\omega_i)e^i$ in that convex combination and set $p^j_i = z^j_i$. Note that $\sum_j z^j_i (x_i/\omega_i) q^j = x_i$ and, therefore, $\sum_j z^j_i q^j = \sum_j p^j_i q^j = p_i \cdot q = \omega_i$. Furthermore, $\sum_i p^j_i = \sum_i z^j_i \leq 1$ for all j.

Clearly,

$$u_i = (x_i/\omega_i)p_i$$

Hence, we have shown (ii). By the display equation above, for any $\hat{q}$ such that $p_i \cdot \hat{q} \leq \omega_i$ we have

$$u_i \cdot \hat{q} = \frac{x_i}{\omega_i} p_i \cdot \hat{q} \leq x_i = u_i \cdot q$$

with equality if and only if $p_i \cdot \hat{q} = p_i \cdot q = \omega_i$. Therefore, q is a least-cost solution to the consumer's problem.

Recall that $\sum_i p^j_i \leq 1$ for all j and that y^j must be on the Pareto frontier of $A_{\omega x}$ whenever $q^j > 0$. Therefore, $\sum_i z^j_i = 1$ whenever $q^j > 0$ and, hence, $\sum_i p^j_i = 1$ for all j such that $q^j > 0$. This proves the optimality of q for the auctioneer and completes the proof of (i). Set $c_i = 0$ for all i and note that (iv) is satisfied. Note also that $u^j_i > 0$ if and only if $p^j_i > 0$ and therefore (iii) is satisfied.

Next, consider the case $o = d(B_u) \neq d(B)$. Since $B_u \leq B \subset A_{\omega \hat{x}} + d(B)$, we must have $d(B_u) \leq d(B) = d(A_{\omega \hat{x}} + d(B))$. Let $c = d(B)$ and define a set of outcomes, $\hat{K}$, as follows: for each $j \in K$, let $v^{\phi(j)} = (v^{\phi(j)}_1, \ldots, v^{\phi(j)}_n)$ where $v^{\phi(j)}_i = \max\{0, y^j_i - c_i\}$. If $v^{\phi(j)} = v^{\phi(l)}$ for $l \neq j$, let $K^1 = K \backslash \{l\}$. Continue in this fashion until reaching K^m such that the mapping $\phi : K^m \to I\!R^n$ is one to one.

Then, repeat the above construction for $\hat{x} = x - d(B)$ to obtain a PLE (p, q) for the collective pseudo market (v, ω) such that $v^{\phi(j)}_i > 0$ if and only if $p^{\phi(j)}_i > 0$ and $v^{\phi(j)}_i / p^{\phi(j)}_i = \hat{x}_i/\omega_i$ if $p^{\phi(j)}_i > 0$. By Lemma A2, (p, q) is also a PLE of the collective pseudo market $(\hat{v}, \omega)$ such that $\hat{v}^j_i = v^j_i + c_i$. It is straightforward to verify that this equilibrium satisfies (ii)-(iv) above.

To convert this PLE into a PLE of the original collective pseudo market (u, ω), set $\hat{p}^j_i = p^{\phi(j)}_i$ for all $j \in K$. Note that $v^{\phi(j)}_i + c_i \geq u^j_i$ for all j. Recall that $q^j > 0$ implies that

y^j is on the efficient frontier of $A_{\omega\hat{x}} + d(B)$ and therefore $y^j \geq d(B) = c$ for all j such that $q^j > 0$. We conclude that $v_i^{\phi(j)} + c_i = u_i^j$ if $q^j > 0$. It is then straightforward to verify that $(\hat{p}, q)$ is a PLE of (u, ω) and that this equilibrium satisfies (ii)-(iv). $\square$

It remains to show that $x \in L_\omega(u)$ implies $x \in N_\omega(B_u)$. Let u and $c = (c_1, \ldots, c_n) \in \mathbb{R}_+$ be such that for all i there is j with $u_i^j > c_i$. Then, define $\bar{u}(c) := (\bar{u}_1(c_1), \cdots, \bar{u}_n(c_n))$ where each $\bar{u}_i(c_i)$ is defined as above.

Lemma A5: *For all $x \in L_\omega(u)$ there are (p, q, α, c) such that*

(i) (p, q) is a PLE for (u, ω) with $x = (u_1 \cdot q, \ldots, u_n \cdot q)$ and $p_i \cdot q = \omega_i$ for all i;

(ii) $\alpha \odot \Delta$ supports $B_{\bar{u}(c)}$ at $\eta_\omega(\alpha \odot \Delta)$ with $(1/\alpha_1, \ldots, 1/\alpha_n)$

(iii) $B_u \leq \alpha \odot \Delta + c$ and $x = \eta_\omega(\alpha \odot \Delta) + c$.

Proof: Let $x \in L_\omega(u)$ and let (p, q) be the corresponding PLE. If $p_i \cdot q < \omega_i$ for some i, let I be the set of all such agents. Let $J_i^* = \{j | u_i^j \geq u_i^m \, \forall m\}$ be the bliss outcomes for i. If $i \in I$ and $q^j > 0$, then $j \in J_i^*$ since otherwise i is not choosing a utility maximizing plan. Furthermore, since q is a least-cost solution for consumer $i \in I$, $p_i^j = p_i^m$ for all j, m such that $q^j > 0$ and $q^m > 0$. Define $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$ as follows: $\bar{p}_i^j = \omega_i$ if $i \in I$ and $j \in J_i^*$; otherwise, $\bar{p}_i^j = p_i^j$. Note that consumer i can afford q at $\bar{p}$. Since q is a least-cost solution for consumer i at prices p, it must be a least-cost solution for consumer i at prices $\bar{p} \geq p$. We conclude that $(\bar{p}, q)$ is also a PLE. Moreover, every consumer satisfies $\bar{p}_i \cdot q = \omega_i$ for all i. Hence, part (i) is satisfied.

So, assume $p_i \cdot q = \omega_i$ for all i. By Lemma A1, for each i there is some $c_i \geq 0$ and $\alpha_i > 0$ such that (q, c_i, α_i) is an ideal solution to (u, p_i, ω_i), $\bar{u}_i(c_i)$ is a utility, and q is a least-cost solution to $(\bar{u}_i(c_i), p_i, \omega_i)$. It follows that (p, q) is a PLE for $(\bar{u}(c), \omega)$ as well. Let $c = (c_1, \ldots, c_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be the corresponding vectors.

Let $r_i = \frac{1}{\alpha_i}$ and $r = (r_1, \ldots, r_n)$. Note that $\alpha \odot \Delta = \mathbb{R}_+ \cap \{w \,|\, r \cdot w \leq 1\}$. Since (u, p_i, ω_i) is an ideal solution for all i, we have $\alpha_i p_i^j \geq u_i^j - c_i$ for all i, j and therefore, $\alpha_i p_i^j \geq \bar{u}_i^j(c_i)$; that is, $p_i^j \geq r_i \bar{u}_i^j(c_i)$. Firm optimality of (p, q) implies $1 = \sum_i \omega_i = \sum_{i=1}^n p_i^j \cdot q \geq \sum_{i=1}^n p_i^j$ for all j and, therefore, $1 \geq \sum r_i \bar{u}_i^j(c_i)$. Thus, we have $\bar{u}^j(c) \in \alpha \odot \Delta$ and therefore, (1) $B_{\bar{u}(c)} \subset \alpha \odot \Delta$.

Clearly, $\eta_\omega(\alpha \odot \Delta) = \alpha \odot \omega$. Since $\alpha_i p_i^j = \bar{u}_i^j(c_i) = u_i^j - c_i$ for all j such that $q^j > 0$, we have $\sum_j \alpha_i p_i^j q^j = x_i - c_i$ for all i. Since $p_i \cdot q = \omega_i$ for all i, we have $\alpha_i \omega_i = x_i - c_i$ for all i; that is (2) $\eta_\omega(\alpha \odot \Delta) = x - c = \bar{u}(c) \cdot q$. That $\alpha \odot \Delta$ supports $B_{\bar{u}(c)}$ at $\eta_\omega(\alpha \odot \Delta)$ with $(1/\alpha_1, \ldots, 1/\alpha_n)$ follows from (1) and (2) above and establishes (ii).

Finally, $x = \eta_\omega(\alpha \odot \Delta) + c$. Since $B_{\bar{u}(c)} \leq \alpha \odot \Delta$ we have $B_{\bar{u}(c)} + c \leq \alpha \odot \Delta + c$ and since $B_u \leq B_{\bar{u}(c)} + c$, we conclude $B_u \leq \alpha \odot \Delta + c$ as desired. This proves (iii). $\square$

To complete the proof of Theorem 1, let $x \in L_\omega(u)$ and let (p, q, α, c) have the properties defined in Lemma A5. Then, by Lemma A3(i) and Lemma A5(iii), $\eta_\omega(\alpha \odot \Delta + c) = \eta_\omega(\alpha \odot \Delta) + c = x$. Moreover, by Lemma A5(iii), $B_u \leq \alpha \odot \Delta + c$ and therefore $x \in N_\omega(B_u)$. $\square$

7.3 Proof of Theorem 2

Let (p, q) be a PLE and let $\bar{p} = \sum_i p_i$. Revenue maximization implies that $\bar{p} \cdot \hat{q} \leq \bar{p} \cdot q \leq 1$ for all $\hat{q} \in Q$. If $\hat{q} \in Q_I$ then $\hat{q}/\omega_I \in Q$ and, therefore,

$$\bar{p} \cdot \hat{q} \leq \omega_I \tag{6}$$

for all $\hat{q} \in Q_I$.

Let $I = \{1, \ldots, n\}$ be the coalition of all agents and assume that there exists $\hat{q}$ with $u_i \cdot \hat{q} \geq u_i \cdot q$ for all i with a strict inequality for at least one agent. Since q is least-cost optimal, it follows that $\bar{p} \cdot \hat{q} > \bar{p} \cdot q$, contradicting revenue maximization.

Let $I \subsetneq \{1, \ldots, n\}$. First, consider the case $p_i \cdot q = \omega_i$ for all $i \in I$. If $\hat{q} \in Q_I$ and $u_i \cdot \hat{q} \geq u_i \cdot q$ for all $i \in I$ with at least one inequality strict, then $\bar{p} \cdot \hat{q} \geq \sum_{i \in I} p_i \cdot \hat{q} > \sum_{i \in I} p_i \cdot q = \omega_I$, contradicting equation (6). Second, consider the case $p_i \cdot q < \omega_i$ for some $i \in I$. Then, utility maximization implies that q is a bliss point for i. That is, $u_i \cdot q = \max_j u_i^j > 0$ and $u_i \cdot q > u_i \cdot \hat{q}$ for all $\hat{q} \in Q_I$. Thus, I cannot block q.

Next, assume that $n = 2$ and q is an element of the pseudo core. Normalize u_i so that $\max_j u_i^j = 1$ for $i = 1, 2$. Let $v_1 = u_1 \cdot q, v_2 = u_2 \cdot q$. Since q is in the pseudo core we have (1) $v_i \geq \omega_i$ for $i = 1, 2$ and (2) there are $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 v_1 + \lambda_2 v_2 \geq \lambda_1 u_1^j + \lambda_2 u_2^j$ for all $j = 1, \ldots k$. Normalize λ_1, λ_2 such that $\lambda_i v_i \geq \omega_i$ for $i = 1, 2$ with equality for either $i = 1$ or $i = 2$. For concreteness, assume that $\lambda_1 v_1 = \omega_1$. By (1) above, $\lambda_i \geq 1$ for $i = 1, 2$. Then, there exists $c_2 \geq 0$ such that $\lambda_2(v_2 - c_2) = \omega_2$. Let $b = (\lambda_1 v_1 + \lambda_2 v_2)/\lambda_2$, $a = \lambda_1/\lambda_2$ and

$$S = \{(x_1, x_2) \,|\, x_1 \geq 0, x_2 \geq c_2 \text{ and } x_2 \leq b - a x_1\}.$$

It is easy to verify that S is the simplex with extreme points $(0, y_2)$, (y_1, c_2) and $(0, c_2)$ where $y_2 = (1 + \lambda_2 c_2)/\lambda_2$ and $y_1 = 1/\lambda_1$. Note that the ω-weighted Nash Bargaining solution of S is $(\omega_1 y_1, \omega_2(y_2 - c_2)) = (v_1, v_2)$. Since $\lambda_i \leq 1$ for $i = 1, 2$, $y_i \geq 1$ for $i = 1, 2$. Then (2) implies $B_u \leq S$. Hence, $(v_1, v_2) \in N_\omega(B_u)$ as desired. □

7.4 Proof of Proposition 1

First, we show that an allocation q is a PLE if and only if, for all j in the support of q,

$$\sum_{\{i:u_i^j=1\}} \frac{\omega_i}{u_i \cdot q} \geq \sum_{\{i:u_i^k=1\}} \frac{\omega_i}{u_i \cdot q} \tag{7}$$

for all $k = 1, \ldots, K$.

To see why (7) is necessary, let (p, q) be a PLE. Since $u_i^j \in \{0, 1\}$, there is some constant $r_i \geq 0$ that does not depend on j such that for all j in the support of q, $p_i^j \in \{0, r_i\}$. If $r_i = 0$ for some i, then i's equilibrium payoff must be 1 and $u_i^j = 1$ for all j in the support of q. Then, replace every price p_i^j for j such that $u_i^j = 1$ with $\hat{p}_i^j = 1$. For i such that $r_i \neq 0$, let $\hat{p}_i^j = p_i^j$ for all j. Clearly, $(\hat{p}, q)$ is also a PLE and the r_is for this equilibrium are all strictly positive.

Consumer optimality implies $u_i \cdot q = \omega_i / r_i$ and, therefore, $r_i = \omega_i/(u_i \cdot q)$. Price $\hat{p}_i^j$ for an alternative j that yields utility 1 to i and is not in the support of q must be at least r_i. Then, auctioneer optimality implies that for j in the support of q,

$$\sum_{\{i:u_i^j=1\}} \frac{\omega_i}{u_i \cdot q} = \sum_i \hat{p}_i^j \geq \sum_i \hat{p}_i^k \geq \sum_{\{i:u_i^k=1\}} \frac{\omega_i}{u_i \cdot q}$$

for all k. Thus, every PLE (p, q) must satisfy condition (7) above. For the reverse direction, let q satisfy condition (7). Then, choose $p_i^j = 0$ if $u_i^j = 0$ and $p_i^j = \omega_i/(u_i \cdot q)$ if $u_i^j = 1$. It is straightforward to verify that q solves the agents' and the auctioneer's optimization problems at these prices.

To complete the proof, we will show that there is a unique x with the following property: there is q such that (i) $x = (u_1 \cdot q, \ldots, u_n \cdot q)$ and (ii) equation (7) is satisfied whenever $q^j > 0$. That such an x exists follows from the fact that pseudo Lindahl equilibria always exist and the necessity argument above.

For uniqueness, note that the argument above implies that for any PLE allocation we may choose the equilibrium price to be $p_i^j = \omega_i/(u_i \cdot q)$ for all j such that $u_i^j = 1$ (and zero otherwise). Thus, for any two pseudo Lindahl outcomes q, q', revenue maximization implies

$$1 = \sum_i \frac{\omega_i}{u_i \cdot q}(u_i \cdot q) \geq \sum_i \frac{\omega_i}{u_i \cdot q}(u_i \cdot q') \tag{8}$$

for all q'. Let $\beta_i = \frac{u_i \cdot q}{u_i \cdot q'}$ and $\beta = \sum_i \beta_i \omega_i$. Then, we can rewrite equation (8) as follows:

$$1 = \sum_i \omega_i \geq \sum_i \frac{\omega_i}{\beta_i}$$

Since $f(t) = \frac{1}{t}$ is a convex function, the inequality above implies $1 \geq \frac{1}{\beta}$. Reversing the roles of q and q' and rewriting equation (8) yields $1 \geq \sum_i \beta_i = \beta$. Hence, $\beta = 1$. Since f is strictly convex, we conclude that $\beta_i = 1$ for all i. □

7.5 Proof of Theorem 3

First we show that N_ω satisfies the four axioms above for any ω such that $\omega_i > 0$ for all i. Lemma A3(iii) implies that N_ω satisfies Efficiency. Scale Invariance follows since the Nash bargaining solution satisfies Scale Invariance. To prove Consistency, let $x \in N_\omega(B)$. Then, there is $A \geq B$ such that $\eta_\omega(A) = x$. Let $B \geq B'$ and $x \in B'$. Since the weak set order is transitive, $A \geq B \geq B'$ implies $A \geq B'$ and $x \in N_\omega(B')$. To prove completeness, consider any bargaining solution $S \subset N_\omega$, and let $x \in N_\omega(B)$. Then, $x = \eta_\omega(A)$ for some $A \geq B$. By Lemma A3(iii), there is a simplex A' such that $A' \geq A \geq B$ and $\eta_\omega(A') = x$. Since S is non-empty, Lemma A3(iii) implies that $S(A') = \{x\}$. Consistency then implies $x \in S(B)$.

For the converse, let S be a bargaining solution that satisfies the axioms and note that Efficiency implies $S(\Delta) = \{\omega\}$ for some weights ω. By Lemma A3(i) and (iii), $N_\omega(\Delta) = \{\omega\}$. Then, Scale Invariance implies $S(a \odot \Delta + z) = a \odot N_\omega(\Delta) + z = \{\eta_\omega(a \odot \Delta + z)\}$; that is, $S(A) = N_\omega(A)$ for any simplex A. Take any bargaining set B and $x \in S(B)$. By Lemma A3(ii), there is a simplex A such that $B \leq A$ and $\eta_\omega(A) = \eta_\omega(B)$. We have already shown that $x = \eta_\omega(A) \in S(A)$. Then, by Consistency, $x \in S(B)$, proving $N_\omega \subset S$. Then, the first part of the theorem and completeness yield $S = N_\omega$. □

7.6 Proof of Theorems 4

Let $(\mathbf{p}, q)$ be a pseudo Walrasian equilibrium for the private pseudo market and define $p = (p_1, \ldots, p_n)$ such that $p_i^j = \mathbf{p}(\mathbf{a}_i^j)$. We will verify that (p, q) is PLE. Let u_i^j be as defined in the text. First, we show that q is a least-cost optimal plan for every agent i. The definition of p_i implies that

$$p_i \cdot \hat{q} = \sum_j \mathbf{p}(\mathbf{a}_i^j)\hat{q}^j = \sum_M \mathbf{p}(M) d_i^{\hat{q}}(M)$$

for every allocation $\hat{q}$. Therefore, $p_i \cdot \hat{q} \leq \omega_i$ implies that $d_i^{\hat{q}} \in \mathbf{B}(\mathbf{p}, \omega_i)$. Thus, $\hat{q}$ is affordable in the collective pseudo market only if $d_i^{\hat{q}}$ is affordable in the private pseudo market. The definition of u_i implies that $u_i \cdot \hat{q} = \sum_M v_i(M) d_i^{\hat{q}}(M)$. We conclude that $\hat{q}$ and $d_i^{\hat{q}}$ yield the same utility. Since d_i^q is least-cost optimal in the private pseudo market, q must be least-cost optimal in the collective pseudo market.

The auctioneer's revenue from $\hat{q}$ in the collective pseudo market is the same as the revenue from $d_T^{\hat{q}}$ in the private pseudo market. This follows since

$$\sum_i p_i \cdot \hat{q} = \sum_{j \in K} \left(\sum_i \mathbf{p}(\mathbf{a}_i^j) \right) \hat{q}^j = \sum_{j \in K} \mathbf{p} \left(\bigcup_i \mathbf{a}_i^j \right) \hat{q}^j = \sum_{F \in \mathcal{F}} \mathbf{p}(F) d_T^{\hat{q}}(F)$$

Since d_T^q maximizes revenue in the private pseudo market, q must maximize revenue in the collective pseudo market.

7.7 Proof of Theorem 5

Let $\mathcal{A} = (H, \mathcal{F}, v)$ and let q be PLE random allocation of the corresponding collective pseudo market. We will show that q is pseudo Walrasian equilibrium random allocation as well. For the (feasible) allocation $j \in K$, define $F^j = \cup_i \mathbf{a}_i^j$ and note that $\mathcal{F} = \{F^j \mid j = 1, \ldots, k\}$. Since agents have unit demand preferences, each $\mathbf{a}_i^j$ either contains an element h_i^j such that $v_i(\mathbf{a}_i^j) = v_i(h_i^j) > 0$ or $v_i(\mathbf{a}_i^j) = 0$. In the latter case, set $h_i^j = \emptyset$.

We claim that there exists a PLE price p such that $p_i^j = p_i^k$ whenever $h_i^j = h_i^k$. To prove this assertion, let $\hat{p}$ be a PLE price for the allocation q. Since q is a least-cost optimal plan for i, $h_i^j = h_i^k$ and $q^j > 0$ implies that $\hat{p}_i^j \leq \hat{p}_i^k$. Thus, if $\hat{p}_i^k > \hat{p}_i^j$, then $q^k = 0$. If we lower the price $\hat{p}_i^k$ so that it is equal to $\hat{p}_i^j$, then the allocation q remains a least-cost optimal plan for consumer i and continues to maximize firm revenue. This proves the assertion. Henceforth, we assume that the PLE price satisfies $p_i^j = p_i^k$ whenever $h_i^j = h_i^k$. Then, the function $r_i : H \to I\!R$, such that

$$r_i(h) = \begin{cases} p_i^j & \text{if } h_i^j = h \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}$$

is well defined.

We say that agent i is the highest bidder for good h if $r_i(h) \geq r_l(h)$ for all $l \in N$. Single-mindedness implies that, for all $F \in \mathcal{F}$, there is an allocation $j(F)$ that assigns every good $h \in F$ to its highest bidder. Hence, for each $F \in \mathcal{F}$, $\sum_i p_i^{j(F)} = \sum_{h \in F} \max_i r_i(h)$.

We claim that $q^j > 0$ implies $j = j(F)$ for some F. If not, then there exists an agent m such that $r_m(h_i^j) > p_i^j = r_i(h_i^j)$. Since $r_m(h_i^j) > 0$, we have $v_m(h_i^j) > 0$. Since m is single-minded, $p_m^j = 0$. Let l denote the allocation that is identical to j except that i and m swap their consumption bundles. Allocation l is feasible because $F^l = F^j$. Moreover, allocation l yields greater revenue than allocation j since $\sum_{i'=1}^n p_{i'}^l = \sum_{i'=1}^n p_{i'}^j - p_i^j + p_m^l > \sum_{i'=1}^n p_{i'}^j$. This contradicts revenue maximization in the collective pseudo market and, therefore, proves the assertion.

Revenue maximization implies $\sum_{i=1}^n p_i^j \geq \sum_{i=1}^n p_i^l$ for all $l \in K$ if $q^j > 0$. Combining this inequality with the arguments in the previous two paragraphs, we conclude that $q^j > 0$ implies

$$\sum_{h \in F^j} \max_i r_i(h) \geq \sum_{h \in F} \max_i r_i(h) \tag{R}$$

for all $F \in \mathcal{F}$.

Define $\mathbf{p}$ as follows: $\mathbf{p}(h) = \max_i r_i(h)$. We claim that $(\mathbf{p}, q)$ is a pseudo Walrasian equilibrium. Inequality (R) implies that d_T^q maximizes the firm's revenue. To prove consumer optimality, first note that $p_i^j = \mathbf{p}(h_i^j)$ if $q^j > 0$ and $p_i^j \leq \mathbf{p}(h_i^j)$ otherwise. Thus, the consumption lottery d_i^q is affordable at prices $\mathbf{p}$.

To prove that d_i^q is optimal, first note that single-mindendess implies that there exists at most one good $h \in F$ such that $v_i(h) > 0$. Let $h_i^F = h$ denote that good if it exists and let h_i^F be the empty set otherwise. Let $\hat{d}$ be any consumption lottery and note that it yields the same utility as any lottery d^* such that $d^*(h_i^F) = \sum_{\{M : h_i^F \in M\}} \hat{d}(M)$ for all $h_i^F \neq \emptyset$. For each F, it is feasible to assign h_i^F to agent i. Therefore, there exists a feasible allocation $\hat{q}$ such that h_i^F is allocated to agent i with probability $d^*(h_i^F)$. From the definition of $\mathbf{p}$ it follows that $\sum_{M \subset H} \mathbf{p}(M)\hat{d}(M) \geq \sum p_i^j \hat{q}^j$. Therefore, any consumption lottery that is affordable in the private pseudo market yields the same utility as an affordable allocation in the collective pseudo market. Since q is least-cost optimal at prices p (in the collective pseudo market), d_i^q must be least-cost optimal at the prices $\mathbf{p}$ (in the private pseudo market). $\square$

References

Abdulkadiroglu A. and T. Sonmez, (1998) "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, 66, 689–701.

Arrow, K. J., E. W. Barankin, and D. Blackwell, (1953) "Admissible Points of Convex Sets," In: Contributions to the Theory of Games (AM-28), Volume II, edited by Harold William Kuhn and Albert William Tucker, Princeton: Princeton University Press, 87–92.

Basteck, C. (2018), "Fair solutions to the random assignment problem," *Journal of Mathematical Economics*, 79, 163–172.

Bogomolnaia, A. and H. Moulin, (2001) "A New Solution to the Random Assignment Problem," *Journal of Economic Theory*, 100, 295–328.

Brandl, F., F. Brandt, D. Peters, C. Stricker, W. Suksompong (2020), "Funding Public Projects: A Case for the Nash Product Rule," working paper.

Buchholz, W. and W. Peters, (2007) "Justifying the Lindahl solution as an outcome of fair cooperation," *Public Choice*, 133, 157–169.

Budish, E. (2011), "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes," *Journal of Political Economy*, 119, 1061-1103.

Chen C. and R. Zeckhauser, (2018) Collective Action in an Asymmetric World, *Journal of Public Economics*, 158, 103–112.

Eisenberg E. and D. Gale (1959), "Consensus of subjective probabilities: The pari-mutuel method." *Annals of Mathematical Statistics*, 30, 165–168.

Fain B., A. Goel, and K. Munagala (2016), "The core of the participatory budgeting problem," In: Proceedings of the 12th International Conference on Web and Internet Economics (WINE), Lecture Notes in Computer Science (LNCS), 384–399.

Foley, D. (1967) Resource Allocation and the Public Sector, *Yale Economic Essays*, 3, 45–98.

Foley, D. (1970), Lindahl's Solution and the Core of an Economy with Public Goods, *Econometrica*, 38, 66–72.

Gul, F, W. Pesendorfer, and M. Zhang (2024), “Efficient Allocation of Indivisible Goods in Pseudo-Markets with Constraints,” *Journal of Political Economy*, 132, 3708–3736.

Harsanyi, J.C. (1959), “A Bargaining Model for the Cooperative n-Person Game,” In: Contributions to the Theory of Games, Volume IV, edited by Harold William Kuhn and Albert William Tucker, Princeton: Princeton University Press, 325–355.

Hylland, A. and R. J. Zeckhauser (1979) “The Efficient Allocation of Individuals to Positions,” *Journal of Political Economy,* 87, 293–314.

Kalai, E. and M. Smorodinsky (1975) “Other Solutions to Nash’s Bargaining Problem,” *Econometrica*, 43, 513–518.

Kelso, A. S. and V. P. Crawford (1982) “Job Matching, Coalition Formation, and Gross Substitutes,” *Econometrica,* 50, 1483–504.

Kaneko, M. 1980, “An extension of the Nash bargaining problem and the Nash social welfare function,” *Theory and Decision*, 12, 135–148.

Lindahl, E. (1919), “Just Taxation - A Positive Solution,” Reprinted in part in: Classics in the theory of public finance, edited by R.A. Musgrave and A. Peacock, Macmillan, 168–176.

Mas-Colell, A., (1992) “Equilibrium Theory with Possibly Satiated Preferences,” In: Equilibrium and Dynamics: Essays in Honour of David Gale, edited by M. Majumdar, London, Macmillan, 201–213.

Miralles A., and M. Pycia (2021), “Foundations of Pseudomarkets: Walrasian Equilibria for Discrete Resources,” *Journal of Economic Theory,* 196, 1–24.

Nash, J.F. (1950) “The Bargaining Problem,” *Econometrica*, 28: 155–162.

Nguyen T., and R. Vohra (2024), “(Near-)Substitute Preferences and Equilibria with Indivisibilities,” *Journal of Political Economy,* 132, 4122–4154.

Perles, M.A. and M. Maschler (1981) “A Super-Additive Solution for the Nash Bargaining Game,” *International Journal of Game Theory*, 10: 163-193.

Raiffa, H. (1953) “Arbitration schemes for generalized two-person games,” In: Contributions to the theory of games, Volume II, edited by H.W. Kuhn and A.W. Tucker, Princeton University Press, 361–387.

Sertel, M. and M. Yildiz (2003) “On the Impossibility of a Walrasian Bargaining Solution,” In: Advances in Economic Design, edited by M. Sertel and S. Koray, Springer, 175–182.

Schmeidler, D., and K. Vind, (1972), “Fair Net Trades,” *Econometrica* 40, 637–642.

Silvestre, J., (2003) “Wicksell, Lindahl and the Theory of Public Goods,” *The Scandinavian Journal of Economics* , 105, 527–553.

Thomson, W., (1994) “Cooperative Models of Bargaining,” In: Handbook of Game Theory, Volume 2, edited by R.J. Aumann and S. Hart, Elsevier Science B.V., 1238–1284.

Sato, T. (1987), “Equity, Fairness, and Lindahl equilibrium,” *Journal of Public Economics*, 33, 261–271.

Varian H. R., (1973), “Equity, Envy, and Efficiency” *Journal of Economic Theory*, 9, 63–91.