Dynamic Demand Estimation in Auction Markets*

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Abstract

We study demand estimation in a large auction market. In our model, a dynamically evolving population of buyers with unit demand and heterogeneous and privately known preferences for a finite set of differentiated products compete in a sequence of auctions that occur in discrete time. We define an empirically tractable equilibrium concept in which bidders behave as though they are competing with the stationary distribution of opposing bids, characterize bidding strategies, and prove existence of equilibrium. Having developed this demand system, we prove that it is non-parametrically identified from panel data. We extend the model to consider a random coefficients demand system akin to workhorse demand models in industrial organization, and show that this too is non-parametrically identified. We apply the model to estimate demand and show how large sellers can exercise market power by using persistent reserve price policies, which induce higher bids and, therefore, revenues. Our analysis highlights the importance of both dynamic bidding strategies and panel data sample selection issues when analyzing these markets.

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1 Introduction

Most goods and services are sold at fixed prices. Yet, auctions are used as the allocation mechanism in a wide variety of contexts, including procurement contracts, Treasury bills, and the granting of oil drilling and spectrum rights. Technology companies such as Facebook, Google, and Microsoft sell advertisers access to online consumers through display and search advertising auctions. And though the majority of eBay's revenue now comes from fixed-price listings of new goods, they still sell a large number of goods (both in absolute and dollar terms) by auction.

These markets share many common features. Buyers get multiple purchase opportunities over time, either for exactly the same product (e.g. a keyword in online search advertising) or for close substitutes (e.g. in treasury bill auctions). This allows bidders to intertemporally substitute, making bidding decisions in light of the option value of waiting for future purchasing opportunities. Bids will reflect individual-specific preferences over the heterogeneous products available. For example, in highway procurement, Lewis and Bajari (2011) document matching between contractors and contract based primarily on distance and contract size, while in online labor markets, employers are more likely to award contracts to workers from their own country (Krasnokutskaya et al., 2016).

In the fixed-price context, an influential discrete choice demand estimation approach has been developed that models how buyers with unit demand and heterogeneous preferences over item characteristics make purchasing decisions (Boyd and Mellman, 1980; Cardell and Dunbar, 1980; Berry, 1994; Berry et al., 1995).[1] However, with the important exception of Jofre-Bonet and Pesendorfer (2003), there has been no analogous work in the empirical auctions literature, which typically considers the identification and econometric analysis of a repeated cross-section of observations of a static auction game with a single product. The goal of this paper is to fill this gap.

Specifically, we consider a large-market model of repeated second-price auctions of differentiated products. Buyers have unit demand and heterogeneous, perfectly persistent, and privately known multidimensional valuations (which may be formulated as latent random coefficients over item characteristics). Their optimal bidding strategy is to shade their bids

[1] Goettler and Gordon (2011) and Gowrisankaran and Rysman (2012) extend these models to consider cases where consumers purchase repeatedly over time, and must form beliefs about future product offerings. Here, we fix the set of products and impose unit demand in the present paper, as in the older literature.

below their valuations to account for the option value of staying in the market and possibly purchasing a better good or the same good at a lower price in the future. We prove the existence of a dynamic equilibrium and show non-parametric identification of the model. We then apply these results in estimating the demand for compact cameras on eBay.

There are four technical roadblocks that we overcome. The first is that allowing for persistent privately known preferences in a dynamic auction game introduces private monitoring, with all its attendant game theoretic complications. We surmount this using a large market equilibrium concept, defining an equilibrium as a symmetric bidding function and a set of beliefs about the distribution of rival bids such that the bidding function is optimal given those beliefs, and the beliefs are consistent with the ergodic distribution of equilibrium play.[2]

The second is a selection problem. Recall that we want to learn the distribution of the *vector* of valuations of bidders for the products in the market. Even if bidders bid their valuations—and they don't—it would still be hard to learn the distribution of valuations, because most of them will exit, either randomly or by winning an auction, well before bidding on every available product and thereby revealing their valuations. Moreover, auction winners are disproportionately those with high valuations, which introduces a selection problem.

We show that we can re-weight the observed distribution of these different types of bidders to learn their "bid-type": their preferred bid vectors.

The third piece of the puzzle is learning an inversion from bid-types to valuations. A useful insight is that a bidder with a given valuation vector faces a Markov decision problem (MDP) when deciding how much to bid on each product, where the payoff functions and transition matrices are observed. We show that the MDP can be inverted: if we know a bidder's bid-type—their optimal bid on each product—we can invert it to get their underlying type, or valuation.[3] By combining this inversion with the selection correction, we can identify the distribution of valuations.

[2]This is similar to ideas elsewhere in the literature, such as the model of belief formation in Krusell and Smith (1998) and the oblivious (mean-field) equilibrium concept of Weintraub et al. (2008) and Iyer et al. (2014). Many other papers in the dynamic games estimation literature have instead assumed that everything persistent is commonly known, but this is counter to the spirit of our project (for examples, see e.g., Jofre-Bonet and Pesendorfer (2003), Aguirregabiria and Mira (2007), Bajari et al. (2007), Pakes et al. (2007); Fershtman and Pakes (2012) is a recent exception.)

[3]This approach—solving the dynamic problem to identify bidder valuations, was pioneered by Jofre-Bonet and Pesendorfer (2003) and has been used subsequently in many of the papers cited in the literature review below. What is novel here, as we discuss in Section 2 below, is that bidder types are latent and persistent, which poses a number of new challenges.

All of these arguments are “pointwise”: they are about how one might learn a particular bidder’s valuations from watching their time series of bids. But in our demand system, we allow individual auction items to have both unobserved characteristics equally valued by all bidders (unobserved heterogeneity) and unobserved characteristics over which bidders have specific tastes (idiosyncratic shocks). The presence of these shocks makes a pointwise identification strategy infeasible. So, the last technical roadblock we overcome is combining our pointwise arguments with an existing statistical literature on deconvolution to develop an approach based on successive measurements of various bid distributions.

Our setup endeavors to situate auctions in a *marketplace*, with opportunities for substitution that are often missing in the auctions literature. However, in reality, substitution is much richer than any one model will capture: it happens through search; it happens as bidders choose which of concurrent auctions to enter, and it also happens intertemporally, as bidders eye future opportunities. We have deliberately chosen to focus on the latter channel. In our setting, a bidder with a strong preference for a particular type of product will shade their bid aggressively until it is auctioned, substituting inter-temporally rather than through search or entry. This loads substitution into the continuation value, which we show to be quite tractable for empirical modeling. Still, like prior work in this domain, a limitation of our approach is that we ignore substitution to alternative mechanisms, such as fixed-price transactions, negotiations, or hybrid mechanisms such as eBay’s auction-buy-it-now format.

The final part of the paper is an application of our framework to data from eBay’s compact camera market. We model preferences as a linear combination of a random taste for a camera and a random taste for camera resolution, and estimate the distribution of these random coefficients. Our estimation approach uses only the first-and-second highest bids in each auction, as it is hard to know how to interpret other bids. We are able to accommodate this data limitation by modifying our selection correction argument, so that the censoring of bids that were neither first nor second highest is explicitly accounted for, which illustrates the flexibility of our approach.

We use these results to consider how a large seller, controlling approximately one-third of the listings, might exercise market power in an auction marketplace. By committing to a persistent reserve price policy (i.e. one that applies to all listings from this seller, versus a transient reserve, which is implemented in only one auction, as a small or one-time seller might), the seller can reduce buyers’ continuation values, which translates into more aggressive bidding and higher revenue. Simulation using our demand estimates shows that this effect can be quite large. When the seller's costs are 90% of the expected sale price under no reserves they are able to use optimal persistent reserves to raise profits by as much as 45.99%, versus 19.39% under a transient reserve. This difference stems from the market power of the seller, and will grow as their market share—and therefore their ability to depress bidders' continuation values—does.

Literature Review. Jofre-Bonet and Pesendorfer (2003) was the first paper to attack estimation in a dynamic auction game in the context of sequential procurement auctions. Subsequent to this, a number of papers have looked at dynamics on the eBay platform specifically. Zeithammer (2006) developed a model with forward-looking bidders, and showed both theoretically and empirically that bidders shade down current bids in response to the presence of upcoming auctions of similar objects. Sailer (2006) estimates participation costs for bidders facing an infinite sequence of identical auctions (see also Groeger (2014) for a dynamic model with participation costs in a procurement environment). Nekipelov (2007) analyzes a model where bidders attempt to prevent learning by late bidding. Ingster (2009) develops a dynamic model of auctions of identical objects, and provides equilibrium characterization and identification results. Finally, Ridinger (2020) incorporates resale and aggregate shocks into a dynamic model that is applied to the market for classic cars.

The original version of our paper, Backus and Lewis (2010), introduced a dynamic auction model with latent, persistent, and high-dimensional types, and showed that under certain restrictions on player beliefs, equilibrium existence and non-parametric identification could be proved. Subsequently, there have been two other notable papers about dynamic auction markets, and eBay specifically. Bodoh-Creed et al. (2017) estimate demand for Kindles, and then simulate an alternative market design where half as many 2-unit auctions were run instead, finding that this would increase efficiency but lower seller revenues. Their model uses the same "large market" belief approximations as our original paper, but they go further by employing a continuum approximation with an infinite number of bidders and products that they show behaves similarly to the finite game and has certain computational advantages. They also allow for participation costs and endogenous entry. Hendricks et al. (2021) keep the finite bidder model and "large market beliefs", but do the analysis in continuous time, and again allow for selective entry: bidders pick which of an upcoming set of available auctions to participate in, depending on the current state (e.g. current posted bids in these auctions). They use their estimates to understand how well the marketplace works relative to a fully efficient competitive benchmark. The main difference between these papers and ours is that their models are of homogeneous goods and focus on efficiency and market design, whereas we have heterogeneous objects and are interested in substitution between products.[4] Note that substitution in our model is captured entirely through the level of the bid; we do not model timing or strategic matching as, for instance, in Hendricks et al. (2021).

2 The Demand System in Context

We begin the paper with a brief review of the relevant portion of the empirical auctions literature and how it relates to similar work on fixed-price markets. The starting point of this literature is a model where the gross payoff (i.e. before subtracting price) of a winning bidder takes the form:

$$u_i = \varepsilon_i.$$

Specifically, short-lived agents i draw independently and identically distributed (iid) real-valued valuations ε_i from some distribution F, and participate in some auction mechanism to win the items (see, e.g., the work of Guerre et al. (2000) on first-price auctions). The econometrician observes bids from independent repetitions of this mechanism and wants to learn F. Subsequent elaborations of this work included treatments of non-iid and asymmetric valuation distributions, incomplete data and other auction mechanisms (see Athey and Haile (2007) for a discussion of identification in these models).

Maintaining the static auction environment but allowing for items with distinct bundles of characteristics yields a gross payoff of the form:

$$u_{i,t} = \boldsymbol{\gamma} \mathbf{z}_t + \varepsilon_{i,t}$$

where items are indexed by the time period in which they are offered t, $\mathbf{z}_t$ is a vector of item characteristics equally valued by all bidders and $\boldsymbol{\gamma}$ is an unknown parameter vector. Haile et al. (2003) show how to estimate the parameter $\boldsymbol{\gamma}$ in a first-stage "normalization" step

[4] Other papers tackle related issues in a static setting. Gentry and Li (2014) and Adams (2012) offered analyses of non-parametric identification in models of costly endogenous buyer entry and optimal entry with multiple competing auctions, respectively, while Marra (2018) models both buyer and seller entry in order to study fee–setting on two–sided auction platforms. And while we focus on heterogeneity in preferences on the buyer side, Krasnokutskaya et al. (2014) consider a world with heterogeneous preferences on the seller side.

prior to learning the distribution F. Notice the resemblance to a utility function assumed in a differentiated product demand model.

The pioneering work of Jofre-Bonet and Pesendorfer (2003) (JBP) tackles some of the issues that arise in a dynamic environment. In their model, there is a fixed set of infinitely long-lived bidders bidding on highway procurement projects, where one auction takes place each period. Each bidder's gross payoff to winning a project in period t can depend on their own time-varying characteristics, as in:

$$u_{i,t} = \boldsymbol{\gamma}(\mathbf{w}_{i,t} \times \mathbf{z}_t) + \varepsilon_{i,t}$$

where $\mathbf{w}_{i,t}$ are bidder characteristics as of time t (e.g. in their application, bidder i's backlog in period t), and $\mathbf{z}_t$ are item characteristics (e.g. project size) for the auction at time t, and again $\varepsilon_{i,t}$ is drawn identically and independently across auctions from an unknown distribution F. The preferences should again feel reminiscent of a fixed-price differentiated products demand system where consumer tastes depend on their observable characteristics up to a scalar unobservable $\varepsilon_{i,t}$. This scalar unobservable is the only source of private information in the model. Players form beliefs about the distribution of competing bids that condition on the observable bidder and item characteristics, and determine their optimal bidding strategy given those beliefs. The goal is to identify the parameter vector $\boldsymbol{\gamma}$ as well as the distribution F.

In the current paper, we study a different dynamic auction game. There is still one auction for an item in each period t, and bidders are long-lived, but now they also have latent persistent preferences over product types j associated with the item offered in period t. The gross payoff to winning in our model is the following:

$$u_{i,j,t} = x_{i,j} + \boldsymbol{\gamma}\mathbf{z}_{j,t} + \xi_{j,t} + \varepsilon_{i,j,t}$$

where j indexes the members of a finite set of products, $x_{i,j}$ is bidder i's perfectly persistent and latent valuation for product j, $\mathbf{z}_{j,t}$ is a vector of item (possibly time-varying) characteristics over which bidders have common preferences $\boldsymbol{\gamma}$, $\xi_{j,t}$ is an item-specific unobservable over which all bidders have common preferences and $\varepsilon_{i,j,t}$ is an idiosyncratic shock. The goal is to identify the distribution $\mathbf{F}$ of the perfectly persistent types $\mathbf{x}$, the parameter $\boldsymbol{\gamma}$, the distribution of unobserved characteristics F_ξ and the distribution of the shocks F_ε.

The demand system in (1) places limited restrictions on the data-generating process. It can be made more or less flexible depending on how the products indexed by j are defined. On one extreme, we could have a single product type, and a single persistent taste shock for purchasing an item. This would load all item heterogeneity into the observable and unobservable attributes, $\mathbf{z}_{j,t}$ and ξ_{jt} respectively, which are common to all bidders' valuations. On the other extreme, we could treat all items with different characteristics as distinct products (i.e., have no characteristics with common preferences), and identify the joint distribution of valuations $\mathbf{F}$. This would allow for bidders to persistently differ in their preferences over all items in ways that cannot be explained by item characteristics. As we will show, the cost of allowing such generality is that the cardinality of the product space may explode, and so do the data requirements for estimation.[5] In our application, we chart a middle course, allowing for camera product types that differ in a key observable attribute — resolution. Customers have heterogeneous preferences over resolution (i.e. the coefficient on resolution) and the overall value of a camera (i.e. the constant), in addition to homogeneous preferences over camera characteristics such as brand, digital zoom, inclusion of an extra battery, etc.

What makes the resulting dynamic auction game challenging to analyze is that players now have two sources of private information: their own persistent type, as well as the transient idiosyncratic shock familiar from prior work in this literature. Because types are persistent, players should refine their beliefs about the distribution of rival types over time, as they observe outcomes in the market. This creates game theoretic complications that we circumvent by the use of large market arguments.

The preference structure analyzed here is reminiscent of the random coefficients model of discrete choice developed by Boyd and Mellman (1980) and Cardell and Dunbar (1980), and well-known in the IO literature due to the work of Berry et al. (1995). To see this, notice that if we restrict valuations to a factor structure, $x_{i,j} = \mathbf{z}_{j,t}\tilde{\gamma}_i$, for idiosyncratic bidder preferences $\tilde{\boldsymbol{\gamma}}_i \equiv \boldsymbol{\gamma}_i - \boldsymbol{\gamma}$ and product characteristics $\mathbf{z}_{j,t}$, we recover the preferences usually seen in random coefficients models. The main difference to that literature is that there is no price term in the (gross) utility. In fixed-price demand systems, an identification challenge is posed by correlation between the prices and the unobserved quality term (here $\xi_{j,t}$). Here this problem doesn't arise, because prices are an outcome of the auction, rather than a choice of an optimizing seller.[6]

[5] A similar phenomenon appears in product space models of differentiated products demand, where the econometrician estimates a matrix of cross elasticities that grows quadratically in the number of products.

[6] That said, we have not allowed for reserve prices in the current paper, and if we had, it would be

3 Model and Equilibrium Analysis

We consider a dynamic auction market in discrete time, in which an item is sold each period by second-price sealed bid auction. Each item is an example of a product (e.g. 8MP cameras), though individual items may differ in some additional characteristics (e.g. whether they come bundled with a battery). There is a discrete set of substitutable products sold in the marketplace. Bidders are assumed to have unit demand, and so these products compete with each other for buyer demand.

Buyers are differentiated in their tastes for different products, entering with private and perfectly persistent preferences over products. They are inattentive, and are active and bid in any particular period with constant probability. We show that they bid their valuation less their continuation value, and assume they assess the latter based on the steady-state distribution of supply and competing bids, rather than on current market conditions (e.g. the number of competing bidders in the current auction). Winning bidders exit the market with certainty, while losing bidders exit with constant probability. The population of bidders is thus constantly evolving.

We have chosen this set of assumptions to match some features of the market for digital cameras on eBay, which is our empirical application. In any eBay category (such as digital cameras), there are many different products sold by auction to a large number of anonymous buyers.

Although these auctions typically last for many days and thus overlap—so that at any given point in time there are many auctions occurring simultaneously—they finish at different ending times, in sequence. As Bajari and Hortaçsu (2004) and Hendricks and Porter (2007) have noted, this timing, combined with the way the proxy bidding system works, implies that eBay is reasonably well approximated by a sequence of second-price sealed bid auctions.

3.1 Environment

We formalize the above description of the environment in what follows:

reasonable to assume they were correlated with $\xi_{j,t}$, creating additional challenges that we do not explore. Existing approaches based on inverting reserve prices to recover unobserved quality may be of use in this case—see for example Roberts (2013) and Decarolis (2018).

Supply. There are J distinct kinds of products sold in a market, indexed by $j = 1 \ldots J$. We denote the set of products by $\mathcal{J}$. In each period t, an item—one of the products—may be available for purchase. Supply of products is exogenous and Markov, with the current item j_t drawn from a stationary multinomial distribution conditional on the lagged product j_{t-1}. We allow for the possibility that no product is available in a given period, and so supply can be summarized by a square transition matrix Q of size $J+1 \times J+1$, where the entry $Q_{j,k}$ gives the probability product k will be supplied next when j is currently offered (and the last row and column give the cases where no product is offered now and will be offered later, respectively). We assume moreover that the multinomial distributions have full support, so that regardless of what was supplied at $t-1$, it is possible that any of the J products (or nothing) is supplied at t. When an item is available, it is sold by second-price sealed-bid auction.

Individual items are instances of one of the J products but may be further differentiated by additional characteristics. We denote characteristics of the item sold at t that are observable to the econometrician by $\mathbf{z}_{j,t}$ and those that are unobservable by $\xi_{j,t}$. We assume that $\xi_{j,t}$ is scalar, exogenous, and drawn independently over t according to a mean-zero distribution F_ξ, yielding a serially independent distribution of unobserved heterogeneity shocks. We assume that the distributions of both observable and unobservable characteristics have bounded support.

Demand. At the beginning of each period, E_t buyers enter the market, where E_t is sampled independently over time from a truncated Poisson distribution with mean parameter λ.[7]

Each buyer has unit demand. Their gross payoff for an item sold in period t is given by:

$$u_{i,j,t} = x_{i,j} + \boldsymbol{\gamma}\mathbf{z}_{j,t} + \xi_{j,t} + \varepsilon_{i,j,t} \tag{1}$$

where i indexes buyers, $x_{i,j}$ is a perfectly persistent and bidder-specific valuation for product j, $\boldsymbol{\gamma}$ is the vector of (common) preferences of buyers for the observable characteristics $\mathbf{z}_{j,t}$, and $\varepsilon_{i,j,t}$ is an idiosyncratic and transient real-valued preference shock. We assume that

[7] To ease some of the proofs, we assume that there exists some maximum number of buyers $\overline{N}$ that can be in the market at any time. This ensures compactness of the state space we will later define, and avoids measure theoretic complications. To ensure this holds, the number of entrants E_t is drawn from a truncated Poisson distribution with support $\{0, 1, \ldots \overline{N} - I_t\}$ where I_t is the number of incumbents at the start of period t. Since $\overline{N}$ can be arbitrarily large, we treat E_t as Poisson for estimation in what follows.

$\mathbf{x}$ is drawn iid across buyers on entry according to a distribution $\mathbf{F}$ with strictly positive density over its support $\mathcal{X} = [0, \overline{x}]^J$; observable characteristics $\mathbf{z}_{j,t}$ are assumed to be iid and exogenous according to a distribution $\mathbf{F}_z$, and the idiosyncratic shock is exogenous and drawn iid from some distribution F_ε with zero mean and strictly positive density over a bounded support.

Note that (1) gives meaning to our distinction between products, indexed by j, and items, which appear in time indexed by t. The distinction hinges on the fact that the persistent, private part of i's preferences, denoted $\mathbf{x}$, varies only over products.

The new entrants combine with the population of buyers from previous periods to form a cohort of bidders of size N_t. New entrants are always active; the remaining incumbent members of the cohort are active in each period with probability τ, which is exogenous, stationary, and iid. Active buyers participate in the current auction (if one is held), observing the product currently under auction, and placing bid $b \in \mathbb{R}$.[8]

Notice that we allow for negative bids. This plays an important simplifying role in the identification arguments below—since otherwise we would have to deal with censored data—but is not important for the theory presented in this section.[9] Agents are risk-neutral and have quasi-linear utility, receiving a total payoff of $u_{i,j,t} - p$ for buying a product j in period t at price p, and zero otherwise. If an auction is held and a bidder wins the auction, they exit the market. All other active bidders exit with probability $(1 - r)$ (i.e. $r \in (0, 1)$ is the survival rate). Inactive bidders do not exit. We assume that agents do not discount future payoffs, although the exogenous exit probability is functionally equivalent to exponential discounting.[10]

3.2 Beliefs and Strategies

We begin our analysis by looking at the buyer's beliefs. In our application to the compact camera market on eBay, it seems reasonable to assume that buyers have simple models of the competition they face. We formalize this idea by assuming that bidders believe that

[8] Note that this rules out an explicit model of search, where buyers endogenously choose which of an upcoming sequence of auctions to bid in. This is a limitation of the model. See Hendricks et al. (2021) for a way to account for this.

[9] If bids must be positive, bidders will simply not participate whenever their desired bid is negative. Their continuation values must be adjusted accordingly.

[10] For this reason the agents' time preferences are not separately identified in this setting.

the distribution of the highest competing bid is equal to the historical average for similar items and best respond accordingly.[11] Let B^1 be a random variable denoting the highest bid placed for the item sold at t, and let $s_t = (j_t, \mathbf{z}_{j,t}, \xi_t)$ be a *public state* variable consisting of information publicly observed by the players: the product being sold and the item's other characteristics. Let $G^1(b|s)$ be the corresponding conditional cumulative distribution function.

Assumption 1 (Beliefs about competing bids). *When the public state is s, bidders believe that the highest rival bid in the current auction B^1 has distribution $G^1(b|s)$*

These are reasonable beliefs to hold: an uninformed bidder, entering a stationary market, should expect the distribution of competing bids to look like the historical distribution of bids on similar products. This is because their own entry into the market is random and thus uninformative as to the current state, which means the distribution of rival bids and historical bids coincide.[12] What the assumption rules out is signaling and learning, e.g. that bidders are more sophisticated, updating their beliefs based either on their own experience after entering the market or additional information they may observe such as the sequence of upcoming auctions.[13] Bidders are "oblivious" in the sense of Weintraub et al. (2008) (though their oblivious equilibrium concept applies only to games of complete information).

We make this modeling choice to simplify the game theory, but it is also attractive as a model of bidder behavior. Being fully rational and conditioning on all available information is presumably costly, and the gains are small. This is because under repeated second-price auctions, the optimal bid today depends on the current state of the market only to the extent that it predicts future competition (the payoff on losing), and the market turns over quickly enough that the current state is not particularly informative.

Under this assumption, the only payoff relevant information at time t is the bidder's persistent valuation $x_{i,j}$, their idiosyncratic valuation $\varepsilon_{i,j,t}$ and the public state s_t. So without

[11] Some version of this assumption is made throughout the related literature (Ingster, 2009; Bodoh-Creed et al., 2013; Hendricks et al., 2021).

[12] Contrast this with a repeated static auction model with N bidders drawing iid valuations; then the historical highest bid distribution G^1 is the maximum of N bids, whereas the highest rival bid distribution is the maximum of $N-1$ bids. When the number of bidders is fixed, the fact that a bidder is in the auction is informative: it means that only $N-1$ people are competing with them. Here there is no fixed number of bidders and the bidder's presence does not inform their beliefs about G^1.

[13] A bidder who lost yesterday might reason that competition was fierce yesterday, and may still be so today. This gives rise to a winner's curse effect (Budish, 2008). There is some evidence that this kind of learning occurs on eBay: Coey et al. (2015) show that bidders tend to (modestly) increase their bids in subsequent auctions for the same product, although they attribute this to deadlines rather than learning.

loss of generality, a pure strategy is a real-valued function $\beta : \mathcal{X} \times \mathbb{R} \times \mathcal{S} \to \mathbb{R}$, where $\mathcal{S}$ is the public state space. In the paper, given the number of agents typically seen on eBay, we choose to restrict attention to symmetric strategies.

3.3 Equilibrium

In an equilibrium of this market, bidders will play optimally given their beliefs about the competition they face. Those beliefs are informed by the long-run distribution of competing bids. So in order to make progress in defining an equilibrium, we need to talk about the long-run evolution of the market.

The environment evolves as a first-order Markov process. The only auto-correlated features of the environment are the supply of products (assumed Markov), and the set of bidder types (perfectly persistent while in the market, but subject to entry and exit). Define a state variable ω_t, distinct from the public state s_t, which consists of the product being auctioned j_t and the persistent valuations of all the incumbent bidders (i.e. $\mathbf{x}_t = \{\mathbf{x}_i\}_{i \in \mathcal{I}_t}$ where $\mathcal{I}_t$ is the set of incumbent bidders). Then next period's state depends on who wins and who loses, which losers survive, what item is sold next period, and the number and valuations of new entrants. Define $T(\mu, \beta)$ as the transition operator for this model. It takes a distribution μ over current states ω into distributions over the next period's states ω'. Then for each β, a stationary distribution is a fixed point: $\mu = T(\mu, \beta)$. We can now define an equilibrium:

Definition 1 (Equilibrium). *A pure strategy equilibrium is a tuple* $(\beta^e(\mathbf{x}, \varepsilon, s), \{G^1(b|s)\}_{s \in S}, \mu^{\beta^e})$ *such that*

(Optimality) $\beta^e(\mathbf{x}, \varepsilon, s)$ *is a best response given beliefs* $G^1(b|s)\ \forall s \in \mathcal{S}$*;*

(Stationarity) μ^{β^e} *is the unique fixed point of the transition operator* $T(\mu, \beta^e)$*, and*

(Consistency) $G^1(b|s) = \int \int 1\left(B^1(\mathbf{x}(\omega), \varepsilon, s) \leq b\right) d\mathbf{F}(\varepsilon) d\mu^{\beta^e}(\omega)$.

The definition mirrors the requirements for a Bayes–Nash equilibrium: play must be optimal given beliefs, and beliefs must be consistent with play. The twist is that here we link strategies to beliefs through the invariant measure μ^{β^e}. This measure gives the long-run distribution of bids. For a market in steady-state, this will correspond to the historical average bid distributions. The consistency requirement says that beliefs about competing bids must be right i.e. calculated from the probability that the maximum rival bid B^1—a function of the set of competing bidders, their idiosyncratic draws and the current state—will be greater than b, on average over the distribution of idiosyncratic draws and the invariant measure.

The stationarity condition disciplines the solution concept. It ensures that the historical and future bid distributions coincide. This motivation depends upon the assumption of uniqueness. If the invariant measure were not unique, then the long-run future of the market would depend on the realization of some future event that splits the Markov chain into non-communicating regions, and the past and future would be quite different. In our context, for any strategy profile there is a corresponding unique invariant measure, so the requirement has no bite (we prove this in Lemma 5 below).

3.4 Best Responses

We find best responses by solving the buyer's decision problem. This problem is dynamic, since losers may have an opportunity to bid at a later date. Each buyer must solve a dynamic program and optimize their bids as a function of their private information $(\mathbf{x}, \varepsilon)$ and the public state s. The public state consists of two transient components (the item characteristics $(\mathbf{z}_{j,t}, \xi_{j,t})$, and one Markov component (the product being sold j_t). As a result, the buyer faces a Markov decision problem.

Define the perceived continuation value of an active bidder $v(\mathbf{x}, \varepsilon, s)$ recursively using a Bellman equation as follows:

$$v(\mathbf{x}, \varepsilon, s) = \max_{b \in \mathbb{R}} \int \Big(1(b \geq B^1)(u(\mathbf{x}, \varepsilon, s) - B^1) + 1(b < B^1) r \int \int v(\mathbf{x}, \varepsilon', s') dP(s'|s) dF(\varepsilon') \Big) dG^1(B^1|s) \tag{2}$$

where s' and ε' are the public state and idiosyncratic shock the next time the bidder is active respectfully, $P(s'|s)$ is the associated transition kernel for the public state and $F(\varepsilon')$ is the distribution of the idiosyncratic shock.

The first term inside the integral represents the case where the bidder submits the largest bid, winning the auction and obtaining surplus equal to their valuation less a payment given by the second-highest bid (potentially zero). The second term represents the case where the bidder loses and survives to bid another day, obtaining their continuation value for the next period in which they will be active. These events are determined by the realization of the highest competing bid, which by Assumption 1, the bidder believes to be distributed according to $G^1(B^1|s)$.

Let x_j denote the persistent part of the valuation for the item currently being auctioned, x_k be their persistent valuation of any of the other products, and let $u(\mathbf{x}, \varepsilon, s)$ correspond to $u_{i,j,t}$ where j, $\mathbf{z}_{j,t}$, and $\xi_{j,t}$ are given by s and $\varepsilon_{i,j,t} = \varepsilon$. Solving the above maximization problem, we get:

Lemma 1 (**Best responses**). *Suppose that beliefs satisfy Assumption 1 and that the bid distributions $G^1(b|s)$ have no gaps in their support. Then the best response function $\beta(\mathbf{x}, \varepsilon, s)$ satisfies:*

$$\beta(\mathbf{x}, \varepsilon, s) = u(\mathbf{x}, \varepsilon, s) - r \int\int v(\mathbf{x}, \varepsilon', s') dP(s'|s) dF(\varepsilon') \tag{3}$$

where the dependence on the distributions $\{G^1(\cdot|s)\}$ is through the continuation value. $\beta(\mathbf{x}, \varepsilon, s)$ is continuous in $(\mathbf{x}, \varepsilon, z, \xi)$, strictly increasing in x_j and decreasing in x_k.

Because this is a second-price auction, bidders bid to be indifferent between marginally winning and losing. Winners who marginally win receive the difference between their value and their bid, while losers receive their expected continuation value (discounted to account for the possibility of exit). Equating these, bids must be equal to valuation $u(\mathbf{x}, \varepsilon, s)$ less expected continuation value. This optimal bid might sometimes be negative, indicating that a bidder would optimally wait for another product rather than win the current product at a positive price.[14]

For what follows, it will be useful to rewrite the best response function in a simpler form. Notice first that the transition kernel $P(s'|s)$ is relatively simple, as both $z_{j,t}$ and $\xi_{j,t}$ are assumed to be transient, so tomorrow's item characteristics are just a random draw from the underlying distributions. The only Markov term is the product being auctioned the next time the bidder is active. Because the time of the next activity is random, the transitions are an average of the supply matrix Q iterated a random number of times, given by $\widetilde{Q} \equiv \sum_{s=1}^{\infty} \tau(1-\tau)^{s-1} Q^s$.

[14]Similar equations appear in many other papers. As Budish (2008), Said (2009) and Zeithammer (2009) all note, this bid characterization is not correct in Bayes-Nash equilibrium, due to a winner's curse effect: on winning, a bidder learns that the remaining types had lower valuations, and therefore regrets winning now rather than later at a lower price. This effect disappears when private valuation shocks are transient (e.g. Jofre-Bonet and Pesendorfer (2003)) or under large market assumptions on beliefs (our Assumption 1 or those made in Iyer et al. (2014), Hendricks et al. (2021) and Bodoh-Creed et al. (2017)).

As a result, we can write:

$$\int\int v(\mathbf{x}, \varepsilon', s')dP(s'|s)dF(\varepsilon') = \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x})$$

for $v_k(\mathbf{x}) = \int\int\int v(\mathbf{x}, \varepsilon, k, \mathbf{z}, \xi)dF_\varepsilon(\varepsilon)d\mathbf{F}_z(z)dF_\xi(\xi)$ (i.e. the expected value of the game tomorrow if the product to be auctioned is k).[15] Define $\widetilde{\beta}_j(\mathbf{x}) = x_j - r\sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x})$. Then the best response in (3) can be written as:

$$\beta(\mathbf{x}, \varepsilon, s) = \widetilde{\beta}_j(\mathbf{x}) + \boldsymbol{\gamma}\mathbf{z}_{j,t} + \xi_{j,t} + \varepsilon_{i,j,t}. \tag{4}$$

Best responses are thus additively separable into a non-linear function of the persistent components, and a linear function of the transient components. For any fixed type $\mathbf{x}$, the MDP thus has a discrete state space (the set of J products). It also has bounded returns, because the maximum payoff is bounded by the maximum valuation, which is finite.[16] It follows that their decision problem has a solution and a unique value function (Blackwell, 1965).

We have thus far established that one best response is to bid valuation less continuation value, and that this continuation value is unique. In the proof of Lemma 1 we further argue that when there are no gaps in the support of the bid distribution, this is the unique best response (i.e. it makes sense to speak of a best response function, rather than a correspondence).

3.5 Existence

To tie up this model, we state the first main result of the paper, an existence theorem:

Theorem 1 (**Existence**). *A pure symmetric strategy equilibrium exists in which (i) bidders bid according to a continuous function $\beta(\mathbf{x}, \varepsilon, s)$ of the form defined in Lemma 1; (ii) the bid distributions form a location family: $G^1(b|s) = \check{G}^1_j(b - \boldsymbol{\gamma}\mathbf{z} - \xi)$ for some distribution $\check{G}^1_j$, where j is the product auctioned in state s.*

The proof of the result is in two parts. First, we establish existence in a simpler game in which there are no item characteristics $(\mathbf{z}, \xi)$. In this case, from (4), strategies take the form

[15] We have expanded out the state s into its components $(k, \mathbf{z}, \xi)$ for this derivation.

[16] This is a consequence of the assumption that all components of the valuation have bounded support.

$\widetilde{\beta}_j(\mathbf{x})+\varepsilon$. We begin by showing that for any $\widetilde{\beta}_j$ function, there exists a corresponding unique invariant measure over the distribution of types $\mathbf{x}$ in the market. Existence of an invariant measure follows from compactness of the state space. Uniqueness follows by showing that all states communicate with one another, ruling out the possibility that the initial state determines the invariant measure.

Notice that in this simpler game, the bid distributions $G^1(b|s)$ vary only with j, so we can work with distributions $\check{G}^1_j$. We must show that there exist equilibrium strategies β^e that are a best response given beliefs $\{\check{G}^1_j\}$ that are consistent with the ergodic measure μ^{β^e}. We prove this by applying Schauder's fixed point theorem on the space of continuous functions on the type space. The proof proceeds by showing that the set of best responses to any continuous bidding function is uniformly equicontinuous and bounded (a compactness condition), and that the best response varies smoothly with the bidding function (a continuity condition). To establish continuity, we prove in Lemma 7 in the appendix that the ergodic measure is continuous in the strategies so that beliefs and best responses are continuous too. Because both $\mathbf{x}$ and ε have no gaps in their support, and the equilibrium bidding function is continuous, the distributions $\check{G}^1_j$ also have no gaps, consistent with the assumptions in Lemma 1.

The second part of the proof notes that if playing $\beta(\mathbf{x},\varepsilon,j)=\widetilde{\beta}_j(\mathbf{x})+\varepsilon$ is an equilibrium of the simpler game, then $\beta(\mathbf{x},\varepsilon,s)=\widetilde{\beta}_j(\mathbf{x})+\boldsymbol{\gamma}\mathbf{z}+\xi+\varepsilon$ remains an equilibrium of the game with characteristics. The intuition is that these characteristics are equally valued by all bidders, their value is simply "bid out", with no other strategic implications. As a consequence, the bid distributions form a location family, where the item characteristics cause shifts in the location but nothing else.

4 Identification

We now turn to non-parametric identification of the dynamic auction model, allowing for persistent private types, listing-specific and idiosyncratic shocks. We assume throughout this section that a single equilibrium is played by the bidders. The data takes the form of a panel, consisting of all bids and the products they bid on from every bidder who was active in the market. Each individual bidder time series will have gaps, corresponding to auctions in which they did not participate because they were not active in that period.

The time series will also be of different lengths since bidders will exit at different times. There are random and non-random reasons for exit: winners are likely to be types with high valuations, but among losers, exit is at random.

In the absence of the time-varying components of the valuation $(\mathbf{z}, \xi, \varepsilon)$, each bidder has a persistent optimal bid for each product that we call their bid-type. We show that there is an inversion from bid-types to valuations, building on an argument first presented in Jofre-Bonet and Pesendorfer (2003). But only bidder time series contain a bid on every product—"complete" bidder time series—can be inverted in this way, and due to endogenous exit the sample of complete bid vectors is selected. An important part of our identification argument is correcting for this selection.

But because bids vary over time due to shifts in item characteristics as well as idiosyncratic shocks, the bid-types are not observable in the data. Instead, we work with bid distributions, showing that the distribution of complete bid vectors is a convolution of the distribution of bid-types with the various shock distributions. A combination of deconvolution, selection correction, and inversion arguments suffices for identification.

We begin by re-stating some of the key equations, and adding additional assumptions. Recall that the gross payoff function takes the form:

$$u_{i,j,t} = x_{i,j} + \boldsymbol{\gamma}\mathbf{z}_{j,t} + \xi_{j,t} + \varepsilon_{i,j,t}. \tag{5}$$

We assume that the Markov supply process, bidder activity, $\xi_{j,t}$ and $\varepsilon_{i,j,t}$ are all mutually independent. Moreover, $\mathbf{z}_{jt}$ is iid conditional on j. The independence of the observed product characteristics from the unobserved shocks allows us to estimate $\boldsymbol{\gamma}$ in a first-stage, without worrying about the endogeneity of the product characteristics. This is consistent with the approach taken in Haile et al. (2003) as well as most demand systems in the industrial organization literature, where product characteristics are usually taken as exogenous.[17] The assumption that $\xi_{j,t}$ and $\varepsilon_{i,j,t}$ are independent allows us to separately identify the distributions F_ξ and F_ε by deconvolution.

[17]Just as in that literature, this independence assumption may be weakened if, for instance, we assume the existence of appropriate instruments for the endogenous observable characteristics. Below, we argue that the coefficients $\boldsymbol{\gamma}$ may be consistently estimated by an OLS regression of the bids on the observable characteristics $\mathbf{z}_{j,t}$, including fixed effects for each of the products $j = 1 \ldots J$. In the presence of endogenous characteristics and instruments, one could instead do an IV regression of the bids on the observables, instrumenting as necessary, and once again including product fixed effects.

We will also assume throughout what follows that the distributions of highest bids $\{G^1(\cdot|s)\}$ have no gaps in their support, so that the best responses are unique; and that the symmetric equilibrium characterized by Theorem 1 is played. This assumption can be empirically validated. Then, expanding out the bidding strategy in (4), optimal bids are given by:

$$b_{i,j,t} = x_{i,j} - r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}_i) + \boldsymbol{\gamma} \mathbf{z}_{j,t} + \xi_{j,t}. + \varepsilon_{i,j,t} \tag{6}$$

We now turn to identification. We assume that in each auction, the econometrician observes the product type being sold, all bids, and the identities of the bidders that made them. We proceed with a sequence of lemmas.

Lemma 2 (**Identification of Dynamic Parameters**). *The parameters* $\{\lambda, \tau, r, Q\}$ *are non-parametrically identified.*

Identification of Q is immediate, as the econometrician sees the sequence of products up for auction. Next, since the entry distribution is Poisson, λ is just the mean number of entrants each period. The survival rate r is identified by the probability that the bidder ever returns to the market following a loss:

$$r = \mathbb{P}(\text{bidder i is observed in any period} > t|\text{bidder i bid and lost in period } t). \tag{7}$$

Moreover, the probability that a bidder bids in the very next auction after losing is:

$$r\tau = \mathbb{P}(\text{bidder i is observed in period t + 1}|\text{bidder i bid and lost in period t}). \tag{8}$$

From this pair of equations, one can solve for the activity rate τ.

Next, we argue that the parameter vector $\boldsymbol{\gamma}$ can be identified in a first-stage regression (cf. Haile et al. (2003)). Define a bid-type for each product j by $\check{b}_{i,j} = x_{i,j} - r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}_i)$. This is the *persistent* part of the optimal bid made by bidder i on product j. Then the bidding function in (6) can be written as:

$$b_{i,j,t} = \boldsymbol{\gamma} \mathbf{z}_{j,t} + \nu_j + \xi_{j,t} + \varepsilon_{i,j,t} + u_{i,j}$$

where ν_j is the average bid-type for product j (average across bidders), and $u_{i,j} = \check{b}_{i,j} -$

ν_j is the difference between bidder i's bid-type for product j and the average.[18] Now by assumption $\mathbf{z}_{j,t}$ is independent of $\xi_{j,t}$ and $\varepsilon_{i,j,t}$. Moreover, because bidders do not select into auctions on the basis of their characteristics $\mathbf{z}_{j,t}$ (activity is random), the error $u_{i,j}$ is uncorrelated with $\mathbf{z}_{j,t}$.[19] The only term potentially correlated with $\mathbf{z}_{j,t}$ is ν_j, since certain products may tend to have certain characteristics. In view of this, $\boldsymbol{\gamma}$ is identified and may be consistently estimated by OLS regression of the bids on the observables, with product fixed effects.

For the rest of this section we therefore assume that $\boldsymbol{\gamma}$ has been identified and all bids have been "normalized" by subtracting the term $\boldsymbol{\gamma}\mathbf{z}_{j,t}$ (i.e. we will simply ignore the term $\boldsymbol{\gamma}\mathbf{z}_{j,t}$ in the identification arguments that follow). Let the full bid-type be $\check{\mathbf{b}}$ (i.e. the J vector consisting of $\check{b}_{i,j}$ for each $j = 1 \dots J$). Also let $B^1_{j,t}$ be the winning bid in each auction, and define $\check{G}^1_j$ as the distribution of $\check{B}^1_{j,t} \equiv B^1_{j,t} - \xi_{j,t}$ (i.e. the distribution of the highest bidder-specific part of the bids). Let $u(\check{\mathbf{b}})$ be defined for each j as follows:

$$u_j(\check{\mathbf{b}}) = \int \check{G}^1_j(\check{b}_j + \varepsilon) \left(\check{b}_j + \varepsilon - E_{\check{G}^1_j}[B^1_j | B^1_j < \check{b}_j + \varepsilon]\right) dF_\varepsilon(\varepsilon). \tag{9}$$

This is the expected static payoff that a type with valuation $\check{\mathbf{b}}$ would receive when bidding $\check{b}_j + \varepsilon$ on an auction for j, averaged over potential realizations of ε.[20] Now in fact each bid-type $\check{\mathbf{b}}$ has a corresponding valuation $\mathbf{x}$, and they are related through this function $u(\check{\mathbf{b}})$:

Lemma 3 (**Inversion**). *There exists an inverse bidding function $\zeta : \mathbb{R}^J \to \mathbb{R}^J$ mapping bid-types into persistent valuations:*

$$\zeta(\check{\mathbf{b}}) = \check{\mathbf{b}} + r\widetilde{Q}(I - r\widetilde{Q})^{-1}u(\check{\mathbf{b}}) \tag{10}$$

We obtain this result in three steps. First, we note that since optimal bids are valuations less continuation values (Lemma 1), valuations are bid-types (first term on RHS of (10)) plus continuation values (second term on RHS). Next, to get an expression for the continuation

[18] Notice that ν_j is a random variable whose value depends on the realization of j (i.e. which product is auctioned that period).

[19] Even with selective entry there is likely to be no correlation: since all the bidders have common preferences for $\mathbf{z}_{j,t}$, the bidders expect the value of those characteristics to be bid out, and so there is no strategic reason to enter auctions with particular values of the observables. This is true of our application, where even though we only observe bids from the top two bidders—and therefore the bid-types in auctions for high camera resolution are positively selected for their preferences for high resolution—there is still no correlation between the other observable characteristics and bid-type.

[20] The shock ξ doesn't enter because it is "bid out" by all players and thus has no effect on payoffs.

value, notice that in each period the bidder either wins, earning their valuation less payment; or loses, getting the continuation value discounted by the probability of exit. Since valuations are equal to bid-type plus continuation value, the bidder's payoff could equivalently be thought of as bid-type less payment, if they win; and the discounted continuation value with certainty:

$$v_j(\mathbf{x}) = u_j(\check{\mathbf{b}}(\mathbf{x})) + r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}) \tag{11}$$

Finally, iteratively expanding the continuation value yields a geometric series, where the expected per-period payments are given by $u(\check{\mathbf{b}})$. Multiplication by $r\widetilde{Q}(I - r\widetilde{Q})^{-1}$ sums and discounts the series.

The problem is that the inverse bid function ζ is not easily identified from the data, since from the final line of (9) the $u(\check{\mathbf{b}})$ vector can be constructed only with the knowledge of the $\check{G}_j^1$ bid distributions and the distribution $F_\varepsilon(\varepsilon)$.

To proceed, notice that G_j^1—the observed distribution of highest bids on product j—is a convolution of the distribution we want $\check{G}_j^1$ and F_ξ, the distribution of unobserved item characteristics. If we can identify F_ξ, then under some technical conditions, we can also identify $\check{G}_j^1$ by deconvolution. To identify F_ξ, we look at the joint distribution of pairs of bids placed by bidders that have just entered the market.[21] Since both the persistent part of their valuations and the idiosyncratic shocks $\varepsilon_{i,j,t}$ are mutually independent on entry, any correlation we observe in their bids must be due to unobserved heterogeneity.[22] This pins down F_ξ.[23] To identify F_ε we look instead at the joint distribution of pairs of bids of the same bidder in two successive auctions of the same product. This joint distribution of bids can be decomposed into a persistent part that induces all correlation, $\check{b}_j$, and a transient part, the sum of $\xi_{j,t}$ and $\varepsilon_{i,j,t}$.[24] This allows us to identify the distribution of the sum of $\xi_{j,t}$ and $\varepsilon_{i,j,t}$. Since F_ξ is already identified another round of deconvolution identifies F_ε.

Lemma 4 (**Deconvolution**). *Suppose (i) that the densities of unobserved characteristics and idiosyncratic shocks satisfy a thin tails condition: there exist positive constants* c_1 *and*

[21] One could allow F_ξ to depend on j, as we will in the application, by doing the deconvolution separately by bidders who have just entered the market for each product type j.

[22] Conditioning on having just entered is important, since selection on having repeatedly lost may be another source of correlation.

[23] This is similar to the argument made in Krasnokutskaya (2011) regarding the identification of unobserved heterogeneity in procurement auctions.

[24] The transient part has a different distribution in the second auction than the first, since appearing in the second auction implies selection on the idiosyncratic error in the first - we account for this in the proof.

c_2 *such that* $f(u) < c_1 e^{-c_2|u|}$ *for all* $u \in \mathbb{R}$, *and (ii) that the characteristic functions of* ε *and* ξ *have isolated real zeros. Then* $\{\check{G}_j^1\}_{j\in\mathcal{J}}$, F_ξ *and* F_ε *are all identified, and thus so is* $\zeta(\check{\mathbf{b}})$.

The thin tails condition (i) suffices to identify F_ε and F_ξ, by application of the results in Evdokimov and White (2012), who extend the deconvolution results of Kotlarski (1967). To subsequently recover $\{\check{G}_j^1\}_{j\in\mathcal{J}}$ requires deconvolution of the highest bid distributions $\{G_j^1\}_{j\in\mathcal{J}}$ with F_ξ, which is permitted by condition (ii).[25] Both conditions are weak and apply to most distributions commonly encountered including the normal, truncated normal, and uniform.

Having identified the inverse bidding function, the next step of the identification argument is arguing that the distribution of bid-types $\widetilde{\mathbf{F}}$ is identified, and thus can be inverted to recover the valuations. Let us call a bidder time series *complete* if it includes at least one bid on each product. To learn $\widetilde{\mathbf{F}}$ we will need to look at the distribution of complete bid vectors. For each bidder who bids on every product, define a bid vector $\mathbf{b}$ by selecting from their bid data the bid corresponding to the first time they bid on each product, with missing entries for any product they did not bid on. A complete bid vector has no missing entries, and from equation (6) can be written as:

$$\mathbf{b} = \check{\mathbf{b}} + \boldsymbol{\xi} + \boldsymbol{\varepsilon} \tag{12}$$

where $\check{\mathbf{b}}$ is their bid-type, $\boldsymbol{\xi}$ is the J-vector of item-specific shocks in the auctions they participated in, and $\boldsymbol{\varepsilon}$ are the J-vector of idiosyncratic shocks. Each of the random variables on the RHS is mutually independent, and the LHS is a convolution of these random variables.

This suggests an identification argument based on deconvolution. Let $\mathbf{G}$ be the distribution of complete bid vectors, and let $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{F}}^S$ be the distribution of bid-types, and the distribution of bid-types conditional on bidding at least once on every product respectively. The relationship between $\widetilde{\mathbf{F}}^S$ and $\mathbf{G}$ is given by:

$$\mathbf{G} = \widetilde{\mathbf{F}}^S * \mathbf{F}_\xi * \mathbf{F}_\varepsilon \tag{13}$$

where $*$ denotes the convolution operator, $\mathbf{F}_\xi$ is the distribution of $\boldsymbol{\xi}$ and similarly $\mathbf{F}_\varepsilon$ is the distribution of $\boldsymbol{\varepsilon}$. Since $\mathbf{G}$ is a distribution of bid vectors, it is observable. By Lemma 4, the univariate F_ε and F_ξ are both identified and have isolated real zeros, and hence so do their vector counterparts $\mathbf{F}_\xi$ and $\mathbf{F}_\varepsilon$, which are just iid samples of size J from those distributions.

[25] Let $\phi_X(\cdot)$ denote the characteristic function of a random variable X. By independence, we have $\phi_{\check{B}_j^1}(t) = \phi_{B_j^1(t)}/\phi_\xi(t)$, which is undefined whenever $\phi_\xi(t) = 0$. Still $\phi_{\check{B}_j^1}(t)$ remains integrable as long as the set of such zeros is of (Lebesgue) measure zero, so that $\check{G}_j^1$ is identified by inverse Fourier transform.

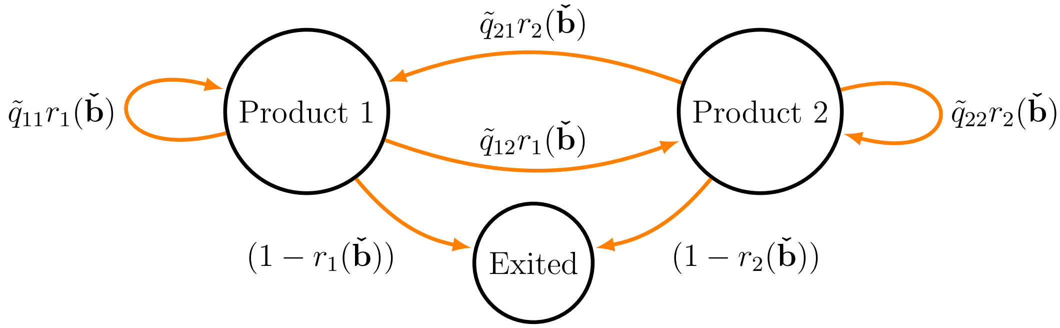


Figure 1: Bidder Path as a Markov Chain. The bidder starts by bidding on either product 1 or product 2. From there they can either exit, bid on the same product again, or bid on the other product.

This implies that $\widetilde{\mathbf{F}}^S$ can be identified by deconvolution from (13).

We are left with one last challenge. The distribution $\widetilde{\mathbf{F}}^S$ is *selected* by the event of bidding at least once on every product. This is a serious selection problem, as bidders with high valuations will disproportionately tend to win and exit early, so that $\widetilde{\mathbf{F}}^S$ is overly representative of low bid-types. We need to correct for this.

To illustrate how one might do this, consider Figure 1. It depicts the bidder's path between products as a Markov Chain. There are only two products, and correspondingly three states the bidder can be in: bidding on product 1, bidding on product 2, or exiting from the market. The first two of these states are transitory; exit is absorbing. A bidder starts off either by bidding on product 1 or product 2. With probability $r_j(\check{\mathbf{b}}) = r\left(1 - \int \check{G}_j^1(\check{b}_j + \varepsilon)dF_\varepsilon(\varepsilon)\right)$, they do not exit following the auction (i.e. do not win and then survive).[26] If they don't exit, the probabilities they bid on each of the products when next active are given by the entries $\tilde{q}_{ij}$ of the $\widetilde{Q}$ matrix defined earlier. With probability $1 - r_j(\check{\mathbf{b}})$ they exit.

Define $s(\check{\mathbf{b}})$ to be the probability that bid-type $\check{\mathbf{b}}$ generates a complete bid vector i.e. bids on both products. This event occurs if the bidder visits all the products prior to exit. This selection probability $s(\check{\mathbf{b}})$ is identified because the quantities needed to calculate it — the transition probabilities between the states of this Markov chain — are identified. For an explicit calculation, see the footnote.[27]

[26] This probability is prior to the realization of ε and the item characteristics, though the latter have no effect on the probability of winning since they are bid out.

[27] Let $p_j(\check{\mathbf{b}})$ be the probability of *only* bidding on product j conditional on entering the market in an auction for j. This probability satisfies the recursive equation: $p_j(\check{\mathbf{b}}) = (1 - r_j(\check{\mathbf{b}})) + r_j(\check{\mathbf{b}})\widetilde{Q}_{ii}p_j(\check{\mathbf{b}})$, where the first term is the probability of exit following the auction and the second is the probability of surviving and entering another auction for product j, and thus having probability $p_j(\check{\mathbf{b}})$ of the event again. Re-arranging:

There is one complication: this argument assumes that the probability of winning for any particular bidder is $\check{G}^1_j(\check{b}_j)$, *independent of the bidder's past bids and losses*, i.e. independent of their personal history. We formalize this assumption as follows. Let $h \in \mathcal{H}$ be a bidder history: a set of tuples, one tuple for each time period in which the bidder was active, where each tuple consists of the time period they were active in, what product they bid on in that time period, what they bid, and whether they won or lost.[28] Bidder histories are observable.

Assumption 2 (Independence of competing bids). *The distribution of highest rival bids is independent of bidder history i.e.* $G^1_j(b|h) = G^1_j(b)\ \forall j\ \forall h \in \mathcal{H}$.

This is an empirical counterpart to Assumption 1. Assumption 1 says that bidders don't condition on their own personal experience in forming beliefs about rivals. Assumption 2 says that doing so would not be informative. This holds when the activity rate τ is small, since then the number of auctions between each time a bidder participates is large.[29]

What this assumption buys us is the Markov formulation of the selection problem, as illustrated for the two product case in Figure 1. Now the transition matrix between the states of the Markov chain depends on the matrix $\widetilde{Q}$ and functions $\{r_j(\check{\mathbf{b}})\}$. By Lemma 2 the matrix $\widetilde{Q}$ and survival probability r are identified. By Lemma 4 the distributions $\check{G}^1_j$ and F_ε are identified, which together imply the functions $\{r_j(\check{\mathbf{b}})\}$ are identified. It follows that the selection probabilities $s(\check{\mathbf{b}})$ are identified (see Lemma 8 in the appendix for an iterative method for calculating them).

Assumption 2 is not necessary for identification, but without it, the transition probabilities are history dependent and involve terms like $\check{G}^1_j(b|h)$. These are observable in the data but occur infrequently.

Next, we can re-weight $\widetilde{\mathbf{F}}^S$ according to the selection probability to get $\widetilde{\mathbf{F}}$; and then apply the bid inversion of Lemma 3 to get the distribution of types $\mathbf{F}$. Summarizing:

Theorem 2 (Non-parametric identification). *Let Assumption 2 and the conditions of Lemma 4 hold. Then* $\mathbf{F}$*,* F_ξ *and* F_ε *are all non-parametrically identified.*

$p_j(\check{\mathbf{b}}) = \frac{(1-r_j(\check{\mathbf{b}}))}{1-r_j(\check{\mathbf{b}})\widetilde{Q}_{ii}}$. Let π_j be the probability of entering the market when product j is auctioned. Now for this two-product example, if the bidder doesn't get stuck bidding on one product they will bid on both products, thus the probability of a complete bid is: $s(\check{\mathbf{b}}) = 1 - \pi_1 p_1(\check{\mathbf{b}}) - \pi_2 p_2(\check{\mathbf{b}})$.

[28] For example, one entry in the history might be "bid b on product 1 in period 12, lost".

[29] Lemma 5 in the appendix shows that the market converges back to the ergodic measure at geometric rate from any state. In our empirical application, if we assume that all bids are observed, then on average 305 auctions pass between bids by the same bidder, implying $\tau = 0.00327$.

We close the section by examining a slight modification: a random coefficients demand system, where products are described by observable characteristics and bidders have preferences over characteristics rather than products.

$$u_{i,j,t} = \mathbf{x}_j \boldsymbol{\alpha}_i + \boldsymbol{\gamma} \mathbf{z}_{j,t} + \xi_{j,t} + \varepsilon_{i,j,t} \tag{14}$$

where now $\mathbf{x}_j$ is the jth row of a $J \times K$ matrix X of product characteristics (where $K \leq J$ is the number of characteristics) and $\boldsymbol{\alpha}_i$ is the bidder's type, a $K \times 1$ persistent vector of random coefficients drawn from $\mathbf{F}_\alpha$ on entry (the definitions of all other expressions remain the same). This is similar to the demand system in (5) except that the persistent valuations for each product have been replaced with persistent preferences (random coefficients) over a lower-dimensional set of product characteristics X (which are different from the additional item characteristics $\mathbf{z}_{j,t}$). This suggests that we could go ahead and learn the distribution of valuations $\mathbf{F}$ and then project down onto X to learn the distribution of random coefficients $\mathbf{F}_\alpha$. Under a rank condition on X, this is possible:

Corollary 1 (**Identification with Random Coefficients**). *Let Assumption 2 and the conditions of Lemma 4 hold. In addition, assume that the design matrix X has full rank. Then $\mathbf{F}_\alpha$, F_ξ and F_ε are all non-parametrically identified.*

5 Empirical Application

We apply our identification result to estimate demand in the auction market for compact cameras on eBay.com. Most consumers purchase only a single camera, so the unit demand assumption seems reasonable. This choice of application is motivated by our ambition to show how one might bridge the gap from identification to estimation in a particularly difficult setting with data limitations, unobserved heterogeneity, and random preferences; i.e., to develop an estimator that takes advantage of much of the flexibility of the identification result.

In our empirical model, compact cameras are measurably differentiated in a salient characteristic—namely resolution—which consumers may value differently, and therefore we estimate a random coefficients variation of our model, with random consumer preferences for resolution. We also allow for common preferences over other observable characteristics $\mathbf{z}_{j,t}$, and a scalar item-specific unobservable $\xi_{j,t}$. We choose not to allow for an idiosyncratic error $\varepsilon_{i,j,t}$ for two reasons. First, because we will restrict attention only to the top two bids in each auction, there's no need for the flexibility of a model with both item unobservables and idiosyncratic errors to fit the data.[30] Second, the identification theory developed above does not extend to the case with both selected data and idiosyncratic shocks; selection on being in the top two bids induces correlation between the idiosyncratic errors of observed bidders, violating an independence condition on which our deconvolution arguments rest. To summarize, we believe this is the right modeling choice because we are more concerned with the behavioral plausibility of the interpretation of the bids as bid-types, which the restriction to the top two bids affords us, than we are with unnecessary flexibility. Thus we adopt a "pure characteristics" specification (Berry and Pakes, 2007).

As in all empirical work, modeling choices come with tradeoffs. Here we have taken the supply process that determines which camera is auctioned next as exogenous, and in addition, we have assumed that sellers do not use reserve prices strategically (there is little evidence that they do in the data). These are limitations of our approach.

We put our estimates to use by documenting the effects of dynamic bidder behavior on optimal reserve prices. As we show, if sellers have the power to commit to persistent reserves over a long time horizon, as a seller controlling a large share of auctions might, this diminishes the bidders' option value of losing, which means higher bids. In our simulations, this effect can as much as double or triple the profit gains to using reserve prices, offering insight into how market power might manifest in an auction marketplace with large sellers.

5.1 Data and Market Overview

The eBay Marketplace. eBay is widely considered to be the world leader in online auctions. Various elements of its platform design, such as the use of proxy bidding agents, feedback scores and "buy-it-now" offers have been widely copied. At any time, eBay hosts a large number of items from a variety of sellers. Buyers can browse these, either by navigating through categories delineated by the site, or by directly searching for key phrases. For example, a search for "digital compact camera" will typically bring up thousands of listings.

[30] When we observe all bids, we could see the same two bidders in successive auctions for the same object, and without an idiosyncratic shock, the theory would find it impossible to rationalize any change in the difference of their two bids $b_{i,j,t} - b_{i',j,t}$ across the two auctions. But once we restrict the data to only the top two bids, this potential falsification of the theory can never arise, since one of those bidders—the winner—exits with certainty.

Unsurprisingly, there is substantial heterogeneity in the cameras offered, in terms of brand, resolution, zoom, and accessories (to name some of the most salient features).

Restricting to auctions yields a list of items, ordered by time until auction end.[31] These auctions all end at different times, so bidders face a set of *sequential* auctions. Bidders have the option to bid using eBay's proprietary "proxy bidding" system, however much prior work has found that most bids are placed on the last day of the auction (Roth and Ockenfels, 2002; Lewis, 2007; Backus et al., 2015).[32] The combination of late and proxy bidding suggest that eBay's auction market is well approximated as a sequence of second-price sealed bid auctions, so our model can be applied to this setting.

Data. We purchased a dataset concerning all sales of digital compact cameras over a 2-year period from TeraPeak, a data analytics company (Terapeak, 2009). The data includes attributes of the camera auctioned (resolution, zoom, brand, product name, bundling of a tripod, extra battery, etc), attributes of the listing (starting price, secret reserve, listing title), and the outcome of the auction. Each listing may be associated with several bids, all of which we observe—including the highest bid, which is not visible on the website and typically unavailable in "scraped" auction datasets from the platform. Market participants are persistent in our dataset—as in our model—and we observe their attributes (feedback, location) and construct measures of experience and activity from observed behavior.

We work with a subset of the data, consisting only of *new* compact cameras, sold in the 3-month period between February 5th and May 6th of 2007. We restrict attention to new cameras to limit the influence of unobserved heterogeneity, though as we will see, this is still a substantial problem. We analyze this particular time period because supply was relatively

[31] Some items are offered at "buy-it-now," i.e. fixed, prices. As Einav et al. (2016) have documented, over 60% of items on eBay are now at fixed prices (though auctions were more common at the time our data was collected). Backus et al. (2019) and Backus et al. (2020) study a subset of these items are "best offer"-enabled, and therefore subject to alternating sequential offers bargaining. We will ignore the presence of both fixed-price (including rarer auction-buy-it-now variants), and negotiated markets in what follows, effectively ignoring substitution opportunities between the two markets in favor of focusing on substitution within the auction market.

[32] In the proxy bidding system, bidders enter the maximum they are willing to pay for the item, and then eBay's proxy bidding system will bid up from the current standing price in standardized increments on their behalf until either their bid is the highest yet entered in the system, or an additional increment would take them over their maximum. For example, if bidder A enters a bid of $8000 on a camera where the standing price is $6000 and the highest bid placed in the system by a rival is $7000, then the system will update the standing price to $7100 ($7000 + $100 increment), and will record this bidder as the currently high bidder. Under unit demand high bidders become "committed" to the auctions they enter, in the sense that if they bid in another auction, there is a risk of winning a second object they don't need.

stable over those 3 months, so the stationarity assumptions implicit in our calculations of the continuation values are reasonable. We pick cameras with the most common resolution levels: those with (rounded) resolution between 5 megapixels and 10 megapixels (MP). We clean the data by excluding auctions with missing data, potential shill bidding, outlying bids and auctions that include a buy-it-now option. See Online Appendix Section B.2 for further details on sample construction, and B.3 for summary statistics of the dataset.

Although, as we have noted, the eBay environment is well-approximated as a series of second-price auctions, that approximation has limitations (Haile and Tamer, 2003; Song, 2004).

For bidders who are outside of the top two, we may not observe the highest amount they would have been willing to bid (for instance, if they arrive after the standing price of the auction already exceeds the bid they intended to place). Therefore we restrict attention to the top two bids. This introduces selection that we will have to correct for.

A first look at the dataset yields four stylized facts that guide the design of the empirical model in what follows. We document each of these claims more carefully in Online Appendix Section B.3. First, bidders do substitute across auctions for different camera types. This is important for identifying cross-elasticities. Second, consistent with our theoretical model, the vast majority of sellers set non-binding starting prices and do not use secret reserve prices. This is demonstrably suboptimal, but setting an optimal reserve is a difficult problem in practice, and the platform encourages sellers to set low starting prices.[33] Third, bidders have substantial option value. We observe substantial price fluctuations over time, as well as substantial delay in bidder re-arrivals. Fourth and finally, there is substantial unobserved product heterogeneity, as the observables explain little of the fluctuations. Therefore we endeavor a model that can accommodate these salient features of the data.[34]

5.2 Estimation

We construct a demand system in which consumers obtain value from the purchase of a single camera, a value which has an idiosyncratic component and a common component.

[33] See, e.g., https://www.ebay.com/help/selling/selling/pricing-items?id=4133. Accessed July 23, 2020.

[34] It is also worth noting that many of the assumptions and implications of the model are testable. For instance, the model implies a constant re-arrival rate of bidders, and stationary bidding strategies. In exploratory work we found no strong evidence of violations.

Formally, bidder i's valuation for product j offered in auction t is given by:

$$u_{i,j,t} = \underbrace{\alpha_{i,c} + \alpha_{i,r}\text{res}_j}_{\text{idiosyncratic}} + \underbrace{\boldsymbol{\gamma}\mathbf{z}_{j,t} + \xi_{j,t}}_{\text{common}}. \tag{15}$$

Here $\mathbf{z}_{j,t}$ captures observed characteristics, while $\xi_{j,t}$ captures unobserved heterogeneity that is observable to all bidders, but not the econometrician. We assume that $\xi_{j,t}$ is distributed normally with mean zero and variance $\sigma_{\xi,j}$ that varies freely with the resolution type of the camera. Note that this is a "pure characteristics" specification (Berry and Pakes, 2007), and does not include an idiosyncratic error term $\varepsilon_{i,j,t}$.

Bidder i's type in our model, α_i, is a double: their fixed utility draw for obtaining any camera on eBay ($\alpha_{i,c}$) as well as an idiosyncratic preference shock for resolution ($\alpha_{i,r}$).[35] The bidder type α_i is drawn, iid upon bidder entry, from the distribution $\mathbf{F}_\alpha$, which is our main estimation target. We assume mutual independence of $(\boldsymbol{\alpha}_i, \mathbf{z}_{j,t}, \xi_{j,t})$. On the supply side, there are five products: 5, 6, 7, 8, and 10 megapixel cameras, so $J = 5$. We make the simplifying assumption that the distribution of arriving auctions is iid multinomial over the product space with a J-vector of probability weights $\boldsymbol{\pi}$ (note that this implies that $\tilde{Q} = Q$ and that Q is a square matrix in which every row is equal to $\boldsymbol{\pi}$).[36]

First-Stage Estimation. We follow the constructive identification argument of Lemma 2 to recover estimates of the supply process ($\boldsymbol{\pi}$), the exit rate (r), the desired bid normalization ($\boldsymbol{\gamma}$), as well as the distributions of bid-types ($\{\check{G}^1_j, \check{G}^2_j\}_{j\in\mathcal{J}}$). $\hat{\boldsymbol{\pi}}$ is given by the empirical frequencies; $\hat{r}$ is estimated by maximum likelihood using a censored geometric process to characterize the survival rate of bidders, censored both if they win or if the last bid is in the final six weeks of our dataset; $\hat{\boldsymbol{\gamma}}$ is obtained by FGLS,[37] $\hat{\sigma}_{\xi,j}$ is the standard deviation of within bidder-product differences in bids for product type j, and finally, we fit a Gamma distribution to $\{\check{G}^1_j, \check{G}^2_j\}_{j\in\mathcal{J}}$, choosing the shape and scale parameters to match the mean

[35] Equivalently, we could interpret $\alpha_{i,c}$ as describing variation in the value a bidder's outside option. As one anonymous referee observed, this could be decomposed if we had also had data on bidder behavior outside of eBay.

[36] We have made these choices to fit the salient features of the data, however, we acknowledge there are other models that may be of interest, depending on the application. See Online Appendix Section B.4 for a fuller discussion on this point. In particular, we consider the incorporation of aggregate fluctuations and public states.

[37] We use FGLS, allowing the error to be correlated with resolution, because idiosyncratic types α_r introduce heteroskedasticity, rendering OLS (which we used in a prior version) inefficient. We thank our editor Aureo de Paula for suggesting this.

and variance of the deconvoluted distribution. As controls in the first-stage regressions we include brand fixed effects, listing attributes including shipping options, seller feedback, and optional listing features (e.g., sellers may pay a fee for their results to be highlighted in search results), as well as a set of dummies for resolution, optical zoom, and digital zoom levels. See Online Appendix Section B.5 for further details, as well as precise values in Table B.3. Also, at this point, it is possible to estimate consumer surplus in each auction, which is substantially larger than would be predicted by the static model. We discuss this in Online Appendix Section B.8.

Estimation of Demand. At this stage, we have estimates of $\boldsymbol{\pi}$, r and the distributions $\{\check{G}_j^1, \check{G}_j^2\}_{j\in\mathcal{J}}$ of first and second order statistics, after adjusting for observed and unobserved heterogeneity. We also have a dataset of normalized bids that were themselves first or second highest bids in auctions, which, in our model, are equal to valuation less continuation value. In principle, we could continue by applying our nonparametric identification strategy from Theorem 2 directly in estimation, for bidders who are observed in the top two bids on at least two different products (since the random coefficient is two-dimensional, two different products suffices for identification). But there are 232 such bidders in our estimation sample, and this makes the necessary deconvolution analysis unattractive in view of the slow convergence properties of such estimators (Carroll and Hall, 2004). Moreover, we would like to also use the 13,392 bidders in our estimation sample who appear in the top two bids on only one camera type.

So instead we take a parametric approach, assuming that α_c and α_r are distributed $N(\mu_c, \sigma_c)$ and $N(\mu_r, \sigma_r)$, respectively. Therefore $\boldsymbol{\theta} \equiv \{\mu_c, \sigma_c, \mu_r, \sigma_r\}$ makes up the set of parameters of the demand system we ultimately hope to recover. Under our parametric assumptions, we have the following likelihood of each observation:

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{b}_i) = \int \underbrace{\mathbb{P}\{\mathcal{B}_i|\tilde{\beta}(\boldsymbol{\alpha})\}}_{\text{selection probability}} \left(\underbrace{\prod_{j\in\{\mathcal{B}_i\}} f_{\xi,j}(b_{i,j} - \tilde{\beta}_j(\boldsymbol{\alpha}))}_{\text{unobserved heterogeneity}} \right) d\,\underbrace{\mathbf{F}(\boldsymbol{\alpha}|\boldsymbol{\theta})}_{\text{demand}}. \tag{16}$$

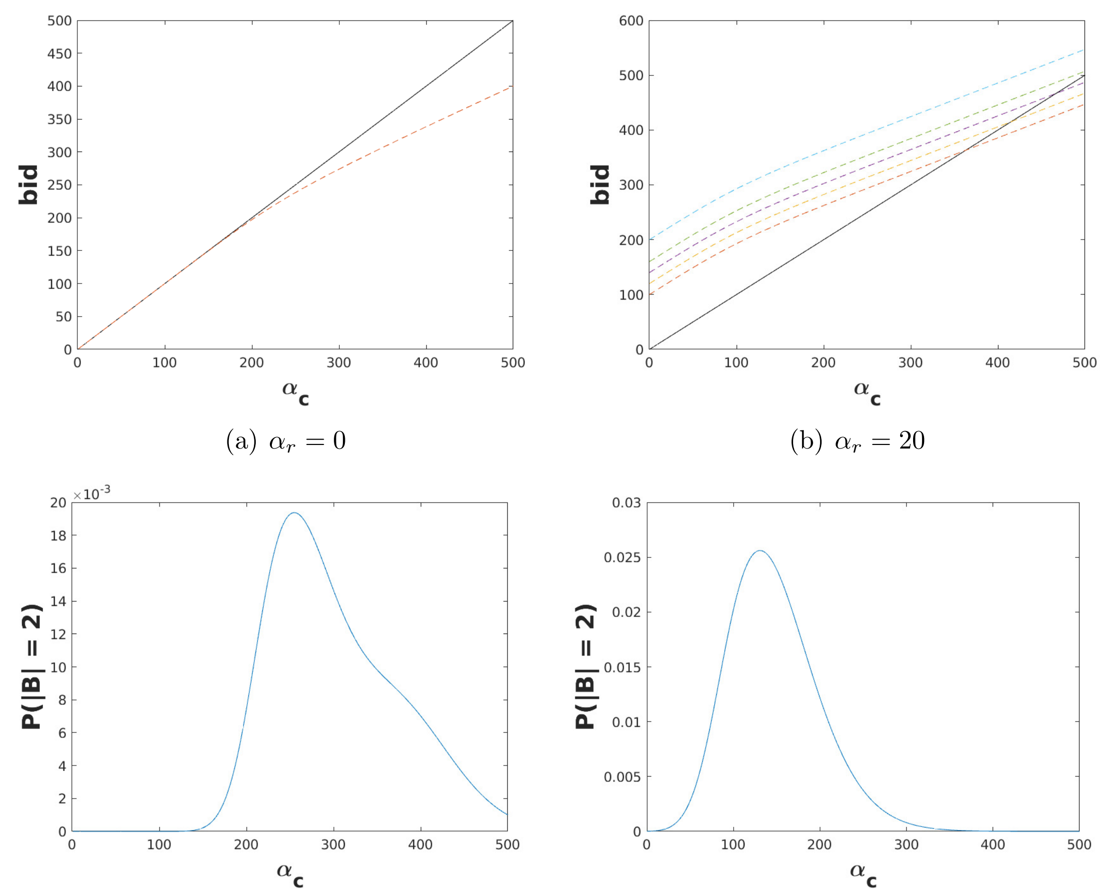
where $\mathbf{b}_i$ is a vector of normalized bids by i; $\mathcal{B}_i$ is the set of products bidder i bids on; $\beta(\boldsymbol{\alpha})$ is type $\boldsymbol{\alpha}$'s bid-type (i.e. what they would bid on each product in the absence of unobserved heterogeneity); $f_{\xi,j}$ is the density of $\xi_{j,t}$, assumed normal with mean zero and variance $\sigma^2_{\xi,j}$ and $F(\boldsymbol{\alpha}|\boldsymbol{\theta})$ is the distribution of $\boldsymbol{\alpha}$ at parameter vector $\boldsymbol{\theta}$.

This likelihood function has three components: the first, $\mathbb{P}\{\mathcal{B}_i|\tilde{\beta}(\boldsymbol{\alpha})\}$ is the selection probability; the likelihood that a bidder of type $\boldsymbol{\alpha}$ is observed in a subset of the product space $\mathcal{B}_i$. The second component of the likelihood function is the deviation of the normalized bid $\mathbf{b}$ from the predicted bid $\tilde{\beta}(\boldsymbol{\alpha})$ on the components $j \in \mathcal{B}_i$, which can be accounted for by unobserved heterogeneity. Finally, we integrate with respect to the type distribution $F(\boldsymbol{\alpha}|\boldsymbol{\theta})$, the only point at which the parameter vector $\boldsymbol{\theta}$ enters.

Estimating bid functions and selection probabilities. In order to compute the likelihood of any observation at a parameter vector $\boldsymbol{\theta}$, we will need to compute the optimal bidding function $\tilde{\beta}(\boldsymbol{\alpha})$ and the selection probability $\mathbb{P}\{\mathcal{B}_i|\beta(\boldsymbol{\alpha})\}$. See Online Appendix Section B.6 for details on this computation, which follows closely the identification arguments of Section 4. There we also show how we modify the selection correction to account for the fact that we only observe the top two bids.

Figure 2 illustrates the resulting bid functions and example selection probabilities when these procedures are applied to our data. The selection probability is for the event that a bidder bids on exactly two different products ($|\mathcal{B}| = 2$). Panel (a) shows the bid function for a bidder with $\alpha_r = 0$, i.e. a bidder who places no value on resolution, and consequently bids the same on every product. Their bid function "peels away" from the 45-degree line. This makes intuitive sense: as α_c increases, their continuation value increases as well, causing them to shade their bids. Panel (b) repeats this exercise for the case where $\alpha_r = 20$. They now bid differently on each product, with the five dotted lines showing their bids on the lowest resolution camera (bottom line) up to the top resolution camera (top line), as α_c varies. The shape is largely preserved, except that now there is a positive intercept because valuations are positive even with $\alpha_c = 0$.

In panels (c) and (d) we show the selection probabilities for these two types of bidders ($\alpha_r = 0$ in (c) and $\alpha_r = 20$ in (d)) as α_c varies. The "hump-shape" arises because the probability of $|\mathcal{B}| = 2$ is increasing at first as the likelihood that the bidder is ever first or second rises, but later declines as the probability that they win their first auction and exit before bidding again goes to one.



(a) $\alpha_r = 0$

(b) $\alpha_r = 20$

(c) $\alpha_r = 0$

(d) $\alpha_r = 20$

Figure 2: Estimated Bid Functions and Selection Probabilities. This figure presents bid functions and selection probabilities as a function of α_c, holding fixed α_r at two levels: 0 and 20.

Optimization. See Online Appendix Section B.7 for a detailed discussion of how we solve the ML optimization problem. We show that, because the parameter vector $\boldsymbol{\theta}$ enters the likelihood (16) only through the distribution of random coefficients $\mathbf{F}(\boldsymbol{\alpha}|\boldsymbol{\theta})$, this problem is particularly well-suited to importance sampling (Kloek and van Dijk, 1978; Ackerberg, 2009), which yields dramatic computational gains in estimation.

Table 1: Demand System Estimates

	α_c	α_r
Mean	61.2106	27.6985
	(1.8244)	(0.2693)
Standard Deviation	40.0459	5.3430
	(0.9061)	(0.1203)

Notes: This table presents estimates for $\{\mu_c, \sigma_c, \mu_r, \sigma_r\}$, the parameters governing the distribution of random coefficient preferences for compact cameras according to equation (15).

5.3 Results

Estimates from this ML exercise are presented in Table 1.[38] Our results suggest that camera resolution is the main determinant of consumer utility, but that there is substantial heterogeneity in how consumers value this attribute. This variation in the two components of utility for a camera predicts that the winner of an auction will depend not merely on the valuations of the bidders but also which camera is up for auction. A bidder with a high α_c and low α_r will bid more aggressively in an auction for a 5MP camera, while a bidder with a low α_c and high α_r will bid more aggressively in an auction for a 10MP camera. Recall that we have estimated $\mathbf{F}$, whereas the observed bids in the data (i.e., first- or second-highest bids) will tend to come from bidders sampled the right tail of the distribution.

This substantive heterogeneity in persistent tastes means that bidders will also vary in their continuation value, and therefore the amount they shade their bids. In Online Appendix Section B.8 we estimate an average continuation value for winning bidders of \$15.08. For comparison, we find that $\sigma_{\xi,j}$ varies between 7.29 and 13.76 (see Table B-4 of the Online Appendix).

5.4 Counterfactual Reserve Prices

With our estimates of the demand system in hand, we consider how a large seller with market power, who controls approximately a third of the market, should optimally set reserves to

[38]The ML estimator yields standard errors that treats the first-stage parameters as known, and will therefore be too small (Murphy and Topel, 2002; Karaca-Mandic and Train, 2003). Following the suggestion of Petrin and Train (2003), we augment these standard errors by bootstrapping the first-stage estimates, clustered by bidder, and re-estimating the model for each bootstrap.

maximize profits. We are interested in this scenario because, for lack of models that allow substitution between auctions, little is understood about how market power manifests in auction marketplaces.[39] In particular, how does the position of a one-time seller, who can only set a reserve for one auction (a *transient* reserve), differ from that of a large seller, who is able to commit to setting a high reserve both today and in their future auctions (a *persistent* reserve)? Our framework is uniquely situated to answer this question. A transient reserve price does not affect the stationary distribution of bids, the continuation value of bidders, or their optimal bidding function. However, following Myerson (1981), it may raise revenue by increasing the expected payment when it binds, at the cost of fewer sales when no bid exceeds the reserve. A persistent reserve price policy has the same effect, but in addition, it will also depress bidders' continuation values by raising expected prices in the future, which in turn pushes out the bid function and raises revenues still more.[40] It is the market power of the large seller that gives them the ability to shift bidder expectations, and therefore benefit more from a reserve policy.

In order to simulate reserve price policies and identify an optimum, we assume that the large seller sells all of the 7 megapixel cameras.[41] We further assume that the seller lists each of the cameras once, and if they do not sell in that auction sells them elsewhere receiving some fixed outside option payoff.[42] This outside sales option can be interpreted as the opportunity cost of selling at auction, and we will refer to it as the seller's cost in what follows. Accordingly, we will define the seller's per auction expected profit as the expectation of the difference between the winning bidder's payment and the seller's cost, times an indicator for the item selling. We will refer to a reserve policy as optimal if it maximizes this quantity.

As a baseline, we first simulate the expected sales prices for each kind of product, assuming no reserves. Then we allow the seller to set reserves that are parameterized by a single parameter f, the fraction of the baseline sales price (e.g. $f = 0.8$ implies that the seller

[39] Notable exceptions are McAfee (1993) and Peters and Serevinov (1997), however, they focus on the competitive case, predicting that reserve prices in equilibrium will be equal to sellers' costs.

[40] There is a second-order effect as well: the persistent reserve price policy will change the ergodic distribution of bidders—on average there will be more bidders per auction, with more mass just beneath the reserve point cutoff. This, however, will not raise revenues for that particular product (although it may for others).

[41] Throughout, we assume that sellers of other cameras (5, 6, 8, and 10 megapixel cameras) do not use reserve prices, consistent with the low starting prices we see in the data. In table B-1 of the Online Appendix, we see that the mean starting price across auctions is 27.63, with a standard deviation of 64.36. For 7 megapixel cameras specifically, this is \$19.90, with a standard deviation of \$54.22. See Online Appendix Section B.3 for further discussion of reserves in the data.

[42] For simplicity, we do not allow sellers the possibility of re-listing items or scheduling their auctions. These would be interesting directions for future research.

sets a reserve price at 80% of the baseline sales price).[43] We consider two different kinds of sellers: a one-time seller, who sets an optimal reserve for a single auction (the transient case); and a large seller, who commits to a single persistent reserve price for all of their auctions. The optimal reserve in the transient case is exactly that in the theory developed by Myerson (1981), and we can directly compute it from the distribution of the winning bid.

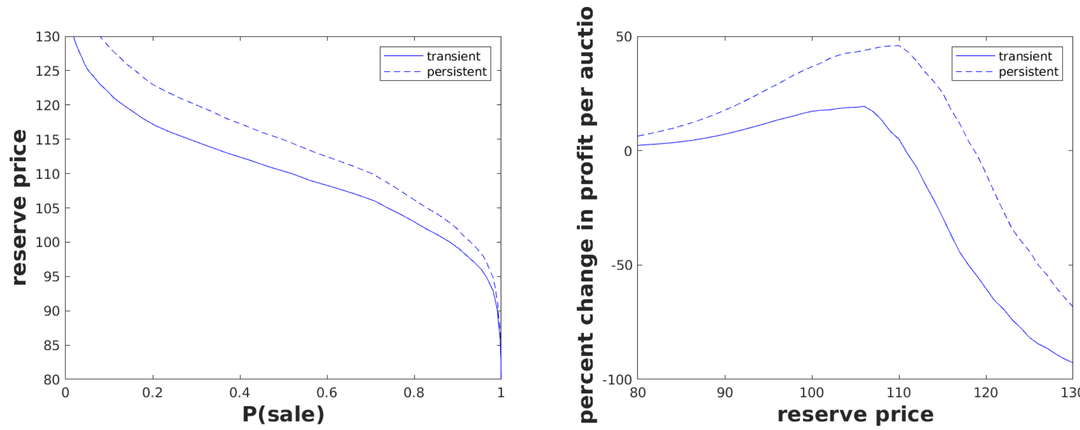
In the persistent case, the presence of a reserve price changes the continuation values and thus the bidding functions themselves. So we need to compute a fixed point iterating between the computation of the bidding function conditional on a distribution of highest rival bids and computation of the distribution of the highest rival bids induced by that bidding function. To do this, we start from the estimated bid distribution under zero reserves, derive the bid function, simulate a long sequence of auctions, estimate the parameters of the distribution of the highest competing bids from the simulated auctions, re-derive the bid function, and repeat this process until it converges. When we simulate auctions, we follow the Markovian logic of the rest of the paper, drawing a good from the supply distribution, new entrants, active bidders from the set of incumbents, optimal bids, and exit. We fix the entry process to be $E = 2 + X$, where X is Poisson with $\lambda = 1.9$, so that the mean of 3.9 matches the empirical ratio of observed bidders to auctions in our dataset, and there are always at least two bidders.[44]

While computing an optimal bid in the transient case is straightforward, computing profits in the persistent case is challenging due to the need to reconstruct the distribution of highest rival bids, as discussed above. So as a practical matter, we find an approximation to the optimal reserve in both cases by grid search over the parameter f, where the grid has interval 0.01. We repeat this optimization exercise over a range of potential values of seller costs (again expressed as fractions of the baseline sales price) to see how optimal reserves vary with costs.

The effects of the persistent reserve policy are shown in Panel (a) of Figure 3. Here we plot the probability of sale against the reserve price for both transient and persistent reserve price policies, and we see that the persistent reserve price policy has rotated the demand curve outwards, particularly for higher reserve prices that are more likely to be pivotal. This rotation of the demand curve implies that the seller will have an incentive to set higher reserve

[43] This is without loss of generality within the class of scalar reserve policies; any absolute reserve could be re-interpreted as a fraction of the baseline sales price. What it does rule out is more complex reserve policies that depend on the current state of the market.

[44] We do this because we almost never observe fewer than two bidders in the data.



(a) (b)

Figure 3: Demand and Profit Per Auction for 7 Megapixel Cameras. Panel (a) of this figure presents simulated demand functions for 7 megapixel cameras under transient (solid line) and persistent (dashed line) reserve price policies using the estimates of Table 1. Panel (b) depicts the corresponding profit function under an assumption that the seller's costs are equal to 90% of the baseline price.

prices, and will achieve higher profits. In Panel (b) of Figure 3 we plot the corresponding profits for different reserves under an assumption that the seller's costs (i.e. outside option) are equal to 90% of the baseline sale price. We see that profits are consistently higher under the persistent reserve price policy. Note that the profits achieved by a transient reserve are available to any seller, regardless of size. But it is only by virtue of the large market share of the seller that, through a persistent reserve price, they are able to depress bidders' continuation values, raise bids, and thereby achieve higher profits. The profit difference is the market power of the large seller.

In Table 2 we consider the optimal reserve price problem of the seller for a range of seller costs. Using grid search over our simulated outcomes, we identified optimal reserves under both the transient reserve price policy as well as the persistent reserve price policy. We find that for low costs, the use of persistent reserves is of little consequence for the seller. However, for high costs, persistent reserves can have a substantial effect on seller profits, raising them by up to 45.99% over the no-reserve baseline in the range that we consider. The percentage change in profits for persistent reserves over the range is consistently double to triple the size of that predicted for transient reserves.

Stepping back a moment, Figure 3 documented substantial screening of low-value bidders from the use of a transient reserve; at the optimal transient reserve of $106 when seller costs are 90% of ASP, 26.07% percent of listings fail to sell. This is profitable because the

Table 2: Optimal Reserve Prices

	Seller Costs as a Percent of ASP									
	0	10	20	30	40	50	60	70	80	90
Transient										
Optimal Reserve	83	83	83	85	86	90	90	93	96	106
Pct. Change in Profits	0.24	0.27	0.32	0.38	0.49	0.74	1.14	2.09	4.40	19.39
Pct. Change in Cons. Surplus	-1.44	-1.44	-1.44	-1.78	-2.01	-3.33	-3.33	-4.94	-7.54	-31.19
Persistent										
Optimal Reserve	91	91	93	94	95	95	95	98	102	110
Pct. Change in Profits	1.22	1.45	1.75	2.22	2.88	3.85	5.30	8.28	15.03	45.95
Pct. Change in Cons. Surplus	-8.36	-8.36	-9.73	-10.48	-11.22	-11.22	-11.22	-13.09	-14.92	-21.11

Notes: This table presents optimal reserves for the large (all 7mp cameras) seller, under transient and persistent reserve price policies, as well as percentage changes in profits induced by those policies, relative to the no-reserve baseline.

reserve price intensifies competition: if the listing does sell, there is some chance that the reserve will determine the sale price. That intensification of competition is amplified when the seller commits to a persistent reserve price for all 7MP camera auctions. Bidders shade less, and therefore bid more, because they expect less surplus from future auctions. So, if the seller were to set a persistent reserve at the same level ($106) we would see *less* screening of low-value bidders: only 17.15% percent of listings would fail to sell (see Figure 3). But optimally, the seller would set a higher persistent reserve of $109, resulting in a higher level of screening (24.27% of listings fail). This is reflected in the effects on consumer surplus displayed in the bottom row of Table 2.

Persistent reserve prices are more effective at transferring rents from consumers to producers because they elicit higher bids. This highlights the importance of dynamics for understanding reserve prices, and the leverage that a large seller in an auction marketplace possesses. Although the level of the optimal reserve changes only a little, the incentives to implement it are dramatically larger when the seller can commit to future reserves, eroding the buyers' continuation values, and therefore raising their optimal bids. This is, to the best of our knowledge, a novel observation in the literature on reserve prices.[45]

[45] Hickman et al. (2017), who like us find little use of reserve prices among eBay sellers, argue that the reason for this is that the gains from optimal reserves are quite small—on the order of pennies, in their application to laptops on eBay. While we find larger gains to transient reserves, on the order of dollars, we view our result as qualitatively consistent. However we also find that the gains are much larger for sellers with market power, which is consistent with casual empiricism: reserve prices tend to be used by long-run players with market power, e.g. procurement agencies and advertising platforms.

A limitation of our analysis is that we take seller participation as exogenous. This is consistent with our focus on demand, however, a fuller model that captured both sides of the market might incorporate the possibility that sellers who fail to sell—because they have set a reserve price—re-enter the market, thereby endogenizing their outside option.

6 Conclusion

This paper offers a flexible demand system for the study of auction markets *as markets*. We developed a notion of equilibrium in such markets and proved its existence, and in turn, were able to characterize the conditions under which bidders' actions can be inverted to infer their private type. While this is sufficient for identification if we treat auctions as independent, isolated draws from a distribution, in an auction marketplace we also need to account for the selection of bidders into the observed and identified set. Subject to the constraints of non-participation, we are able to partially identify the distribution of types by explicitly modeling this selection as a function of observable equilibrium objects.

This selection correction turns out to be an important source of flexibility in the model, allowing us to accommodate standard limitations of auction data, such as only observing the two highest bids.

A second important source of flexibility in the model is that we have made all the substitution between products occur *inter-temporally*. High-dimensional preferences are thus projected down to a valuation for the current product and a continuation value which, thanks to the Markov dynamics of the game, we can model in a straightforward way. These Markov dynamics also give us the flexibility to extend the state space of the game to incorporate arbitrary public signals about the state of the market on which bidder behavior may depend. Of course, this comes at a cost, in that it restricts the way we think about bidder search and entry. A fruitful direction for future work, given a compelling model of consumer search, would be to use those search and entry choices to learn more about bidder preferences.

We illustrated much of this flexibility in an application to the auction market for compact cameras on eBay. There we estimated the distribution of types for a utility function that looked a lot like the kind you might estimate in a fixed-price market using standard methods in industrial organization. We applied the estimates to simulate a counterfactual world in which a large seller, controlling approximately one-third of the market, could set reserve prices (expressed as a fraction of the baseline expected sales price) for all products, thereby depressing continuation values and raising current bids. We found that this effect can be quite large. It rotates the demand curve outwards and raises profits; for example, if the seller's costs are 90% of the baseline, then the profits of the seller are 45.99% higher under optimal persistent reserves rather than no reserve, versus 19.39% under an optimal transient reserve. This difference stems from the market power of the large seller.

Our contribution—the development of a demand system for auction markets—is meant to mirror similar work in fixed-price markets, and to a similar end: the structural estimation of demand allows us to do counterfactuals of both private and public interest. There remain several open directions for future work. This framework could potentially extend naturally to multi-unit demand, in which bidders may shade against the opportunity cost of moving further down their marginal utility curve. This is an important direction for the modeling of advertising auctions and treasury auctions. There are also unmet challenges in the modeling of substitution across auctions within-period rather than inter-temporally. Auction markets are a pervasive mechanism for the allocation of goods and services, and there remains much work to be done to understand competition between and substitution among them.

7 Data Availability Statement

The data underlying this article were provided under license by Terapeak, a company that has since been acquired by eBay. The code that generated the results is available at https://doi.org/10.5281/zenodo.10416565.

References

Ackerberg, D. A. (2009): "A new use of importance sampling to reduce computational burden in simulation estimation," *Quantitative Marketing and Economics*, 7, 343—376.

Adams, C. P. (2012): "Identification in the Roy Model with an Application to Auctions," Working Paper, Federal Trade Commission.

Aguirregabiria, V. and P. Mira (2007): "Sequential Estimation of Dynamic Discrete Games," *Econometrica*, 75, 1–53.

Athey, S. and P. Haile (2007): "Empirical Models of Auctions," in *Advances in Economics and Econometrics, Theory and Applications: Ninth World Congress of the Econometric Society*, ed. by R. Blundell, W. Newey, and T. Persson, Cambridge University Press.

Backus, M., T. Blake, B. Larsen, and S. Tadelis (2020): "Sequential Bargaining in the Field: Evidence from Millions of Online Bargaining Interactions," *Quarterly Journal of Economics*, 135, 1319–1361.

Backus, M., T. Blake, D. Masterov, and S. Tadelis (2015): "Is Sniping a Problem for Online Markets?" *Proceedings of the 24th ACM International World Wide Web Conference.*

Backus, M., T. Blake, and S. Tadelis (2019): "On the Empirical Content of Cheap-Talk Signaling: An Application to Bargaining," *Journal of Political Economy*, 127, 1599–1628.

Backus, M. and G. Lewis (2010): "An Estimable Demand System for a Large Auction Platform Market," Working Paper.

Bajari, P., L. Benkard, and J. Levin (2007): "Estimating Dynamic Models of Imperfect Competition," *Econometrica*, 75, 1331–1370.

Bajari, P. and A. Hortaçsu (2004): "Economic Insights from Internet Auctions," *Journal of Economic Literature*, 42, 457–486.

Berry, S. (1994): "Estimating Discrete Choice Models of Product Differentiation," *Rand Journal of Economics*, 25, 242–262.

Berry, S., J. Levinsohn, and A. Pakes (1995): "Automobile Prices in Market Equilibrium," *Econometrica*, 63, 841–890.

Berry, S. and A. Pakes (2007): "The Pure Characteristics Demand Model," *International Economic Review*, 48, 1193—1225.

Blackwell, D. (1965): "Discounted Dynamic Programming," *Ann. Math. Statist.*, 36, 226–235.

Bodoh-Creed, A., J. Böehnke, and B. Hickman (2013): "Optimal Design of Two-Sided Market Platforms: An Empirical Case Study of Ebay," Working Paper.

——— (2017): "How Efficient Are Decentralized Auction Platforms," Working Paper.

Boyd, J. H. and R. E. Mellman (1980): "The Effect of Fuel Economy Standards on the U.S. Automotive Market: An Hedonic Demand Analysis," *Tranportation Research Part A: General*, 14, 367–378.

Budish, E. (2008): "Sequencing and Information Revelation in Auctions for Imperfect Substitutes: Understanding eBay's Market Design," Working paper, Harvard University.

Cardell, N. S. and F. C. Dunbar (1980): "Measuring the Societal Impacts of Automobile Downsizing," *Transportation Research Part A: General*, 14, 423–434.

Carroll, R. J. and P. Hall (2004): "Low order approximations in deconvolution and regression with errors in variables," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 66, 31–46.

Coey, D., B. Larsen, and B. Platt (2015): "A Theory of Bidding Dynamics and Deadlines in Online Retail," *American Economic Review*, 110, 3748–3785.

Decarolis, F. (2018): "Comparing Public Procurement Auctions," *International Economic Review*, 59, 391–419.

Einav, L., C. Farronato, J. Levin, and N. Sundaresan (2016): "Auctions versus Posted Prices in Online Markets," Working Paper.

Evdokimov, K. and H. White (2012): "Some Extensions of a Lemma of Kotlarski," *Econometric Theory*, 28, 925–932.

Fershtman, C. and A. Pakes (2012): "Dynamic Games with Asymmetric Information: A Framework for Empirical Work," *Quarterly Journal of Economics*, 127, 1611–1661.

Gentry, M. and T. Li (2014): "Identification in Auctions with Selective Entry," *Econometrica*, 82, 315–344.

Goettler, R. L. and B. Gordon (2011): "Does AMD Spur Intel to Innovate More?" *Journal of Political Economy*, 119, 1141 – 1200.

Gowrisankaran, G. and M. Rysman (2012): "Dynamics of Consumer Demand for New Durable Goods," *Journal of Political Economy*, 120, 1173–1219.

Groeger, J. R. (2014): "A Study of Participation in Dynamic Auctions," *International Economic Review*, 55, 1129–1154.

Guerre, E., I. Perrigne, and Q. Vuong (2000): "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 68, 525–574.

Haile, P., H. Hong, and M. Shum (2003): "Nonparametric Tests for Common Values in First-Price Sealed-Bid Auctions," NBER Working Paper 10105.

Haile, P. A. and E. Tamer (2003): "Inference with an Incomplete Model of English Auctions," *The Journal of Political Economy*, 111, 1–51.

Hendricks, K. and R. Porter (2007): "A Survey of Empirical Work in Auctions," in *Handbook of Industrial Organization*, ed. by M. Armstrong and R. Porter, North-Holland, vol. III.

Hendricks, K., A. Sorensen, and T. Wiseman (2021): "Dynamics and Efficiency in Decentralized Online Auction Markets," Working Paper, University of Wisconsin.

Hickman, B., T. P. Hubbard, and H. J. Paarsch (2017): "Identification and Estimation of a Bidding Model for Electronic Auctions," *Quantitative Economics*, 8, 505–551.

Ingster, A. (2009): "A Structural Study of a Dynamic Auction Market," Working Paper, NYU Stern.

Iyer, K., R. Johari, and M. Sundararajan (2014): "Mean Field Equilibria of Dynamic Auctions with Learning," *Management Science*, 60, 2949–2970.

Jofre-Bonet, M. and M. Pesendorfer (2003): "Estimation of a Dynamic Auction Game," *Econometrica*, 71, 1443–1489.

Karaca-Mandic, P. and K. Train (2003): "Standard Error Correction in Two-Stage Estimation with Nested Samples," *Econometrics Journal*, 6, 401–407.

Kloek, T. and H. van Dijk (1978): "Bayesian Estimates of Equation System Parameters: An Application of Integration by Monte Carlo," *Econometrica*, 46, 1–19.

Kotlarski, I. (1967): “On characterizing the gamma and the normal distribution.” *Pacific J. Math.*, 20, 69–76.

Krasnokutskaya, E. (2011): “Identification and Estimation in Highway Procurement Auctions under Unobserved Auction Heterogeneity,” *Review of Economic Studies*, 71, 293–327.

Krasnokutskaya, E., K. Song, and X. Tang (2014): “The Role of Quality in On-Line Service Markets,” Working paper, Johns Hopkins.

Krasnokutskaya, E., C. Terwiesch, and L. Tiererova (2016): “Trading across Borders in Online Auctions,” Working Paper, Johns Hopkins.

Krusell, P. and A. A. Smith (1998): “Income and Wealth Heterogeneity in the Macroeconomy,” *Journal of Political Economy*, 106, 867—896.

Lewis, G. (2007): “Asymmetric Information, Online Disclosure and Adverse Selection: The Case of eBay Motors,” Working paper, Harvard.

Lewis, G. and P. Bajari (2011): “Procurement Contracting with Time Incentives: Theory and Evidence,” *Quarterly Journal of Economics*, 126.

Marra, M. (2018): “Pricing and fees in auction platforms with two-sided entry,” Working Paper.

McAfee, P. (1993): “Mechanism Design by Competing Sellers,” *Econometrica*, 61, 1281–1312.

Murphy, K. M. and R. H. Topel (2002): “Estimation and Inference in Two-Step Econometric Models,” *Journal of Business and Economic Statistics*, 20, 88–97.

Myerson, R. B. (1981): “Optimal Auction Design,” *Mathematics of Operations Research*, 6, 58—73.

Nekipelov, D. (2007): “Entry Deterrence and Learning Prevention on eBay,” Mimeo, UC Berkeley.

Pakes, A., M. Ostrovsky, and S. Berry (2007): “Simple Estimators for the Parameters of Discrete Dynamic Games, with Entry/Exit Examples,” *Rand Journal of Economics*, 38, 373—399.

Peters, M. and S. Serevinov (1997): "Competition Among Sellers Who Offer Auctions Instead of Prices," *Journal of Economic Theory*, 75, 141–179.

Petrin, A. and K. Train (2003): "Omitted Product Attributes in Discrete Choice Models," NBER Working Paper No. 9452.

Ridinger, W. (2020): "Sequential Auctions and Resale," Ph.D. thesis, The London School of Economics and Political Science.

Roberts, J. W. (2013): "Unobserved heterogeneity and reserve prices in auctions," *The RAND Journal of Economics*, 44, 712–732.

Roth, A. E. and A. Ockenfels (2002): "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review*, 92, 1093–1103.

Said, M. (2009): "Auctions with Dynamic Populations: Efficiency and Revenue Maximization," Working paper, Yale University.

Sailer, K. (2006): "Searching the eBay Marketplace," Working Paper, University of Munich.

Song, U. (2004): "Nonparametric Estimation of an eBay Auction Model with an Unknown Number of Bidders," Working Paper, University of British Columbia.

Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press.

Terapeak (2009): "Dataset of eBay Bids and Listing Characteristics for Compact Cameras," Accessed June 25, 2009.

Weintraub, G., C. L. Benkard, and B. Van Roy (2008): "Markov Perfect Industry Dynamics with Many Firms," *Econometrica*, 76, 1375–1411.

Zeithammer, R. (2006): "Forward-looking bidding in online auctions," *Journal of Marketing Research*, 43, 462–476.

——— (2009): "Sequential auctions with information about future goods," Working paper, UCLA Anderson.

Appendix

A Proofs

Proof of Lemma 1

Proof.

$$\begin{aligned}\beta_j(\mathbf{x},\varepsilon,s) = \arg\max_{b\in\mathbb{R}} \int \Bigg(& 1(b \geq B^1)(u(\mathbf{x},\varepsilon,s) - B^1) \\ & + 1(b < B^1) r \int\int v(\mathbf{x},\varepsilon',s') dP(s'|s) dF(\varepsilon') \Bigg) dG^1(B^1|s)\end{aligned} \tag{1}$$

Consider the RHS of (1). The integrand takes the value $u(\mathbf{x},\varepsilon,s) - B^1$ when $B^1 \leq b$ and the value $r\int\int v(\mathbf{x},\varepsilon',s')dP(s'|s)dF(\varepsilon')$ when $B^1_j > b$ and so can be maximized by choosing b so that the event $B^1 \leq b$ occurs iff $u(\mathbf{x},\varepsilon,s) - B^1_j \geq r\int\int v(\mathbf{x},\varepsilon',s')dP(s'|s)dF(\varepsilon')$ or equivalently iff $B^1 \leq u(\mathbf{x},\varepsilon,s) - r\int\int v(\mathbf{x},\varepsilon',s')$. This can be achieved by choosing $b^* = u(\mathbf{x},\varepsilon,s) - r\int\int v(\mathbf{x},\varepsilon',s')$, since then $B^1 \leq b \Rightarrow B^1 \leq b^*$ and $B^1 > b \Rightarrow B^1 > b^*$. Moreover, under the assumption that there are no gaps in the support of B^1, this is the unique best response. For example for any putative best response $b > b^*$, there is a positive probability of the event $b > B^1 > b^*$, in which case the bidder wins and gets a payoff of $u(\mathbf{x},\varepsilon,s) - B^1 < u(\mathbf{x},\varepsilon,s) - b^* = r\int\int v(\mathbf{x},\varepsilon',s')$ i.e. lower than their discounted continuation value, which is sub-optimal (the case $b < b^*$ is similar).

Now from (4) in the main text:

$$\beta(\mathbf{x},\varepsilon,s) = \widetilde{\beta}_j(\mathbf{x}) + \boldsymbol{\gamma}\mathbf{z}_{j,t} + \xi_{j,t} + \varepsilon_{i,j,t} \tag{2}$$

Continuity of the bidding function in (z,ξ,ε) immediately follows. We must show continuity of $\widetilde{\beta}_j(\mathbf{x}) \equiv x_j - r\sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x})$ in $\mathbf{x}$. Fix an initial state j, and consider two types $\mathbf{x}^1$ and $\mathbf{x}^2$. Suppose wlog that $v_j(\mathbf{x}_2) > v_j(\mathbf{x}_1)$ and for the purposes of this argument that $\mathbf{x}^2$ bids optimally but type $\mathbf{x}_1$ simply copies their strategy (i.e. makes the same bids as type $\mathbf{x}_2$ in every state). Let the expected utility that each type earns from this common strategy be U^1_j and U^2_j. Now $|U^2_j - U^1_j| \leq \max_k |x^2_k - x^1_k|$ for all j, since the maximum difference in expected utility is upper bounded by the maximum difference in payoffs when successfully winning an item (because the types are playing the same strategy, expected allocations and payments are the same).

Also $U_j^1 < v_j(\mathbf{x}^1)$ and $U_j^2 = v_j(\mathbf{x}^2)$, since $\mathbf{x}^1$ is behaving sub-optimally while $\mathbf{x}^2$ is optimizing. Then we have $0 < v_j(\mathbf{x}^2) - v_j(\mathbf{x}^1) < v_j(\mathbf{x}^2) - U_j^1 = U_j^2 - U_j^1 \leq \max_k |x_k^2 - x_k^1|$.

But since it was arbitrary that $v_j(\mathbf{x}^2) > v_j(\mathbf{x}^2)$, in fact we have: $|v_j(\mathbf{x}^2) - v_j(\mathbf{x}^1)| \leq \max_k |x_k^2 - x_k^1|$ for all j, and therefore: $\max_j |v_j(\mathbf{x}^2) - v_j(\mathbf{x}^1)| \leq \max_j |x_j^2 - x_j^1|$ This immediately proves that the continuation values are continuous in $\mathbf{x}$, with a modulus of continuity of 1: if $\max_j |x_j^2 - x_j^1| \leq \varepsilon$ then $\max_j |v_j(\mathbf{x}^2) - v_j(\mathbf{x}^1)| \leq \varepsilon$. For bids, notice that: $\|\widetilde{\beta}_j(\mathbf{x}^1) - \widetilde{\beta}_j(\mathbf{x}^2)\| = \max_j |x_j^1 - r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}^1) - x_j^2 + r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}^1) \ = \ \max_j |x_j^2 - x_j^1 - r \sum_k \widetilde{Q}_{j,k}(v_k(\mathbf{x}^2) - v_k(\mathbf{x}^1))\| \leq \max_j |x_j^1 - x_j^2| + r \max_j |\tilde{v}_j(\mathbf{x}^1) - \tilde{v}_j(\mathbf{x}^2)| \leq (1+r) \max_j |x_j^1 - x_j^2|$ where the first inequality follows by the triangle inequality. Thus bids are continuous in $\mathbf{x}$ with a modulus of continuity of $(1+r)$.

For monotonicity, again it suffices to show monotonicity of the functions $\widetilde{\beta}_j(\mathbf{x})$. Consider two types $\mathbf{x}^1$ and $\mathbf{x}^2$ with $x_k^1 = x_k^2$ for all $k \neq j$ and $x_j^1 < x_j^2$. Now $v_k(\mathbf{x}^2) - v_k(\mathbf{x}^1) \leq x_j^2 - x_j^1$ for all k, since $\mathbf{x}^1$ can follow the strategy of $\mathbf{x}^2$ and get the exact same payoff as type $\mathbf{x}^2$ except when bidding on product j, when they get a payoff at most $x_j^2 - x_j^1$ lower. Thus: $\widetilde{\beta}_j(\mathbf{x}^2) - \widetilde{\beta}_j(\mathbf{x}^1) = x_j^2 - x_j^1 - r \sum_k \widetilde{Q}_{j,k}(v_k(\mathbf{x}^2) - v_k(\mathbf{x}^1)) \geq x_j^2 - x_j^1 - r(x_j^2 - x_j^1) > 0$ proving that β_j is strictly increasing in x_j. Also $v_k(\mathbf{x}^2) - v_k(\mathbf{x}^1) \geq 0$ for all k since $\mathbf{x}^2$ can follow the strategy of $\mathbf{x}^1$ and get at least as high a payoff. It follows that $\beta_k(\mathbf{x}^2) - \beta_k(\mathbf{x}^1) = -r \sum_l \widetilde{Q}_{k,l}(v_l(\mathbf{x}^2) - v_l(\mathbf{x}^1)) \leq 0$ which (together with analysis of other cases) shows that β_j is decreasing in x_k for any $k \neq j$. □

Proof of Theorem 1.

Proof. For ease of exposition, we defer some parts of this proof to a sequence of lemmas that directly follow the proof. Lemma 5 establishes the existence of a unique ergodic measure for *any* strategy β. Thus to prove existence it will suffice to show that there is *some* strategy β^e that satisfies the optimality and consistency conditions laid out in Definition 1.

Next, define a modified game in which there are no item characteristics (neither observed nor unobserved). In Lemma 6 below we argue that if $\check{\beta}^e(\mathbf{x}, \varepsilon, j)$ is an equilibrium strategy for this modified game, then $\beta^e(\mathbf{x}, \varepsilon, s) = \check{\beta}^e(\mathbf{x}, \varepsilon, j) + \mathbf{z}\boldsymbol{\gamma} + \xi$ is an equilibrium strategy in the original game, and the bid distributions form a location family, with $G^1(b|s) = \check{G}_j^1(b - \mathbf{z}\boldsymbol{\gamma} - \xi)$ for $\check{G}_j^1$ the equilibrium bid distributions in the modified game. In view of this, to prove the theorem we need only show existence of an equilibrium strategy $\check{\beta}^e(\mathbf{x}, \varepsilon)$ for the modified game. For simplicity, we will just write β^e rather than $\check{\beta}^e$ in what follows.

Let the support of the idiosyncratic shock be E. The strategies are thus defined on $\mathcal{X} \times E$. Let $C(\mathcal{X} \times E)$ be the set of all continuous functions on that space metrized by the sup norm. Let $\mathcal{L}_2 = \{f \in C(\mathcal{X}) : L(f) \leq M\}$ where $L(f)$ is the Lipschitz constant of the function f. Define the best response function $\Gamma(\beta)$ to any strategy $\beta \in \mathcal{L}_2$ as in Lemma 1, i.e. $\Gamma(\beta)(\mathbf{x}, \varepsilon, j) = x_j - r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}) + \varepsilon$. A fixed point $\beta = \Gamma(\beta)$ is a pure strategy equilibrium. We will prove the existence of a fixed point by Schauder's fixed point theorem: if Γ is a continuous mapping from a non-empty convex and compact subset K of a Banach space X into itself, then it has a fixed point.

Convexity, Compactness. $\mathcal{L}_2$ is non-empty and convex. Moreover, since $\mathcal{L}_2$ has a finite Lipschitz constant, the set of functions $f \in \mathcal{L}_2$ are uniformly equicontinuous, and so by Arzelà-Ascoli, $\mathcal{L}_2$ is relatively compact. Since $\mathcal{L}_2$ is also complete it is compact. It remains to prove that $\Gamma(\mathcal{L}_2) \subseteq \mathcal{L}_2$. Now, every $\beta \in \Gamma(C(\mathcal{X}))$ admits a modulus of continuity of $1+r$, by the argument made in the proof of Lemma 1. This implies $\Gamma(\mathcal{L}_2) \subseteq \mathcal{L}_{1+r} \subseteq \mathcal{L}_2$.

Continuity. Next, we need to show that Γ is continuous in β when $\beta \in \mathcal{L}_2$ i.e. if $\beta^n \to \beta$ then $\Gamma(\beta^n) \to \Gamma(\beta)$. The first step is to show that the distribution of highest bids $\check{G}^1_j$ is continuous in β in the weak-* topology i.e. $\beta_n \to \beta$ implies $\mu^n_j \to \mu_j$ in the sense of weak convergence of measures (convergence in the weak topology is equivalent in this case). By Lemma 7 directly below, the ergodic distribution of types is weakly continuous in β. Now given an ergodic measure μ^β the highest bid distribution $\check{G}^1_j$ is defined by:

$$\check{G}^1_j(b) = \frac{\int \mathbb{P}_{\mu^\beta}\left(\max_{i>1} \beta(\mathbf{x}_i(\omega), \varepsilon_i, j) \leq b \wedge \omega_1 = j\right) d\boldsymbol{F}(\boldsymbol{\varepsilon})}{\mathbb{P}_{\mu^\beta}\left(\omega_1 = j\right)}$$

since the first entry ω_1 indicates the product under auction and the highest bid is $\max_{i>1} \beta(\mathbf{x}_i(\omega), \varepsilon_i, j)$ (where we adopt the convention $\beta(-1, \varepsilon, j) = -\infty \; \forall j$ i.e. non-existent bidders, notated by $\mathbf{x} = -1$, don't count towards the maximum). The distribution is continuous in β since max is a continuous operation, and by Lemma 5, μ^β is absolutely continuous with respect to (wrt) Lebesgue measure in any states where there are incumbent bidders in the market.

The second step is to show that if the highest bid distributions $\mu = \{\mu_j\}$ are close in the sense of weak convergence, then so are the best responses. Recall from the main text that when playing optimally (i.e. best responding) the value of the game is just the value of an infinite sequence of annuities, each of which pays $\mathbb{E}_{\tilde{G}_j^1}[\max\{0, \beta(\mathbf{x}, \varepsilon, j) - B_j^1\}]$ whenever j is auctioned. The term inside the expectation is a continuous function, and so it follows from the definition of weak convergence that the value function $v_j(\mathbf{x})$ is smooth in μ. Then since the best responses $\Gamma(\beta)(\mathbf{x}, \varepsilon)$ are by definition equal to value less discounted continuation value, they too must be smooth in the bid distributions. Putting this all together, we get the required continuity of Γ in β.

□

Lemma 5. *For any strategy β, there exists a unique corresponding ergodic measure μ^β, converged to at geometric rate. This measure is absolutely continuous with respect to Lebesgue measure except at the atoms corresponding to states in which there are no bidders in the market.*

Proof. Let ω index states, Ω be the state-space, $\mathcal{B}(\Omega)$ be the Borel σ-algebra over the state space, and $P_\beta(\omega, A)$ be the one-step transition probability when the players play according to β (i.e. the probability of reaching set $A \in \mathcal{B}(\Omega)$ from state ω in a single period). By Theorem 11.12 in Stokey et al. (1989), uniform geometric convergence in total variation norm will be achieved if their "condition M" holds: $\exists \varepsilon > 0$ such that for every $A \in \mathcal{B}(\Omega)$, either $[P_\beta(\omega, A) > \varepsilon \; \forall \omega \in \Omega]$ or $[P_\beta(\omega, A^c) > \varepsilon \; \forall \omega \in \Omega]$.

We claim the following is sufficient for condition M: there exists some $\omega_0 \in S$ and some $\varepsilon > 0$ such that $P_\beta(\omega, \omega_0) > \varepsilon \; \forall \omega \in \Omega$. To prove this, notice that for any $A \in \mathcal{B}(\Omega)$ either $\omega_0 \in A$ or $\omega_0 \in A^c$. If the former, then for any $\omega \in \Omega$, $P_\beta(\omega, A) \geq P_\beta(\omega, \omega_0) > \varepsilon$. If the latter, then $P_\beta(\omega, A^c) \geq P_\beta(\omega, \omega_0) > \varepsilon$.

Let ω_0 be the state where there are no bidders and no supply. We denote this state as $\omega = (-1, -1 \cdots - 1)$, where the -1 in the first entry indicates that there is no item being sold and the remaining -1 terms indicate no bidders in each of the remaining $\bar{N}$ positions.

This state is reachable in a single step: if at the end of a period everyone exits and no-one enters, and in the following period no products are available for auction, the state occurs. The probability of this occurring is at least $\underbrace{(1-r)^{N_{t-1}-1}}_{\text{all losers exit}} \underbrace{e^{-\lambda}}_{\text{no entry}} \underbrace{\min_j Q_{j,J+1}}_{\text{no supply}}$. So the required condition holds with $\varepsilon = (1-r)^{N_{t-1}-1} e^{-\lambda} \min_j Q_{j,J+1}$.

Next, by the Lebesgue decomposition theorem, it will suffice to show that the singular part of the ergodic measure μ^β consists only of atoms at the points ω_j. We assume that the singular part is discrete, and argue by contradiction: suppose there are other atoms and let the state $\tilde{\omega}$ be a selection from the set of atoms with the property that there are no other atomic states with more incumbent bidders. By the definition of ergodicity $\mu_j^\beta(\tilde{\omega}) = T(\mu_j^\beta, \beta)(\tilde{\omega})$ and because $\tilde{\omega}$ is an atom $\mu_j^\beta(\tilde{\omega}) > 0$, so $T(\mu_j^\beta, \beta)(\tilde{\omega}) > 0$. But for $T(\mu_j^\beta, \beta)(\tilde{\omega}) > 0$ it is necessary that there are other atoms that reach $\tilde{\omega}$ in a single step with strictly positive probability. Notice that no state with the same or fewer incumbent bidders than $\tilde{\omega}$ reaches $\tilde{\omega}$ with positive probability, since at least one incumbent bidder must exit (the winner), and then reaching $\tilde{\omega}$ requires drawing entrants with the exact same valuations as the exiting incumbents. Since $\mathbf{F}$ is atomless wrt Lebesgue measure, this is a zero probability event. By similar logic, no state ω' with more bidders reaches $\tilde{\omega}$ with positive probability *unless* it includes all the incumbents in $\tilde{\omega}$ plus some additional incumbents (in this case there is a positive probability that these additional incumbents exit and no one enters, reaching $\tilde{\omega}$). But since by assumption $\tilde{\omega}$ was the atom with the largest number of incumbent bidders, any such state ω' must have zero measure under μ^β. But then we have shown that $T(\mu_j^\beta, \beta)(\tilde{\omega}) = 0$, a contradiction. □

Lemma 6. *Let* $\left(\check{\beta}^e(\mathbf{x}, \varepsilon, j), \{\check{G}_j^1(b)\}, \mu^{\beta^e}\right)$ *be an equilibrium of the modified game. Let* j *be the product auctioned in state* s*. Then there exists an equilibrium of the original game* $\left(\beta^e(\mathbf{x}, \varepsilon, s), \{G^1(b|s)\}, \mu^{\beta^e}\right)$ *in which bidders bid according to* $\beta^e(\mathbf{x}, \varepsilon, s) = \check{\beta}^e(\mathbf{x}, \varepsilon, j) + \mathbf{z}\boldsymbol{\gamma} + \xi$*, and* $G^1(b|s) = \check{G}_j^1(b - \mathbf{z}\boldsymbol{\gamma} - \xi)$

Proof. We must verify the conditions for an equilibrium. Suppose player play according to β^e i.e. $\beta^e(\mathbf{x}, \varepsilon, s) = \check{\beta}^e(\mathbf{x}, \varepsilon, j) + \mathbf{z}\boldsymbol{\gamma} + \xi$ Then all bidders shift their bids according to the item characteristics, having no effect on the rank order of bids and thus on who wins the auction. Since all the state transitions are unaffected, μ^{β^e} is the ergodic measure for both the modified and the original game.

Next, $G^1(b|s) = P(\max_{i>1} \check{\beta}^e(\mathbf{x}_i, \varepsilon_i, j) + \mathbf{z}\boldsymbol{\gamma} + \xi < b|s) = P(\max_{i>1} \check{\beta}^e(\mathbf{x}_i, \varepsilon_i, j) < b - \mathbf{z}\boldsymbol{\gamma} - \xi) = \check{G}_j^1(b - \mathbf{z}\boldsymbol{\gamma} - \xi)$, where the first equality is by definition, the second uses the fact that $\check{\beta}$ is independent of $(\mathbf{z}, \xi)$ to drop the conditioning event, and the last step makes use of the fact that the ergodic measures for the two games coincide.

This covers stationarity and consistency. Finally, we must establish that β^e is a best response when others play according to β^e. By Lemma 1 optimal bids are equal to current valuations less continuation value: $\beta^e(\mathbf{x}, \varepsilon, s) = x_j + \varepsilon + \mathbf{z}\boldsymbol{\gamma} + \xi - r\sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x})$. Now the prescribed bids take the form: $\beta^e(\mathbf{x}, \varepsilon, s) = \check{\beta}(\mathbf{x}, \varepsilon, j) + \mathbf{z}\boldsymbol{\gamma} + \xi = x_j + \varepsilon - r\sum_k \widetilde{Q}_{j,k} \check{v}_k(\mathbf{x}) + \mathbf{z}\boldsymbol{\gamma} + \xi$ where $\check{v}$ is the continuation value in the modified game, and the first equality is by definition while the second uses the fact that bids in the modified game must also be optimal. Examining the expression for the optimal and prescribed bids, it is clear that β^e is optimal iff $v_j(\mathbf{x}) = \check{v}_j(\mathbf{x})$ for every j (i.e. the continuation values coincide). Now in any state s, the probability of winning and expected payoff conditional on winning remain unchanged, since all bidders are adjusting their bids by $\mathbf{z}\boldsymbol{\gamma} + \xi$ relative to the original game. Since the expected allocations and payments are unchanged in every state, so too is the continuation value from any state. So β^e is a best response. $\square$

Lemma 7. *Let $\beta \in \mathcal{L}_2$. The ergodic measure μ^β is weakly continuous in β i.e. for all sequences $(\beta_n) \in \mathcal{L}_2$ converging to $\beta^* \in \mathcal{L}_2$ we have : $\mu^{\beta_n}(\cdot) \xrightarrow{\text{weakly}} \mu_{\beta^*}(\cdot)$.*

Proof. Let (β_n) be a sequence in $\mathcal{L}_2$ that converges to $\beta^* \in \mathcal{L}_2$. Towards a contradiction suppose that (μ^{β_n}) does not converge to μ^{β^*}. Now since the state space Ω is compact, the set of measures (μ^β) on Ω are tight. Moreover, $\mathcal{L}_2$ is compact, and so by Prokhorov's theorem the set of measures $(\mu^\beta)_{\beta\in\mathcal{L}_2}$ is sequentially compact. Therefore we know that there is a subsequence (β_{m_n}) and an element $\beta' \in \mathcal{L}_2$ such that

$$\mu_{\beta_{m_n}} \xrightarrow{weakly} \mu_{\beta'}.$$

Let $d = \bar{N} + 1$ and let $A \in \mathbb{R}^d$ be a Borel set. We define $P_\beta(\omega, A)$ as the probability of reaching the set A from the state ω in a single step when strategies β are played. Let $\epsilon > 0$ be a positive real. We have

$$\begin{aligned}
&\Big|\mu_{\beta_{m_n}}(A) - \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta_{m_n}}(\omega)\Big| \\
&\overset{(a)}{\leq} \Big| \int_{\mathbb{R}^d} \Big[P_{\beta_{m_n}}(\omega, A) - P_{\beta^*}(\omega, A)\Big] d\mu_{\beta_{m_n}}(\omega)\Big| \\
&\overset{(b)}{\leq} \epsilon + \mathbb{P}_{\omega\sim\mu_{\beta_{m_n}}}(\big|[P_{\beta_n}(\omega, A) - P_{\beta^*}(\omega, A)\big| > \epsilon).
\end{aligned}$$

where to get (a) we used the fact that $\mu_{\beta_{m_n}}$ is the ergodic measure of $P_{\beta_{m_n}}$, and (b) splits the realization of the state into small and large deviations.

Next, we will show that $\mathbb{P}_{\omega\sim\mu_{\beta_{m_n}}}(\big|[P_{\beta_{m_n}}(\omega, A) - P_{\beta^*}(\omega, A)\big| > \epsilon) \to 0$ as $\beta_{m_n} \to \beta^*$ (since $\beta_n \to \beta^*$, the subsequence does too). It is sufficient to show that $P_\beta(\omega, A)$ is continuous in β almost everywhere. Notice that next period's state depends only on the strategies β through determining the winner of the current auction — entry, activity and exit through losing are constant in β. Fix a state ω. Then the identity of the winner is continuous in β at ω, unless there is a tie

since then a small perturbation in β can affect the winner (and therefore the next state) discontinuously. But the set of such states is of lower dimension than the state space, and therefore Lebesgue measure zero, and (by absolute continuity of the ergodic measures) zero probability. It follows that:

$$\Big|\mu_{\beta_{m_n}}(A) - \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta_{m_n}}(\omega)\Big| \to 0. \tag{3}$$

Again using the fact that all the ergodic measures are assumed to be absolutely continuous with respect to the Lebesgue measure we have that: $\mu_{\beta_{m_n}}(A) \to \mu_{\beta'}(A)$. Moreover as $\omega \to P_{\beta^*}(\omega, A)$ is measurable and bounded we know that for all $\delta > 0$ there are N_δ different Borel sets (B_i) and different reals c_i such that

$$\sup_{\omega} \big|P_{\beta^*}(\omega, A) - \sum_{i \le N_\delta} c_i \mathbb{I}(\omega \in B_i)\big| \le \delta$$

Moreover using the fact that $\mu_{\beta_{m_n}}(\cdot)$ converges weakly to $\mu_{\beta'}(\cdot)$ we have:

$$\int_{\mathbb{R}^d} \big[\sum_{i \le N_\delta} c_i \mathbb{I}(\omega \in B_i)\big] d\mu_{\beta_{m_n}}(x) - \int_{\mathbb{R}^d} \big[\sum_{i \le N_\delta} c_i \mathbb{I}(\omega \in B_i)\big] d\mu_{\beta'}(x) \to 0.$$

Therefore by a triangle inequality, we have:

$$\int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta_{m_n}}(\omega) - \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta'}(\omega) \to 0. \tag{4}$$

Now since $\mu_{\beta_{m_n}}(A)$ weakly converges to $\mu_{\beta'}(A)$, we can combine this fact with (3) and (4) to deduce that for all Borel sets A we have

$$\mu_{\beta'}(A) = \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta'}(\omega) \tag{5}$$

But by Lemma 5 the ergodic measure is unique, so we must have that: $\mu_{\beta'} = \mu_{\beta^*}$. Contradiction. □

Proof of Lemma 2.

Proof. See the argument in the text. □

Proof of Lemma 3.

Proof. We begin by writing out an expression for the continuation value, prior to the realization of the transient shocks.

$$
\begin{aligned}
v_j(\mathbf{x}) &= \int\int G_j^1(\check{b}_j+\xi+\varepsilon|\xi)\left(x_j+\xi+\varepsilon-E_{G_j^1|\xi}[B_j^1|B_j^1<\check{b}_j+\xi+\varepsilon,\xi]\right)dF_\varepsilon(\varepsilon)dF_\xi(\xi)\\
&+\int\int(1-G_j^1(\check{b}_j+\xi+\varepsilon|\xi))dF_\varepsilon(\varepsilon)dF_\xi(\xi)\left(r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x})\right)\\
&=\int\int\check{G}_j^1(\check{b}_j+\varepsilon)\left(x_j+\varepsilon-E_{\check{G}_j^1}[\check{B}_j^1|\check{B}_j^1<\check{b}_j+\varepsilon]\right)dF_\varepsilon(\varepsilon)dF_\xi(\xi)\\
&+\int\int(1-\check{G}_j^1(\check{b}_j+\varepsilon))dF_\varepsilon(\varepsilon)dF_\xi(\xi)\left(r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x})\right)\\
&=\int\check{G}_j^1(\check{b}_j+\varepsilon)\left(x_j+\varepsilon-E_{\check{G}_j^1}[B_j^1|B_j^1<\check{b}_j+\varepsilon]\right)dF_\varepsilon(\varepsilon)\\
&+\int(1-\check{G}_j^1(\check{b}_j+\varepsilon))dF_\varepsilon(\varepsilon)\left(r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x})\right)
\end{aligned}
\tag{6}
$$

The first line is the definition of the continuation value, as the probability of winning times the surplus conditional on winning, plus the discounted continuation value when losing, integrated out over the realizations of both shocks. In this expression the distribution of competing bids conditions on the commonly known shock ξ_l, which affects the bids of all bidders. The second line follows from the location family result of Theorem 1. The final line removes the redundant outer integral.

Now the bid-types are equal to the valuation less the discounted continuation value: $\check{b}_j =$

$\check{x}_j - r\sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}})$. Solving for x_j in the above expression and substituting gives:

$$\begin{aligned}
v_j(\mathbf{x}) = & \int \check{G}^1_j(\check{b}_j+\varepsilon)\left(\check{b}_j + r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x}) + \varepsilon - E_{\check{G}^1_j}[B^1_j|B^1_j < \check{b}_j+\varepsilon]\right) dF_\varepsilon(\varepsilon) \\
& + \int (1-\check{G}^1_j(\check{b}_j+\varepsilon))dF_\varepsilon(\varepsilon)\left(r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x})\right) \\
& = \int \check{G}^1_j(\check{b}_j+\varepsilon)\left(\check{b}_j + \varepsilon - E_{\check{G}^1_j}[B^1_j|B^1_j < \check{b}_j+\varepsilon]\right) dF_\varepsilon(\varepsilon) \\
& + \int (\check{G}^1_j(\check{b}_j+\varepsilon) + 1 - \check{G}^1_j(\check{b}_j+\varepsilon))dF_\varepsilon(\varepsilon)\left(r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x})\right) \\
& = u_j(\check{\mathbf{b}}) + r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x})
\end{aligned} \tag{7}$$

where in the second line we collect terms in $r\sum_k \widetilde{Q}_{j,k}v_k(\mathbf{x})$, and in the final line we use the fact that $u_j(\cdot)$ is a function of the bid-type only. For each state j (including the no-supply state) we have an equation of this form. Stacking the equations we get a system of equations:

$$v(\mathbf{x}) = u(\check{\mathbf{b}}) + r\widetilde{Q}v(\mathbf{x})$$

where $v(\mathbf{x})$ is $J \times 1$, $u(\check{\mathbf{b}})$ is $J \times 1$, r is a scalar and $\widetilde{Q}$ is $J \times J$.

Solving the system yields:

$$v(\mathbf{x}) = (I - r\widetilde{Q})^{-1}u(\check{\mathbf{b}})$$

where since $r \in (0,1)$ and $\widetilde{Q}$ is right-stochastic, the matrix $(I - r\widetilde{Q})$ is strictly dominant diagonal and therefore invertible. Re-arranging the definition for the bid-types, the underlying valuation is given by $\mathbf{x} = \check{\mathbf{b}} + r\widetilde{Q}v(\mathbf{x})$. Substituting in for the continuation value:

$$\mathbf{x} \equiv \zeta(\check{\mathbf{b}}) = \check{\mathbf{b}} + r\widetilde{Q}(I - r\widetilde{Q})^{-1}u(\check{\mathbf{b}})$$

completing the proof. □

Proof of Lemma 4.

Proof. We will repeatedly invoke (a minor modification of) Lemma 2 of Evdokimov and White (2012): suppose the joint distribution of (Y_1, Y_2) is observed, $Y_1 = M + U_1$ and

$Y_2 = M + U_2$ and (M, U_1, U_2) are mutually independent, with $E[|Y_1| + |Y_2|] < \infty$ and either $E[M] = 0$ or $E[U_1] = 0$. Suppose also that there exist positive constants c_1 and c_2 such that $f_{U_1}(u) < c_1 \exp(-c_2\|u\|)$ ("the tail condition"). Then the distributions of M, U_1, and U_2 are identified.

Take as Y_1 and Y_2 the joint distribution of pairs of bids placed by bidders that have just entered the market. From (6), these bids are a sum of two components: a component that is common ($\xi_{j,t}$) and a component that is bidder-specific ($\check{x}_{i,j} - r \sum k\widetilde{Q}_{j,k} v_k(\check{\mathbf{x}}_i) + \varepsilon_{i,j,t}$). Because $\xi_{j,t}$, $\varepsilon_{i,j,t}$ and $\mathbf{x}_i$ are mutually independent, with finite second moments, the two components are mutually independent with finite absolute first moments (the latter following from Cauchy-Schwarz). Take $U_1 = \check{x}_{i,j} - r \sum k\widetilde{Q}_{j,k} v_k(\check{\mathbf{x}}_i) + \varepsilon_{i,j,t}$. It satisfies the tail condition since the valuations have finite support and by assumption $\varepsilon_{i,t,t}$ satisfies the tail condition. Finally, taking $M = \xi_{j,t}$ we have $E[M] = 0$. So the lemma's conditions are satisfied and we can identify F_ξ.

Next, consider the joint distribution of pairs of bids of the same bidder in two successive auctions of the same item. Again from (6), these are composed of a persistent bidder-specific element, $M = \check{x}_{i,j} - r \sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}}_i)$, and idiosyncratic shocks $U_1 = \xi_t + \varepsilon_{i,j,t_1}$ and $U_2 = \xi_t + \varepsilon_{i,j,t_2}$. Now since both ξ and ε satisfy the tail condition, so U_1, and moreover $E[U_1] = 0$.[46] So the conditions of the theorem are satisfied, allowing identification the distribution $F_{\xi+\varepsilon}$. Now since the distribution of ξ has isolated real zeros, and F_ξ is identified per the above, so is the distribution F_ε as per the remark in footnote 25.

Finally, notice that $B^1_{j,t} = \check{B}^1_{j,t} + \xi_{j,t}$, where the distribution of the highest bid G^1_j on the LHS and the distribution of F_ξ on the RHS are known, and $\check{B}^1_{j,t}$ and $\xi_{j,t}$ are independent. The missing distribution $\check{G}^1_j$ is thus identified by inverse Fourier transform. Finally, given that all these objects are identified, $u_j(\check{\mathbf{b}})$ is identified according to (9). □

Lemma 8. *The function $s(\check{\mathbf{b}})$ is identified.*

Proof. We provide a procedure for iteratively computing the selection correction function, providing a constructive proof of identification. Bidding on each of the products $j = 1 \ldots J$ is a transitory state, as is bidding in a period with no supply. Exit is an absorbing state. The probability of transition from any transitory state to another one is $r_j(\check{\mathbf{b}})\widetilde{Q}_{j,k}$; the probability

[46] The tail condition holds iff the moment generating functions (mgf) for ξ and ε exist. Since they do, the mgf of $\xi + \varepsilon$ also exists (and is equal to the product of the individual mgfs), and so the tail condition holds for the sum.

of exit is $(1 - r_j(\check{\mathbf{b}}))$. All of these are identified from knowledge of $\widetilde{Q}$, F_ε and $\{\check{G}_j^1\}$.

Define for each subset A of the transient states $1 \dots J+1$, $p(\check{\mathbf{b}}, A, j)$ to be the probability that bid-type $\check{\mathbf{b}}$ only visits states *within* A prior to exit, when entering in state j. We have the following recursive representation:

$$p(\check{\mathbf{b}}, A, j) = 1(j \in A)\left((1 - r_j(\check{\mathbf{b}})) + r_j(\check{\mathbf{b}})\sum_k \widetilde{Q}_{j,k} p(\check{\mathbf{b}}, A, k)\right)$$

since the probability of staying in A requires (i) starting in A (hence the indicator) and (ii) either exiting or surviving and entering another state k and once again having the chance of staying within A given by $p(\check{\mathbf{b}}, A, k)$. This recursive expression satisfies the Blackwell conditions for a contraction, and so can be identified and quickly computed for any A.

Now the probability of visiting *every* state in a set A, denoted $P(\check{\mathbf{b}}, A, j)$, is equal to the probability of staying within A less the probability of staying within any strict subset of A:

$$P(\check{\mathbf{b}}, A, j) = p(\check{\mathbf{b}}, A, j) - \sum_{B \subset A} p(\check{\mathbf{b}}, B, j)$$

Given this, we can compute $P(\check{\mathbf{b}}, \mathcal{J}, j)$ for each j, and then sum over the steady-state probability of entering in each of the states j, denoted π_j, to get $s(\check{\mathbf{b}}) = \sum_{j=1}^{J} \pi_j P(\check{\mathbf{b}}, \mathcal{J}, j)$. $\square$

Proof of Theorem 2.

Proof. We supplement the argument from the main text with additional details. First, we argue that the ergodic distribution $\mathbf{G}(\mathbf{b})$ is identified. We can rewrite it as follows:

$$\begin{aligned}\mathbf{G}(\mathbf{b}) = \mathbb{P}(\mathbf{B} \leq \mathbf{b} | \mathbf{B} \text{ is complete}) =& \frac{\mathbb{P}(\mathbf{B} \leq \mathbf{b} \ \wedge \mathbf{B} \ \text{ is complete})}{\mathbb{P}(\mathbf{B} \text{ is complete})} \\ =& \lim_{T \to \infty} \frac{\frac{1}{T}\sum_t 1(\mathbf{B}_t \leq \mathbf{b} \ \wedge \mathbf{B}_t \ \text{ is complete})}{\frac{1}{T}\sum_t 1(\mathbf{B}_t \ \text{ is complete})}\end{aligned}$$

where $\mathbf{B}$ is a random bid vector and $\mathbf{B}_t$ is sampled by selecting a bidder at random from each cohort of entrants, and then tracking their bids until they exit the market. The first two equalities in the display follow by definition and the definition of conditional probabilities. The last equality follows from the pointwise ergodic theorem: the averages in the numerator and the denominator tend asymptotically to the corresponding probabilities with respect to the ergodic measure of market states ω defined in Lemma 5. By expanding it in this way we can see the empirical objects required to estimate $\mathbf{G}(\mathbf{b})$: the probability that a randomly selected bidder from a cohort has a complete bid vector less than $\mathbf{b}$ and the probability that a randomly selected bidder has a complete bid vector, both of which are observable.

Second, we work through the deconvolution argument. $\mathbf{G}$ is a convolution of $\widetilde{\mathbf{F}}^S$ and $\mathbf{F}_\xi$ and $\mathbf{F}_\varepsilon$, where the (selected) bid-type $\check{\mathbf{b}}$ and $\boldsymbol{\xi}$ and $\boldsymbol{\varepsilon}$ are all mutually independent. Letting $\phi_X(\cdot)$ denoted the characteristic function of the random variable X, it follows that we have:

$$\phi_{\mathbf{b}}(t) = \phi_{\check{\mathbf{b}}}(t)\phi_{\boldsymbol{\xi}}(t)\phi_{\boldsymbol{\varepsilon}}(t)$$

Re-arranging yields:

$$\phi_{\check{\mathbf{b}}}(t) = \frac{\phi_{\mathbf{b}}(t)}{\phi_{\boldsymbol{\xi}}(t)\phi_{\boldsymbol{\varepsilon}}(t)}$$

where the numerator and denominator are both known. Moreover by the assumptions of Theorem 2 the denominator has only isolated real zeros so that the characteristic function $\phi_{\check{\mathbf{b}}}(t)$ is integrable with respect to Lebesgue measure. Integrating the characteristic function of $\check{\mathbf{b}}$ permits recovery of the distribution $\widetilde{\mathbf{F}}^S$. Finally, the density of bid-types is $\tilde{f}(\check{\mathbf{b}}) = k\tilde{f}^S(\check{\mathbf{b}})/s(\check{\mathbf{b}})$ where k is the observable probability that a randomly chosen bidder submits a complete bid. Finally, by Lemma 3 each bid-type can be inverted to recover the underlying valuation, and we have:

$$\mathbf{F}_X(x) = \mathbb{P}_{\widetilde{\mathbf{F}}}(\{\check{\mathbf{b}} : \zeta(\check{\mathbf{b}}) \leq \mathbf{x}\})$$

□

Proof of Corollary 1.

Proof. From (14) the permanent part of the each bidder's valuation is $\mathbf{x} = X\boldsymbol{\alpha}$ where α is sampled from $\mathbf{F}_\alpha$. Now since X has full rank, we can write $\boldsymbol{\alpha} = X^+\mathbf{x}$ where X^+ is the Moore-Penrose pseudo-inverse of X. So the mapping is invertible: i.e. each $\mathbf{x}$ corresponds to a unique $\boldsymbol{\alpha}$. It follows that the densities of $\mathbf{F}_\alpha$ and $\mathbf{F}$ are related: $f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = f_{\mathbf{x}}(X\alpha)$. Now from Theorem 2 the distribution $\mathbf{F}$ is identified, so we are done. □