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Diagnostic Business Cycles

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A large psychology literature argues that, due to selective memory recall, decision-makers' forecasts of the future are overly influenced by the perceived news. We adopt the diagnostic expectations (DE) paradigm (Bordalo et al. (2018)) to capture this feature of belief formation, develop a method to incorporate DE in business cycle models, and study the implications for aggregate dynamics. First, we address (i) the theoretical challenges associated with modeling the feedback between optimal actions and agents' DE beliefs and (ii) the time-inconsistencies that arise under distant memory (i.e. when news is perceived with respect to a more distant past than just the immediate one). Second, we show that under distant memory the interaction between actions and DE beliefs naturally generates repeated boom-bust cycles in response to a single initial shock. We also propose a portable solution method to study DE in dynamic stochastic general equilibrium models and use it to estimate a quantitative DE New Keynesian model. Both endogenous states and distant memory play a critical role in successfully replicating the boom-bust cycle observed in response to a monetary policy shock.

Key words: Diagnostic expectations, Boom-bust cycles, Imperfect memory recall.

1. Introduction

A large psychology and experimental literature documents that decision-makers' forecasts of the future appear overly influenced by news. In economics, this feature of belief formation has been captured by the diagnostic expectations (DE) paradigm, formulated recently by Bordalo et al. (2018), building on the representativeness heuristic by Kahneman and Tversky (1972). While promising in the breadth of its potential implications, so far the DE paradigm has focused on environments where news is exogenous and defined with respect to the immediate past. However, these two characteristics appear overly restrictive in applications. First, decisions often involve a feedback between agents' beliefs

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and endogenously determined economic states. Second, empirical evidence indicates that the selective memory recall may be based on more distant information sets.[1]

Motivated by these observations, we make three contributions. First, we develop micro-foundations to jointly address the theoretical challenges associated with modeling (i) the feedback between optimal actions and agents' DE beliefs over both exogenous and *endogenous* variables, and (ii) the time-inconsistencies in those optimal actions that arise when selective memory recall is based on a more *distant* past, rather than just the immediate past. Second, we show that under distant memory, but not under recent memory, the interaction between actions and DE beliefs naturally generate *endogenous* repeated boom-bust cycles in response to a single initial shock. Third, we propose a *portable solution method* to study DE in linear recursive macroeconomic models, which can thus accommodate large-scale dynamic stochastic general equilibrium models. We leverage the tractability of our proposed method to incorporate DE into an estimated quantitative New Keynesian model of the type widely used for policy analysis. We find that DE about endogenous states and distant memory recall play a critical role in replicating the empirically documented boom-bust cycle in response to a monetary policy shock.

In the recent formulation of Gennaioli and Shleifer (2010) and Bordalo et al. (2018), under the assumption of normality of the data generating process, DE distort current rational expectations (RE) with a term that depends on the difference between current RE and lagged RE (the reference, or comparison group). Thus, the size of distortion is proportional to the revision in RE. In Bordalo et al. (2018), this idea is formalized in terms of two parameters. A parameter θ controls the severity of the distortion, while a parameter J controls the time lag over which the RE revision is defined.

Applied theory contributions. In the first part of the paper, we use a three-period consumption-savings model to analyze the implications of DE beliefs under endogenous states and distant memory recall. We identify two theoretical challenges.

First, in an economic model, a future uncertain object (such as consumption) typically depends both on future exogenous forces (such as future income shocks) and on the agent's current endogenous actions (such as current savings). We emphasize how this latter, additional source of dependence, which we label *endogenous predictability*, can have important implications under selective memory. For instance, when the agent increases savings, she objectively raises the likelihood of high resources available for the next period and thus of future consumption levels. This increase in likelihood cues memory to oversample high future consumption scenarios, so the agent ends up being overly optimistic. Since these memory-distorted consumption beliefs in turn matter for the optimal savings choice, a feedback between distortions and actions arises. We micro-found this joint determination as the outcome of an intra-personal interaction between a *Memory* and *Deliberation* self and argue that this approach is consistent with recent evidence provided by Bordalo et al. (2022) showing that memory is to some extent spontaneous.

Second, when the reference point for the DE distortion depends on the distant past ($J>1$), as opposed to the immediate past ($J=1$), *the law of iterated expectations (LOIE) fails*. In a multi-period model, the failure of the LOIE is important because it leads to

1. For example, Bordalo et al. (2020) find that a reference belief based on the four quarters ago information set seems to account well for the empirical over-reaction observed in the surveys of professional forecasters, while Bordalo et al. (2019) argue that the sluggishness in expected returns is best explained by a reference information set eleven quarters in the past.

time-inconsistent choices. Intuitively, when the agent makes plans, she ignores the effect of current actions, such as savings, on future distorted beliefs about endogenous states, such as capital. As a result, when she is subsequently subject to those distorted beliefs, she disregards those past plans. To address the issue of time inconsistency, we adopt the *naïveté* approach (e.g. O'Donoghue and Rabin (1999)). The agent fails to take into account that her preferences are time-inconsistent and thinks that in the future she will make choices under perfect memory recall, or RE. However, when the future arrives, the agent ends up changing behavior and be again subject to her imperfect memory recall. We find the naïveté approach psychologically coherent and more consistent with the underlying foundation of diagnostic beliefs as a heuristic and a mental short-cut than the alternative approach of *sophistication*. Under sophistication, the agent fully understands how imperfect memory recall changes her future preferences. Furthermore, the naïveté approach turns out to be computationally more tractable, since the current naive agent does not need to internalize the life-time indirect effect of the current action on the formation of future comparison groups.

The consumption-savings model yields two critical insights. First, under DE the agent exhibits a *higher* marginal propensity to consume (MPC) out of temporary income shocks than under RE.[2] This is because the current high income innovation leads the agent to increase savings, which, through selective memory recall, cues an overly optimistic view of future consumption. Due to this 'as if' optimistic view induced by DE, the agent optimally consumes more and saves less today than under RE. Second, when the reference point is based on the more distant past ($J>1$), the surprise in the inherited capital relative to those more distant past expectations emerges as a novel informational state affecting decisions. This is because when $J>1$, in contrast to $J=1$, the inherited capital is *not* a sufficient state variable for the reference group in forming DE.

We showcase the importance of this novel informational state dynamics by studying an infinite-horizon extension of our setup: the permanent income hypothesis (PIH) model. Under distant memory, a single, temporary income shock can generate endogenous, repeated boom-bust cycles because past actions feed into current beliefs, that in turn inform new actions. Following a positive iid income shock, the income surprise creates optimism about the future, leading to over-consumption. This initial over-consumption is eventually followed by *disappointment* in the available amount of capital, as the reference group evolves to reflect past good times. Once again the agent over-reacts, cutting consumption more forcefully than what she would under RE, causing a bust. Subsequently, the over-accumulation of capital leads to positive surprises and renewed optimism. As a result of this surprise her consumption recovers, and so on. The larger the lag J of the reference expectation, the longer *and* more severe the under- and over-accumulation of capital.

Methodological contribution and quantitative evaluation. In the second part of the paper we first explain how to solve linear general equilibrium models in the presence of DE (under naïveté) by using standard solution methods, such as Sims (2000). In a

2. Jappelli and Pistaferri (2010), Kueng (2018), Fagereng et al. (2021) and McDowall (2020) find that the MPC out of temporary income shocks is puzzlingly large, even for financially un-constrained agents. Our mechanism differs from two recent related approaches that generate such high MPCs. Lian (2020) shows that (partial) sophistication is key for an agent to decide to save less today out of anticipation of future mistakes. In Ilut and Valchev (2022), agents are similarly naive as in our benchmark model, but have uncertainty over their optimal consumption functions, which endogenously leads to stable beliefs characterized by high MPC.

nutshell, the model is solved under the assumption that agents can observe the current state of the economy, but are subject to DE when they form expectations. In turn, DE are based on a linear transformation of a shadow RE law of motion. Our solution method is *portable, tractable* and, importantly, also allows for *general* forms of how memory recall loads on different past information sets.

We apply this solution method to incorporate DE into a quantitative New Keynesian model (Christiano et al. (2005), Smets and Wouters (2007)). Given our particular interest in the role played by distant memory recall, we model the reference distribution in a flexible manner, as a weighted average of lagged RE. We estimate the model using a Bayesian version of the impulse-response-function (IRF) matching method developed by Christiano et al. (2010), where the empirical IRFs are recovered using a local Jordà (2005) projection to a Romer and Romer (2004) monetary policy shock.

We find that the DE model reproduces the empirical IRFs to a monetary policy shock well, successfully generating a persistent and hump-shaped boom-bust cycle in consumption and other macro variables. In contrast, the RE model fails in delivering the empirical boom-bust and amplitude, indicating that DE are a critical force in the estimated model. As a result, the marginal likelihood, a Bayesian measure of fit that penalizes models with more parameters, heavily favors the estimated DE over the RE model. In addition, the DE model is also able to match remarkably well untargeted empirical responses, including the Survey of Professional Forecasters expectations on inflation and GDP growth.

Distant memory is crucial for the empirical success of the DE NK model. The estimated memory weights are centered on expectations formed five and six quarters ago. A re-estimated model imposing the constraint of recent memory ($J=1$) finds no role for DE, with the estimated IRFs identical to the RE ones. This is because distant memory and the degree of diagnosticity θ are *complements* and interact to amplify the role of DE through the feedback between memory and actions. Counterfactual simulations imposing smaller lags J highlight that distant memory affects both the magnitude and the duration of boom-bust cycles. The longer the lag in memory, the more consequential the actions taken by agents in the meantime.

As in the consumption-savings and the PIH models, this result can be understood in light of surprises in the capital stock. An expansionary monetary policy shock stimulates consumption and investment so the stock of capital increases and the agent is positively surprised by the resources available. Spending further rises, which in turn leads to more capital stock and further positive capital surprise. This virtuous feedback loop continues until the reference expectation of capital begins to catch up to the realized capital. The agent is then less optimistic about the future and begins cutting back on spending. Eventually, she becomes *disappointed* in the level of capital relative to the reference distribution formed at the height of the boom, leading her to over-correct. Consumption is now reduced below the steady-state level, pushing down the level of aggregate demand and capital accumulation. A bust period arises, where the feedback between beliefs and actions leads to further economic declines and disappointment.

In contrast to the consumption-savings and the PIH models, where the interest rate was constant, in the New Keynesian model the consumption boom-bust must now be accompanied by a corresponding movement in the interest rate. Perceived consumption growth under DE is linked to the perceived real interest rate. The perceived real rate is negatively related to the perception of changes in the price level and can be decomposed into (i) the one-step-ahead expectation of inflation under DE and (ii) the surprise in the current price level. We label this second term the *perceived innovation in cumulative* *inflation*, because the surprise in today's price level reflects the cumulative inflation between the current period and the time at which reference expectations were formed. On impact, because of an increase in utilization, inflation declines. This determines a negative surprise in the price level and a lower than usual expected future price level that is consistent with a perceived acceleration in consumption. Inflation eventually starts picking up, leading to a reduction in the negative surprises for the price level and then to positive surprises. This path determines a *reversal* in the perceived innovation in cumulative inflation, which moves into the positive territory during the bust part of the cycle, when agents find the resulting high perceptions of future price level consistent with their pessimism about future consumption.

Our paper is closely related to some recent contributions that study DE in macro models. Bordalo et al. (2019) analyze DE about a TFP process to account for credit cycles, Maxted (2020) builds a He and Krishnamurthy (2019) style macro-finance model featuring DE, while d'Arienzo (2020) introduces DE into a term-structure model to study bond market puzzles. L'Huillier et al. (2021) further share a similar interest with us in introducing DE into linear, dynamic general equilibrium models. We contribute to the literature in two key ways. First, we address the conceptual challenges of modeling the role of endogenous states and distant memory recall in jointly affecting DE beliefs and optimal actions. In particular, compared to this existing work, we confront the problem of time inconsistency by providing a behavioral foundation of naïveté and sophistication and show that distant memory is necessary to generate repeated boom-bust cycles in response to a single initial shock.[3] Second, in quantitative terms, we propose and use an easily portable solution method to estimate a New Keynesian model with DE to show that the feedback between actions and distant memory is critical in replicating the boom-bust cycle we recover from the data.

2. DE with endogenous states and distant memory

Consider an environment where the only source of stochasticity is a univariate process Y_t. Time is discrete and indexed by t. Let Y^t denote the history $\{Y_0,...,Y_t\}$ of Y_t realizations up to, and including, time t. The true data generating process is

$$Y_{t+1} = \mu_Y(Y^t) + \varepsilon_{t+1}, \tag{2.1}$$

where ε_{t+1} are mean-zero iid normal shocks with variance $\sigma^2 > 0$. Here $\mu_Y(.)$ gives the time t conditional mean, as a function of current and past Y_t realizations.

Endogenous predictability. We now introduce the process through which selective memory recall distorts subjective forecasts in the presence of endogenous states. Let C_{t+1} be the random variable that the agent is interested in forecasting (e.g. future consumption or future wealth). Suppose that its underlying data generating process is

$$C_{t+1} = \mu_{C,Y}(Y^t) + \mu_{C,K}(Y^t) + \delta\varepsilon_{t+1}. \tag{2.2}$$

3. In this context, L'Huillier et al. (2021) study the role of endogenous states in driving DE beliefs, but their analysis and solution method applies only when memory is based on the immediate past. d'Arienzo (2020) explores the LOIE failure as a mechanism for a maturity increasing overreactions of expectations to news. Here we connect this failure to time-inconsistency and study it in models with endogenous states.

In the expression above, $\mu_{C,Y}(.)$ captures the *exogenous predictability* in C_{t+1} that arises because of the exogenous state Y^{t+1} while $\mu_{C,K}(.)$ reflects the *endogenous predictability* component. In an economic model, in which C_{t+1} is partly endogenously-determined, this component arises from actions that are optimally set as a response to Y^t. Finally, $\delta \neq 0$ reflects the exposure of C_{t+1} to ε_{t+1}, ensuring that the former is also non-predetermined as of time t. The two sources of predictability can be summarized as an overall predictability:

$$\mu_C(Y^t) \equiv \mu_{C,Y}(Y^t) + \mu_{C,K}(Y^t). \tag{2.3}$$

To illustrate how the two sources of predictability interact with each other, let C_{t+1} be future consumption determined by a budget constraint as $Y_{t+1}+K_t$, where Y_{t+1} is a stochastic labor income with a conditional mean $\mu_Y(Y^t) = \rho Y_t$ and K_t is accumulated savings. For the sake of the argument, assume that K_t is chosen as αY_t. Then, the underlying, true density for C_{t+1} has $\delta = 1$ and overall predictability

$$\mu_C(Y^t) = (\rho+\alpha) Y_t. \tag{2.4}$$

Diagnostic beliefs. We now discuss how selective memory may distort the agent's probability judgments over C_{t+1}. For this purpose, we build on Gennaioli and Shleifer (2010) and Bordalo et al. (2018), who formulate a behavioral model of *diagnostic expectations* (DE). The psychological first-principle basis in this model is limited and selective memory retrieval: subjective probability assessments overweigh event realizations that easily come to mind because they are "representative," in the sense of the Kahneman and Tversky (1972). The "representative" heuristic has been documented by a large psychology and experimental literature (e.g. Bordalo et al. (2018), Bordalo et al. (2021), Bordalo et al. (2022)).

The basic intuition of DE is that the judged probability of an otherwise uncertain event partly reflects its 'true,' objective, frequency (the 'kernel of truth') as well as a subjective element that reflects the accessibility of that event in the agent's memory. When new information arrives, memory selectively recalls more vividly past events that are more associated with, or representative of, the current news. In our context, we describe the DE model as the distorted density

$$h_t^\theta(\widehat{C}_{t+1}) = h(\widehat{C}_{t+1}|\mu_C(Y^t) = \mu_C(\widehat{Y}^t)) \left[\frac{h(\widehat{C}_{t+1}|\mu_C(Y^t) = \mu_C(\widehat{Y}^t))}{h(\widehat{C}_{t+1}|\mu_C(Y^t) = \mathbb{E}_{t-J}[\mu_C(Y^t)])} \right]^\theta \frac{1}{a} \tag{2.5}$$

where a is an integration constant that ensures that $h_t^\theta(\widehat{C}_{t+1})$ integrates to one. We use 'hats' when needed to emphasize the specific realization of any given random variable.

There are several important elements in this distorted distribution $h_t^\theta(\widehat{C}_{t+1})$, all of which appear in some form in the earlier formulations of DE (e.g. Bordalo et al. (2018)). First, $h(\widehat{C}_{t+1}|\mu_C(Y^t) = \mu_C(\widehat{Y}^t))$ is the true density in equation (2.2) of a future realization $\widehat{C}_{t+1}$ for a given current realization of the conditional mean, $\mu_C(\widehat{Y}^t)$. Second, $\mathbb{E}_{t-J}[\mu_C(Y^t)]$ is the *comparison group* for the random variable $\mu_C(Y^t)$. $\mathbb{E}_{t-J}[.]$ denotes the expectation operator for any arbitrary random variable conditional on $t-J$ information (i.e. conditional on Y^{t-J}) under the true law of motion for Y_t in equation (2.1). This comparison group gives the state prevailing if there is no news, compared to the immediate ($J=1$), or more distant past ($J>1$). Third, $\theta > 0$ captures the strength of the impact of representativeness on judgments. If $\theta = 0$, memory recall is frictionless and the heuristic has no effects. Fourth, the distorted density in equation (2.5) applies if and only if the conditional variance $\delta^2\sigma^2 > 0$ in equation (2.2). When $\delta^2\sigma^2 = 0$, the conditional likelihood of observing a scenario for $\widehat{C}_{t+1}$ other than the one that the agent is now fully informed of (given by $\mu_C(\widehat{Y}^t)$) has become equal to zero. As noted by Gennaioli and Shleifer (2010), the lack of such conditional (or "residual") uncertainty leaves no room for memory to distort conditional forecasts.[4]

Bordalo et al. (2018) show how the normality assumption of $h(.)$ leads to a tractable characterization of $h_t^\theta(.)$. Specifically, compared to $h(.)$, under the DE density $h_t^\theta(.)$ the random variable C_{t+1} remains conditionally normally distributed with the same variance $\delta^2\sigma^2 > 0$, but a distorted conditional mean

$$\mathbb{E}_t^\theta(C_{t+1}) = \mu_C(\widehat{Y}^t) + \underbrace{\theta[\mu_C(\widehat{Y}^t) - \mathbb{E}_{t-J}[\mu_C(Y^t)]]}_{=\text{Memory distortion}} \tag{2.6}$$

The DE *memory distortion* term captures the *over-reaction* of the conditional mean to the new information. This distortion is proportional by a factor of θ to the 'surprise' in the realized conditional mean. The surprise in the RE mean, in turn, depends on the new information arising between time $t-J$ and current t, i.e. the realized path for $Y_{t-J+1},...Y_t$, through the function μ_C. Embedding (2.6) in economic models presents two challenges: (i) the joint determination of actions and DE beliefs and (ii) the failure of LOIE under distant memory. We address each of these issues in turn.

The 'kernel of truth' $\mu_C(Y^t)$ that appears in the distorted density captures the overall statistical predictability. Under this proposed representation, all sources of predictability are treated symmetrically and jointly. In our example, it means that memory recall does not treat the component due to *exogenous predictability*, $\mu_{C,Y}(Y^t)$, as different from the component due to *endogenous predictability*, i.e. $\mu_{C,K}(Y^t)$. As we discuss below, this assumption, which conforms with recent evidence provided by Bordalo et al. (2021) and Bordalo et al. (2022) showing that memory is in critical ways spontaneous, has important implications.

Joint endogenous determination. The distorting effects of DE discussed above, motivated by empirical and conceptual appeal, take as given the endogenously-determined predictability through $\mu_{C,K}(Y^t)$ in equation (2.2). When such predictability arises from actions that are in turn affected by DE beliefs, a first theoretical challenge arises — optimal actions and DE beliefs need to be jointly determined. For example, in a consumption-savings model, optimal savings is (i) endogenously affected by DE beliefs over future consumption (like $\mathbb{E}_t^\theta(C_{t+1})$ in equation (2.6)) and (ii) in turn affects the underlying statistical process for consumption, over which selective memory recall distorts those DE beliefs (like the conjectured response coefficient α in equation (2.4)).[5]

4. This natural property also implies that if the agent is *only* interested in forecasting (or 'now-casting') a random variable like $\mu_C(Y^t)$, conditional on observing Y^t, then $\mathbb{E}_t^\theta[\mu_C(Y^t)] = \mathbb{E}_t[\mu_C(Y^t)] = \mu_C(\widehat{Y}^t)$. In the language of Bordalo et al. (2018), to compute $\mathbb{E}_t^\theta[\mu_C(Y^t)]$ the realization $\mu_C(\widehat{Y}^t)$ constitutes its infinitely representative state (see appendix in Bordalo et al. (2018) on Corollary 1).

5. For endogenous predictability to arise, a feedback between beliefs and actions is necessary: If $\alpha = 0$ in equation (2.4), there is no endogenous predictability. The Online Appendix presents an investment model that illustrates this point.

Distant memory and LOIE. A second conceptual challenge in economic models driven by DE arises because the Law of Iterated Expectations (LOIE) fails when DE is based on distant memory. To see this issue in the environment summarized by equations (2.1) and (2.3), consider some arbitrary periods $t>J$, integer $n\geq 1$, and some comparison group $t-J$ in equation (2.5), where $J\geq 1$. As discussed in Bordalo et al. (2018), Corollary 1, we can establish (proof in the Online Appendix) that

Lemma 1. *LOIE holds generically under DE, i.e.* $\mathbb{E}_t^\theta\left[\mathbb{E}_{t+1}^\theta[C_{t+1+n}]\right]=\mathbb{E}_t^\theta[C_{t+1+n}]$*, if and only if memory is based on the immediate past (i.e.* $J=1$*).*

The key term in $\mathbb{E}_t^\theta\left[\mathbb{E}_{t+1}^\theta[C_{t+1+n}]\right]$ that determines if LOIE holds is the perceived surprise

$$\mathbb{E}_t[C_{t+1+n}]-\mathbb{E}_t[\mathbb{E}_{t+1-J}[C_{t+1+n}]]. \tag{2.7}$$

Intuitively when memory is based on a more distant past ($J>1$), the time t expectation over the $t+1$ DE forecast of C_{t+1+n} introduces an *additional* lagged forecast (here $\mathbb{E}_{t+1-J}[C_{t+1+n}]$) which would not be otherwise included in the time t DE forecast of C_{t+1+n} itself. Indeed, the key term in equation (2.7) becomes generically zero if and only if $J=1$.

We show below that the LOIE failure under distant memory typically leads to time-inconsistency in optimal actions. To confront this issue, we study two separate frameworks that differ in regards to how we model the agent's beliefs about her future actions (i.e, naïveté vs. sophistication, as for example in O'Donoghue and Rabin (1999) and Laibson (1997)).

Plan. The rest of the paper is organized as follows. In Section 3 we use a simple consumption-smoothing problem to present our approach to the joint determination of DE beliefs and optimal actions in the presence of endogenous states, with recent and distant memory. In Section 4 we incorporate DE into a quantitative New Keynesian model of the type widely used for policy analysis, showcasing both the methodological tractability and quantitative success of our approach.

3. Joint determination of beliefs and actions under DE

We start with a three-period consumption-savings model to detail our approach. An agent born at a generic time 1 inherits beliefs from J periods ago and capital K_0 from last period. In each period, the agent receives the exogenous income $Y_t=\overline{Y}+\varepsilon_t$ for $t=1,2,3$, where ε_t are mean zero iid normal shocks with variance $\sigma^2>0$. At time 1, the agent chooses actual savings K_1 and a contingent plan for savings at time 2, K_2, to maximize current utility and the expected discounted sum of future utilities. Here we also assume, for simplicity, a real interest rate $r=0$, a discount factor $\beta=1$, and a quadratic utility function $u(C)=bC-.5C^2$, where $b>0$ and $C<b$ are such that utility is increasing in consumption in that region. Her optimal end-of-life $K_3=0$ in both the RE and DE problems below, since we rule out bequests motives.

3.1. Rational Expectations solution

We first solve the model under RE. At time 1, the agent solves

$$\max_{K_1,K_2} \{u(C_1)+\mathbb{E}_1[u(C_2)+u(C_3)]\} \tag{3.8}$$

$$s.t.\ C_1=Y_1+K_0-K_1;\ \ C_2=Y_2+K_1-K_2;\ \ C_3=Y_3+K_2-K_3.$$

and at time 2 the agent solves, given the inherited capital K_1,

$$\max_{K_2}\{u(C_2)+\mathbb{E}_2[u(C_3)]\} \tag{3.9}$$

$$s.t.\ C_2=Y_2+K_1-K_2;\ \ C_3=Y_3+K_2-K_3.$$

Proposition 1.. *The solution under RE is time consistent and is given by*

$$K_1=\alpha^{RE}_{K_0}K_0+\alpha^{RE}_{\varepsilon_1}\varepsilon_1, \quad K_2=\alpha^{RE}_{K_1}K_1+\alpha^{RE}_{\varepsilon_2}\varepsilon_2,$$

where $\alpha^{RE}_{K_0}=\frac{2}{3}=\alpha^{RE}_{\varepsilon_1}$ *and* $\alpha^{RE}_{K_1}=\frac{1}{2}=\alpha^{RE}_{\varepsilon_2}$.

First, under RE, the agent spreads her resources to achieve a perfectly smooth consumption path (in expectation). Second, the marginal propensity to save is invariant to the origin (savings K_0 and K_1 or income shock ε_1 and ε_2) of available resources. Finally, time consistency means that the planned savings policy $K_2(K_1,\varepsilon_2)$ under time 1 problem (3.8) coincides with the optimal policy $K_2(K_1,\varepsilon_2)$ under time 2 problem (3.9).

3.2. Diagnostic Expectations solution

We now introduce DE. We use θ,p-superscripts and θ-superscripts to denote planned choices and equilibrium choices, respectively, under the DE problem. The time 1 problem is to choose actual savings $K_1^{\theta,p}$ and a contingent plan for $K_2^{\theta,p}$ to maximize

$$\max_{K_1^{\theta,p},K_2^{\theta,p}} \left\{u(C_1^{\theta,p})+\mathbb{E}_1^\theta\left[u(C_2^{\theta,p})+u(C_3^{\theta,p})\right]\right\} \tag{3.10}$$

$$s.t.\ C_1^{\theta,p}=Y_1+K_0-K_1^{\theta,p};\ \ C_2^{\theta,p}=Y_2+K_1^{\theta,p}-K_2^{\theta,p};\ \ C_3^{\theta,p}=Y_3+K_2^{\theta,p}-K_3,$$

where recall that trivially $K_3=0$. The first-order conditions are:

$$C_1^{\theta,p}=\mathbb{E}_1^\theta\left[C_2^{\theta,p}\right], \quad \mathbb{E}_1^\theta\left[C_2^{\theta,p}\right]=\mathbb{E}_1^\theta\left[C_3^{\theta,p}\right]$$

At time 2, conditional on K_1^θ and Y_2, the agent re-optimizes (and thus may exhibit time-inconsistency) over her initially planned $K_2^{\theta,p}$, by looking for a K_2^θ that solves

$$\max_{K_2^\theta}\left[u(C_2^\theta)+\mathbb{E}_2^\theta u(C_3^\theta)\right] \tag{3.11}$$

$$s.t.\ C_2^\theta=Y_2+K_1^\theta-K_2^\theta;\ \ C_3^\theta=Y_3+K_2^\theta-K_3.$$

The first-order condition at time 2 is:

$$C_2^\theta = \mathbb{E}_2^\theta\left[C_3^\theta\right] \tag{3.12}$$

Before solving the model and discussing the differences between recent and distant memory, we find it useful to explain how to interpret the optimality conditions that we just derived.

3.2.1. Deliberation and memory with endogenous states When taking DE of endogenous variables, the primitive data generating process is *not* invariant to the agent's choices. Through her actions, the agent makes some future states of the world more or less likely. This is an *objective* consequence of the actions taken by the agent. Under the representativeness heuristic underlying DE, these states that become objectively more likely are oversampled, in the sense that the agent more easily retrieves them from memory. As a result, she comes to perceive them as even more likely than what they really are. In the consumption-savings problem at hand, when the agent increases savings compared to usual times, she effectively increases the likelihood of high resources available for the next period and thus of future consumption levels. This increase in likelihood cues memory to oversample high future consumption realizations, so the agent ends up being too optimistic. The kernel of truth property is satisfied, as high consumption states are in fact more likely following high savings, but the agent fails to appreciate the exact increase in probability. In particular, due to the representativeness heuristic the agent's memory overstates the increase in this likelihood. In the presence of endogenous states, the kernel of truth underlying the memory distortion is thus in itself a function of the agent's actions.[6]

We tackle the joint determination of the savings action and DE beliefs over consumption by micro-founding it as a *fixed point* between the optimal actions taken by *Deliberation* (self D) and the diagnostic expectations formed by *Memory* (self M). In particular, building on the logic above, self M's selective sampling of consumption states is cued by self D's savings action. In turn, self D optimally chooses savings to smooth out consumption, taking *as given* the current economic states *and* the conditional DE forecasts over future consumption provided by self M. In other words, self D does not internalize the effect of her actions on that selective sampling of self M. The fixed point is the self D's optimal savings derived under self M's selective sampling cued in turn by that particular optimal savings action.

We now detail this interaction between deliberation and memory in the simplest laboratory, namely the time-2 optimization in (3.11). Due to the iid assumption on the exogenous income process, the conditional predictability of future consumption comes here entirely from the current savings choice. Like in Section 2 (see equation (2.6)), the memory distortion over future consumption is proportional to the surprise in the

6. This notion that actions induce oversampling may arise naturally in other settings. For instance, a driver, through the act of wearing a seat belt, makes scenarios in which the seat belt prevents fatal injuries more likely. A diagnostic driver then oversamples these representative scenarios, ending up too optimistic compared to the objective distribution. Interestingly, this example is related to the theory of "risk compensation" in the medical field (see Hedlund (2000) for an overview of the literature). According to the theory, an action that improves one's safety (for example, wearing a seat belt) could cause the person to behave in a riskier manner (for example, speeding) that may, in the extreme case, make the overall act (here, driving) less safe.

conditional mean of consumption, brought upon here entirely by the perceived surprise in the savings action.

Let $K^M_{2,2-J}$ denote self M's reference value for the current time-2 savings, a comparison value based on the information set J periods ago. Let K^M_2 denote the value of current savings as perceived by self M and K^D_2 denote a given value of the current savings choice taken by self D. For any given $(K^D_2, K^M_2, K^M_{2,2-J})$ the distorted conditional expectation over C^θ_3 is

$$\mathbb{E}^\theta_2[C^\theta_3] = \overline{Y} + K^D_2 + \underbrace{\theta\left[K^M_2 - K^M_{2,2-J}\right]}_{=\text{Memory Distortion of Self } M} \tag{3.13}$$

which reflects the over-reaction property of DE to time-2 new information, in the form of the *memory distortion* $\theta\left[K^M_2 - K^M_{2,2-J}\right]$.

Intuitively, the conceptual distinction between K^M_2 and K^D_2 models an intra-personal cognitive process where Deliberation takes as given the memory distortion and does not actively manipulate Memory. In equation (3.13) self D views the current action K^D_2 as the only component of C^θ_3 that she can control, by reasoning that one extra unit of savings today must mean one more unit of available C^θ_3 tomorrow. In contrast, she takes as given the memory distortion part of expected C^θ_3 as a component that self D herself cannot affect.

Formally, in equation (3.13), self D internalizes the (one-to-one) effect on savings K^D_2 on $\mathbb{E}_2[C^\theta_3] = \overline{Y} + K^D_2$ and takes as given the memory objects $(K^M_2, K^M_{2,2-J})$. Thus, self D does not internalize the *equilibrium consistency* restriction that we impose in this intra-personal process, namely that the current memory perception K^M_2 is consistent with deliberation, i.e.

$$K^M_2 = K^D_2. \tag{3.14}$$

As in Section 2, our approach continues to be motivated by the psychological foundation of diagnostic beliefs as spontaneous, less than fully deliberative, selective memory recall (Bordalo et al. (2021), Bordalo et al. (2022)).

Self D's time-2 problem in (3.11) is to set the optimal K^D_2 to maximize current and expected utility, given $(K^M_2, K^M_{2,2-J})$ in (3.13). The first-order condition states that self D chooses an optimal savings action, denoted by K^θ_2, to implement perceived consumption smoothing

$$C^\theta_2 = \mathbb{E}^\theta_2[C^\theta_3]$$

recovering our earlier condition (3.12). In particular, this equality of perceived consumption growth is implemented by an optimal choice K^θ_2 such that

$$Y_2 + K^\theta_1 - K^\theta_2 = \overline{Y} + K^\theta_2 + \theta\left[K^\theta_2 - K^M_{2,2-J}\right]. \tag{3.15}$$

Here $\mathbb{E}^\theta_2[C^\theta_3]$ is evaluated using the deliberation-memory consistency of equation (3.14) where we have set $K^M_2 = K^\theta_2$ to reflect the fact that in equilibrium the size of the memory distortion depends on the specific optimal choice made by Deliberation. As such, equation (3.15) characterizes the feedback between Deliberation and Memory as a fixed point in driving the joint determination of current optimal choices and current memory distortion. Indeed, the latter reflects self M's oversampling of consumption states as induced by the surprise between the current realized optimal K_2^θ and the given reference value $K_{2,2-J}^M$. In turn, the choice K_2^θ is optimal, given the economic states (Y_2, K_1^θ) and the resulting memory distortion.

This analysis has taken as given $K_{2,2-J}^M$. We build on the DE approach discussed in Section 2 to construct the comparison groups as an internally consistent past forecast of current behavior. In particular, we model the time-t comparison group as the conditional RE forecast of time t optimal behavior as anticipated J periods ago. We now analyze the specific qualitative behavioral implications of the feedback between deliberation and memory, allowing for a comparison group influenced by the recent $(J=1)$ or more distant past $(J>1)$.

3.2.2. Recent memory ($J=1$) We prove two properties when memory recall is based on the immediate past, i.e. $J=1$. First, the DE solution is time consistent. Second, DE leads to a higher MPC than under RE.

Proposition 2.. *When $J=1$, the conditional time-2 optimal solution $K_2^\theta(K_1^\theta, \varepsilon_2)$ is identical (‘time-consistent’) to the time-1 optimal contingent plan $K_2^{\theta,p}(K_1^\theta, \varepsilon_2)$.*

To see this, note that while the optimal time-1 plan $K_2^{\theta,p}$ in equation (3.10) is set such that $\mathbb{E}_1^\theta\left[C_2^{\theta,p} - C_3^{\theta,p}\right] = 0$, the conditional optimal K_2^θ solves the time-2 perceived tradeoff $C_2^\theta - \mathbb{E}_2^\theta[C_3^\theta] = 0$. Since the LOIE holds when $J=1$ (Lemma 1), the conditional optimal K_2^θ implements exactly the time-1 desired consumption path under $K_2^{\theta,p}$:

$$\mathbb{E}_1^\theta\left[C_2^\theta - C_3^\theta\right] = \mathbb{E}_1^\theta\left[C_2^\theta - \mathbb{E}_2^\theta C_3^\theta\right] = 0.$$

Guided by the RE solution, we conjecture an optimal policy of a similar form:

$$K_1^\theta = \alpha_{K_0}^\theta K_0 + \alpha_{\varepsilon_1}^\theta \varepsilon_1;\ K_2^\theta = \alpha_{K_1}^\theta K_1^\theta + \alpha_{\varepsilon_2}^\theta \varepsilon_2.$$

Proposition 3.. *When $J=1$, compared to the RE policy functions, the optimal policy functions K_1^θ and K_2^θ feature the same optimal response to the endogenous state but a muted response to the current income innovation, i.e.*

$$\alpha_{K_0}^\theta = \frac{2}{3} = \alpha_{K_0}^{RE}; \quad \alpha_{K_1}^\theta = \frac{1}{2} = \alpha_{K_1}^{RE}; \quad \alpha_{\varepsilon_1}^\theta = \frac{2}{3+\theta} < \alpha_{\varepsilon_1}^{RE}; \quad \alpha_{\varepsilon_2}^\theta = \frac{1}{2+\theta} < \alpha_{\varepsilon_2}^{RE}.$$

The intuition for the muted savings response to the income shock is at the heart of our *endogenous predictability* mechanism. To see this, recall the $t=2$ optimality condition in equation (3.15), and substitute in the time 2 comparison value $K_{2,2-J}^M = \mathbb{E}_1[K_2^\theta]$ to get:[7]

$$\varepsilon_2 + K_1^\theta - K_2^\theta = K_2^\theta + \theta\left[K_2^\theta - \mathbb{E}_1\left[K_2^\theta\right]\right]. \tag{3.16}$$

7. The time-consistency property derived above motivates the time-2 comparison group to be $\mathbb{E}_1[K_2^\theta]$, i.e. the conditional RE forecast of time 2 optimal behavior as anticipated last period.

Given a current unusually high income shock ε_2, and thus (in equilibrium) a higher than usual level of assets K_2^θ, the agent correctly recognizes that her total future resources and consumption are likely to be higher than usual. Since income Y_3 is iid, this conditional predictability of future resources comes just from K_2^θ, which through the response $\alpha_{\varepsilon_2}^\theta$ induces the *endogenous* persistence from ε_2 to C_3^θ. As described in the motivating Section 2, an agent subject to the representativeness heuristic is then *overly influenced* by her perception of the new information contained in this unusual state of high total expected resources $\overline{Y}+K_2^\theta$. Selective memory recall thus associates the current situation of higher than usual savings with *optimism* about total future resources. Given this 'as if' optimistic view induced by DE, the agent optimally consumes more and saves today less than the RE agent.

In the specific language developed in Subsection 3.2.1, Deliberation increases savings K_2^θ to smooth consumption, taking as given the higher than usual current income shock ε_2, the state K_1^θ, and the self M's memory distortion. In turn, self M is spontaneously triggered by the higher than usual savings action K_2^θ to over-inflate high future consumption states. The feedback between Deliberation and Memory thus leads to more current savings than usual (i.e. the response $\alpha_{\varepsilon_2}^\theta$ is positive) but, importantly, to less savings than under RE (i.e. $\alpha_{\varepsilon_2}^\theta < \alpha_{\varepsilon_2}^{RE}$). The observed consumption behavior is thus characterized by a *higher* marginal propensity to consume (MPC) than implied by RE, as well as a lack of consumption smoothing as measured by an external observer.[8]

Let us turn now to the DE response to the endogenous state K_1^θ. The key economic observation here is that when $J=1$, the economic state K_1^θ also serves as the *necessary and sufficient* conditioning information to form the comparison group $\mathbb{E}_1[K_2^\theta]$. Therefore, the DE beliefs' over-reaction to the new information, $K_2^\theta - \mathbb{E}_1[K_2^\theta]$, only contains the current innovation ε_2 and not the endogenous state K_1^θ, that was already known when the reference expectations were formed. Thus, when $J=1$, DE affect the reaction to ε_2, but not to K_1^θ. Indeed, the latter response remains identical to the RE one.

3.2.3. Distant memory ($J>1$)

Now consider the case when memory recall is based on the more distant past, i.e. $J>1$.

Proposition 4.. *When $J>1$, the conditional time-2 optimal solution for K_2^θ is not equal to the time-1 optimal contingent plan $K_2^{\theta,p}(K_1^\theta,\varepsilon_2)$ (i.e. it is not 'time-consistent').*

This result is a direct manifestation of the LOIE failure under distant memory (Lemma 1). As we show in the Online Appendix, the time inconsistency arises because of the information content of $K_1^{\theta,p}$ with respect to the capital expected at time 0 (assuming $J=2$). Between when reference expectations were formed, at time 0, and when a new decision is made, at time 2, an income shock occurred and agents reacted to the shock. As a result, capital is not what the agent expected it to be. Agents do not take into account this surprise in capital when they solve the planning problem at time 1.

8. The latter measurement immediately follows since the true conditional distribution for consumption does not coincide with the distorted one. We also note that our simple economic model does not feature financial constraints, usually viewed as the standard economic reason for high MPC. Thus, our model's implication speaks closer to the challenge posed to standard models by the empirical evidence on high MPC out of temporary income shocks of financially un-constrained agents (see for example evidence in Kueng (2018), Fagereng et al. (2021) and McDowall (2020)).

Faced with this inherent time-inconsistency, we then need to model the agent's current beliefs about her future actions. Here we use insights from existing literature on time-inconsistency (e.g. the seminal work by Strotz (1955) and Pollak (1968)) that point to two different frameworks. The first approach, coined in this literature as *naïveté* in the O'Donoghue and Rabin (1999) sense, used for example in Akerlof (1991), models an agent who does not forecast her future self's behavior to be governed by the representativeness heuristic. The second approach is *sophistication* (e.g. Laibson (1997)), where the agent understands that her future action is dictated by the representativeness heuristic.[9]

Naïveté problem. Under *naïveté* , the time 1 problem is

$$\max_{K_1^\theta}\left\{u(C_1^\theta)+\mathbb{E}_1^\theta\left[u(C_2^{RE})+u(C_3^{RE})\right]\right\} \tag{3.17}$$

where the agent at time 1 believes her time 2 future self will take the action K_2^{RE} so to

$$\max_{K_2^{RE}}\left[u(C_2^{RE})+\mathbb{E}_2[u(C_3^{RE})]\right]. \tag{3.18}$$

The RE-superscript on a time t variable signify choices that are made under an RE policy function, taking as given the state variable entering that period. From the budget constraints, the (forecasted) consumption choices are therefore

$$C_1^\theta=Y_1+K_0-K_1^\theta;\ C_2^{RE}=Y_2+K_1^\theta-K_2^{RE};\ C_3^{RE}=Y_3+K_2^{RE}-K_3^{RE},$$

where K_1^θ (and K_2^{RE}) signify the choice resulting from a DE under naïveté (and RE, respectively) policy function that solve (3.17) (and (3.18), respectively) and trivially $K_3^{RE}=0$.

The optimal solution for K_1^θ in equation (3.17) solves the intertemporal tradeoff

$$C_1^\theta=\mathbb{E}_1^\theta\left[C_2^{RE}+\frac{\partial K_2^{RE}}{\partial K_1^\theta}\left(C_3^{RE}-C_2^{RE}\right)\right], \tag{3.19}$$

This captures the direct effects of the current choice on tomorrow's consumption and the indirect effects through the capital choice at time 2, anticipated to follow K_2^{RE} (where $\frac{\partial K_2^{RE}}{\partial K_1^\theta}=0.5$ by Proposition 1) and the resulting consumption path.

Lemma 2. *Under naïveté, for any $J\geq 1$, the tradeoff for the optimal K_1^θ in equation (3.19) reduces to*

$$C_1^\theta=\mathbb{E}_1^\theta[C_2^{RE}]. \tag{3.20}$$

9. As in the present bias literature, it is possible to think about intermediate cases in which agents are "partially naive," in the sense that they are aware that their beliefs are distorted, but they fail to correctly assess the severity of the distortion. One possible way to model this idea is by introducing a perceived distortion parameter $\hat{\theta}$ that differs from the true distortion parameter θ. This partial naïveté model would nest the pure naïveté case ($\hat{\theta}=0$) and the pure sophistication case ($\hat{\theta}=\theta$). We thank an anonymous referee for pointing out this possible extension.

The key for this result is that the future self is expected to optimally select K_2^{RE}, which conditional on time 2 states achieves $\mathbb{E}_2[C_3^{RE}]-C_2^{RE}=0$. Thus, $(C_3^{RE}-C_2^{RE})$ equals just the income innovation ε_3, unpredictable under $\mathbb{E}_1^{\theta}$ (for any $J\geq 1$). Due to this induced unpredictability, for the naive agent the indirect effect of K_1^{θ} as a relevant state for future conditionally optimal choices can be ignored — a form of envelope-theorem result that makes the problem particularly tractable.

While these are her beliefs at time 1, entering period 2 with the state realization K_1^{θ} and new information determined at time 2, her problem is once again influenced by the representativeness heuristic. Her conditionally optimal action solves

$$\max_{K_2^{\theta}}\left[u(C_2^{\theta})+\mathbb{E}_2^{\theta}[u(C_3^{RE})]\right], \tag{3.21}$$

where $C_2^{\theta}=Y_2+K_1^{\theta}-K_2^{\theta}$ and $C_3^{RE}=Y_3+K_2^{\theta}-K_3^{RE}$. The optimal action implements

$$C_2^{\theta}=\mathbb{E}_2^{\theta}[C_3^{RE}]. \tag{3.22}$$

The behavioral interpretation of equations (3.17), (3.18) and (3.21) is that, at time 1, the agent maximizes assuming that after time 2 the future selves will not be subject to any memory heuristics (i.e. she will act ‘fully rationally’), even though at time 2 the decision maker ends up changing behavior and is in fact subject to her imperfect memory recall.

Joint naive beliefs and actions. To characterize the resulting optimal actions and DE beliefs, we can extend the terminology introduced earlier in Subsection 3.2.1 around Memory and Deliberation selves to account for the naïveté assumption. In particular, similar to the argument around equations (3.13) and (3.15), we look to impose consistency of beliefs for the Memory self of the naive agent.

There are two properties of consistency that arise here. First, the ‘kernel of truth’ component part is

$$\mathbb{E}_2[C_3^{RE}]=\overline{Y}+K_2^{\theta}-\mathbb{E}_2[K_3^{RE}],$$

where K_2^{θ} is the optimal choice by the Deliberation self (which Memory self takes as given) and we make specific the role of perceived future optimal behavior under RE through $\mathbb{E}_2[K_3^{RE}]$.

Second, the comparison group for the time-2 naive Memory self is the RE forecast made by the former (J periods ago) naive self about time-3 perceived optimal behavior. Since naive agents at any given time believe future optimal behavior to be governed by the RE policy, the comparison groups are also built under forecasts of future optimal RE behavior. Therefore, this internal consistency requires that the comparison group is built as

$$\mathbb{E}_{2-J}\left[\mathbb{E}_2[C_3^{RE}]\right]=\mathbb{E}_{2-J}\left[\overline{Y}+K_2^{RE}-\mathbb{E}_2[K_3^{RE}]\right].$$

Put together, the naive Deliberation self solves the trade-off in (3.22), where the naive Memory self’s beliefs are:

$$\mathbb{E}_2^{\theta}[C_3^{RE}]=\mathbb{E}_2[C_3^{RE}]+\theta\left[\mathbb{E}_2[C_3^{RE}]-\mathbb{E}_{2-J}[C_3^{RE}]\right], \tag{3.23}$$

where we used the LOIE under RE to simplify the comparison group expression above.

Solution under naïveté. We focus on $J=2$ in this three-period model. The optimality conditions (3.20), (3.22), and the RE policies of Proposition 1 produce the following solution.

Proposition 5.. *When $J=2$, the optimal time 1 and 2 policy functions under naïveté are*

$$K_1^\theta = -\frac{2\theta}{3(2+\theta)}\mathbb{N}_{-1,0}[K_0] + \frac{2}{3}K_0 + \frac{2}{3+\theta}\varepsilon_1,$$
$$K_2^\theta = -\frac{\theta}{2(2+\theta)}\mathbb{N}_{0,1}\left[K_1^\theta\right] + \frac{1}{2}K_1^\theta + \frac{1}{2+\theta}\varepsilon_2,$$

where $\mathbb{N}_{-1,0}[K_0] \equiv K_0 - \mathbb{E}_{-1}[K_0]$ and $\mathbb{N}_{0,1}\left[K_1^\theta\right] \equiv K_1^\theta - \mathbb{E}_0\left[K_1^{RE}\right]$ represent the surprises in the stock of capital with respect to the expectations formed in the past.

The difference with the $J=1$ case of Proposition 3 is the presence of a novel informational state, given by the surprise in capital.[10] These novel state dynamics arise because the economic states K_0 and K_1^θ are *not* sufficient state variables anymore for the comparison group that matters for decisions at time 1 and 2, respectively. With $J=2$, the relevant comparison groups are built on conditional expectations $\mathbb{E}_{-1}[K_0]$ and $\mathbb{E}_0[K_1^{RE}]$, respectively.

In particular, in Proposition 5 the elasticities on K_0 and K_1^θ continue to recover the role of capital stock as an economic state, which influences decisions as in the RE policy function. The novel informational role is captured by the elasticities on $\mathbb{N}_{-1,0}[K_0]$ and $\mathbb{N}_{0,1}[K_1^\theta]$.

Consider the optimality condition governing the time 2 policy function K_2^θ. This condition resembles the earlier equation (3.15), but now the reference value is $K_{2,2-J}^M = \mathbb{E}_0[K_2^{RE}]$:

$$\varepsilon_2 + K_1^\theta - K_2^\theta = K_2^\theta + \theta\left[K_2^\theta - \mathbb{E}_0\left[K_2^{RE}\right]\right]. \tag{3.24}$$

For example, take a positive innovation in ε_1, which in equilibrium (verified by Proposition 5) causes an increase in K_1^θ. A higher K_1^θ than expected at time 0 under the relevant comparison group leads to a perceived positive innovation in $K_2^\theta - \mathbb{E}_0[K_2^{RE}]$. By equation (3.24), holding everything else constant, since agents are over-influenced by this surprise, they become over-optimistic about future resources and save less. This explains why the innovation $\mathbb{N}_{0,1}\left[K_1^\theta\right]$ enters with a negative sign in the time 2 policy function. Similar intuition explains why the innovation $\mathbb{N}_{-1,0}[K_0]$ enters with a negative sign in the time 1 policy function K_1^θ.

Under recent memory ($J=1$), these news terms collapse to zero: $\mathbb{N}_{0,0}[K_0] \equiv K_0 - \mathbb{E}_0[K_0] = 0$ and $\mathbb{N}_{1,1}\left[K_1^\theta\right] \equiv K_1^\theta - \mathbb{E}_1\left[K_1^{RE}\right] = 0$. In the second relation, we have used $\mathbb{E}_1\left[K_1^{RE}\right] = K_1^\theta$ because a projection of current capital at horizon zero is always equal to the current capital, no matter the data generating process that Memory uses when forming the projection. Intuitively, when $J=1$, the news terms disappear because the

10. When $J=1$, because the LOIE holds, the time-consistent policy functions of Proposition 3 are equivalent to those derived under naïveté (or sophistication). See Proposition B1 in the Online Appendix.

stock of capital inherited from the past is already part of the information set entering the comparison group. Instead, when $J > 1$, the agent makes decisions in the meantime, and these decisions create surprises with respect to the comparison group based on the more distant past.

Naïveté vs sophistication. We conclude this subsection by briefly considering the alternative assumption of sophistication. We provide details in the Online Appendix. For the time 2 policy function we recover the same coefficients as the naïveté case, except that $\mathbb{E}_0[K_1^{\theta}]$ enters into the savings rule instead of $\mathbb{E}_0[K_1^{RE}]$. This occurs because, in order to maintain belief consistency across selves, we assume that the sophisticated agent's comparison group is the expectation formed J periods ago by the former sophisticated self. In turn, at time 1 the agent would choose a different plan than what she anticipates will be her optimal time-2 *conditional* action. Thus, her optimal time 1 action aims to fix that misalignment. The sophistication counterpart of the optimality condition in equation (3.19) contains indirect effects of the current action on the future policy. Since the agent anticipates that she will over-consume at time 2, the consumption-smoothing motive between time 1 and 2 leads the agent to consume more at time 1 out of temporary income shock ε_1 relative to naïveté.

In extending the theoretical framework of this consumption-smoothing model to more realistic and quantitatively relevant business cycle models, we propose to focus on the *naïveté* approach. The key reason is that the sophistication approach's required hyper-rationality runs counter to the motivation of modeling agents' beliefs about their future circumstances as influenced by a heuristic. Indeed, the latter is usually viewed as a cognitive, mental shortcut that allows agents to make judgments quickly and efficiently (Tversky and Kahneman (1975) and Kahneman (2011)).[11] The naïveté approach is arguably psychologically more coherent and consistent with the underlying foundation of diagnostic beliefs as a heuristic reflecting a memory representation affected by imprecise, selective, and less than fully rational recall.[12]

Moreover, computationally the naïveté approach is significantly more tractable, a property that we explain and leverage in the rest of the paper. Therefore, we present the naïveté approach as a 'portable extension of existing models' (as advocated by Rabin (2013)) that tractably incorporates the psychology foundation of the representativeness heuristic and the role of imperfect memory recall in standard business cycle models.

3.3. Distant memory and boom-bust cycles

Our analysis shows that the interaction between endogenous economic states and distant memory introduces novel informational states that affect the model's propagation mechanism. In this subsection we use the simplest infinite-horizon extension of the three-period model to showcase this altered propagation.

We study the permanent income hypothesis (PIH) model under DE with naïveté. We continue to assume quadratic utility and iid income shocks. Households can save by

11. Part of this hyper-rationality is that in infinite-horizon models the current sophisticated agent would internalize the life-time indirect effect of current savings as a future information state, i.e. how current savings affect the formation of comparison groups that will matter in the future selective memory recall of the past.

12. In terms of literature, the naïveté approach is also consistent with how previous contributions such as Bordalo et al. (2019) and Maxted (2020) dealt with the failure of LOIE in exogenous processes.

buying capital at the price $q=(1+r)^{-1}$, where $r>0$ is the exogenous real interest rate and the discount factor is $\beta=(1+r)^{-1}$. The time t budget constraint is then:

$$K_t=(1+r)(K_{t-1}+\overline{Y}+\varepsilon_t-C_t).$$

As before, we first solve the model under RE. The first-order condition (FOC) is:

$$C_t^{RE}=\mathbb{E}_t\left[C_{t+1}^{RE}\right].$$

In the Appendix, we conjecture and verify the RE consumption policy function:

$$C_t^{RE}=\frac{r}{1+r}\left(K_{t-1}^{RE}+\varepsilon_t\right)+\overline{Y} \tag{3.25}$$

and the resulting RE saving decision

$$K_t^{RE}=K_{t-1}^{RE}+\varepsilon_t \tag{3.26}$$

Thus, under RE, capital is a random walk and shocks have a permanent effect on savings.

Similar to Subsection 3.2.3, under DE and naïveté the agent's problem at time t is:

$$\max_{K_t^\theta}\left\{u(C_t^\theta)+\mathbb{E}_t^\theta\left[\mathcal{V}(K_t^\theta)\right]\right\}, \tag{3.27}$$

where the continuation utility $\mathcal{V}(.)$ reflects the maintained assumption that the agent at time t believes her future selves from time $t+1$ on will act under RE. Thus the agent expects that the time $t+1$ self will take the action K_{t+1}^{RE} so as

$$\mathcal{V}(K_t^\theta)=\max_{K_{t+1}^{RE}}\left[u(C_{t+1}^{RE})+\mathbb{E}_{t+1}\left[\mathcal{V}(K_{t+1}^{RE})\right]\right]. \tag{3.28}$$

Proposition 6.. *The optimal savings policy function under DE and naïveté is*

$$K_t^\theta=K_{t-1}^\theta-\frac{r\theta}{1+r(1+\theta)}\mathbb{N}_{t-J,t-1}\left[K_{t-1}^\theta\right]+\frac{1+r}{1+r(1+\theta)}\varepsilon_t.$$

where the news term $\mathbb{N}_{t-J,t-1}\left[K_{t-1}^\theta\right]\equiv K_{t-1}^\theta-\mathbb{E}_{t-J}\left[K_{t-1}^{RE}\right]$.

In the Online Appendix we show that this solution arises from the FOC of the problem in (3.27)

$$C_t^\theta=\mathbb{E}_t^\theta\left[C_{t+1}^{RE}\right],$$

where the agent anticipates future behavior according to the $t+1$ RE policy (see equation (3.25)), similar to the result detailed in Lemma 2. As in the three period model (see Proposition 5), the DE solution presents an additional informational state variable $\mathbb{N}_{t-J,t-1}\left[K_{t-1}^\theta\right]$. This state variable is relevant to the extent that it induces optimism or pessimism about the future as a result of a discrepancy between the resources currently available and those anticipated based on the agent's imperfect memory. Furthermore, this additional state variable is activated only to the extent that memory is based on the distant past ($J > 1$).

Consistent with the discussion in Subsection 3.2.3 (in particular around equation (3.23)), the comparison group is based on the RE solution. This is how the agent perceives capital should have evolved based on the information available at time $t-J$. Thus, using the law of motion for capital under RE in equation (3.26) we have $\mathbb{E}_{t-J}[K_{t-1}^{RE}] = K_{t-J}^{\theta}$, where K_{t-J}^{θ} is the capital in place at $t-J$. The optimal policy in Proposition 6 becomes:

$$K_t^{\theta} = K_{t-1}^{\theta} - \frac{r\theta}{1+r(1+\theta)}\left[K_{t-1}^{\theta} - K_{t-J}^{\theta}\right] + \frac{1+r}{1+r(1+\theta)}\varepsilon_t. \tag{3.29}$$

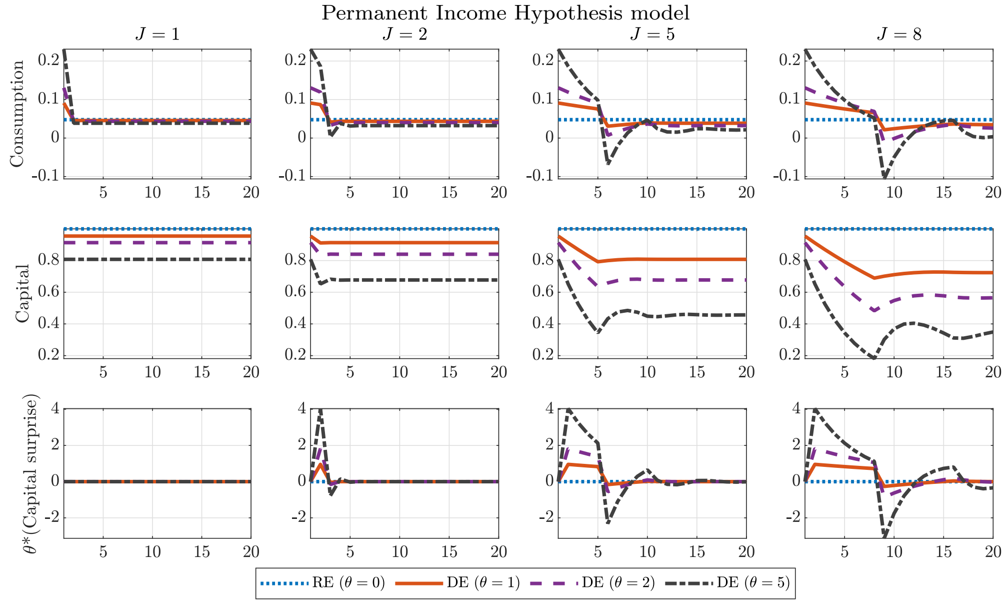
The solution above elucidates that *both* the severity of the DE distortion captured by $\theta > 0$ and the lag in the reference distribution J matter to determine the behavior of consumption and capital in response to a transitory shock. Figure 1 illustrates how these two margins interact with each other in a model in which DE apply to both exogenous and endogenous variables. Specifically, we consider a unitary income shock that dissipates after one period. The three rows report the consumption response, the capital response, and the response of the surprise in capital scaled by the DE parameter θ (i.e., $\theta\left[K_{t-1}^{\theta} - K_{t-J}^{\theta}\right]$), respectively. Across columns, we vary the lag of the reference distribution, J, while the different lines in each panel are obtained by varying the severity of the DE distortion, θ. For each variable, the panels are on the same scale to facilitate the comparison.

Endogenous booms and busts. Under DE, the initial capital response to a transitory shock ε_t is smaller than under RE (evident from equation (3.29)). The initial capital response decreases with θ, but it does not vary with J. This is because the shock is always unpredictable, no matter when the reference expectations were formed. In other words, the information content of the initial shock does not vary with respect to J. The under-accumulation of capital translates in an over-reaction in consumption. Under recent memory ($J = 1$, first column), the path reverts to the one followed under RE after one period. Analytically, with $J = 1$ the news state in equation (3.29) is irrelevant. This is consistent with the fact that when $J = 1$ there are no further perceived surprises, as shown in the third row. Of course, both consumption and capital are permanently lower than under RE because of the initial under-accumulation of capital, but the impulse response is now flat, as under RE.

Moving from left to right, we can appreciate the importance of allowing for distant memory. While the initial response is not affected, the persistence of the initial over-reaction is controlled by J. More importantly, after $J+1$ periods the agent is surprised again. However, this time the surprise is the result of the interaction between Deliberation and Memory. Consider the case $J = 2$ (second column). In period 3, the memory formed at time 1 becomes relevant. Now the agent is disappointed in the current level of capital. This is because Memory recalls the projection based on the stock of capital that was available at time 1, the reference value for period 3. Capital is lower than expected because in the meantime the DE agent has consumed too much. The response of the DE agent at time 3 is to cut consumption more than what an RE agent would do if confronted with the same level of capital. As J increases (third and fourth column), not only it takes longer for the agent to reverse her behavior, but the correction also increases in magnitude. Thus, J does not only affect the lag in the correction, but also its *amplitude*.

As J increases, an additional, crucial feature of distant memory becomes more visible: A single, initial shock can endogenously induce repeated boom-bust cycles. As J increases,

FIGURE 1
Permanent Income Hypothesis model



Notes: The figure reports the impulse responses to a unitary iid shock for the Permanent Income Hypothesis model. The three rows report the responses of consumption, capital, and the surprise in capital scaled by the DE parameter θ (i.e., $\theta\left[K^{\theta}_{t-1}-K^{\theta}_{t-J}\right]$), respectively. Across columns, we vary the lag of the reference distribution, J, while the different lines in each panel are obtained by varying the severity of the DE distortion, θ. For each variable, the panels are on the same scale to facilitate the comparison.

kinks and inflection points occur with lags and magnitudes that depend on J. Consider the case of $J=8$ (fourth column). As before, a first kink occurs in period $J+1=9$, when the agent reacts to a disappointing level of savings by cutting consumption more than what she would have done under RE. As the agent keeps accumulating capital and the reference level of capital progressively declines, consumption recovers. The third row highlights that the agent eventually becomes *positively* surprised by the amount of capital that she has at her disposal. This is the result of her own actions in response to the perceived low stock of capital. She is accumulating capital, while the reference value is constantly declining. As a result, capital and consumption start slowing down, generating an inflection point. Eventually, a second kink occurs (in period $2J+1=17$). Now the reference level of capital is increasing once again, as a result of the agent past behavior. Thus, the agent is disappointed by her current level of capital compared to what she was expecting based on her past behavior.

Overall, distant memory creates rich interactions between the actions taken by the agent and her memory. The extent of the DE distortion captured by the parameter θ and the lag of the reference distribution as captured by J interact to create repeated boom-bust cycles. A large DE distortion implies that agents react forcefully to perceived surprises in the amount of resources. This behavior creates the conditions for future surprises. Kinks in the response of the economy occur every J periods, but inflection points can occur in between these kinks as past actions induce changes in beliefs. Finally, if memory were based on an average of multiple lags of past expectations, as in our quantitative model of Section 4, instead of on a single lag J, the kinks would appear smoother and more similar to turning points.

4. A quantitative DE New Keynesian model

We leverage the previous qualitative insights to incorporate DE into a *quantitative New Keynesian model* of the type widely used for policy analysis. We emphasize the critical role played by endogenous predictability and distant memory recall in this new class of models. Methodologically, we formally rely on the naïveté approach to model beliefs, as argued earlier. This allows us to develop a tractable and recursive *solution method* to characterize equilibrium laws of motion when agents act under DE. We estimate the model and show that it replicates the empirical boom-bust cycle in response to a monetary policy shock.

In deriving our theoretical results, we made use of the tractability arising in a model with Gaussian shocks where perceived tradeoffs are linear, thus maintaining conditional normality. However, in more general cases, the non-linear version of the model will not lead to a conditionally normal distribution. Indeed, in the class of models that we analyze in this section, it is the solution of the *log-linearized* model that has this property. In this setting, we exploit the convenient formulation of the representativeness heuristic based on the density h^θ in equation (2.5) by applying it on the log-linearized perceived tradeoffs. Our primitive approach, in line with what proposed in Bordalo et al. (2018), consists of emphasizing the role of the representativeness heuristic in distorting the perceptions of the marginal tradeoffs. We find the direct modeling of perceptions of linearized marginal tradeoffs as distorted by the density h^θ appealing because: (a) in linearized models these perceptions guide actual (marginally driven) decisions, and (b) in standard Gaussian environments these tradeoffs can be tractably characterized, a feature that we leverage throughout the paper. In our final remarks, we briefly discuss directions for future research, including how to allow for non-linearities while preserving tractability.

4.1. The model

The model features monopolistic competition in the labor and goods market, subject to adjustment costs in setting nominal prices. Consumption-investment decisions are influenced by real rigidities, in the form of habit formation and investment adjustment costs, and monetary policy follows a Taylor rule.

Household. The representative household chooses capital K_t^θ, investment I_t^θ, capital utilization rate u_t^θ, bonds B_t^θ, consumption C_t^θ, labor $N_{h,t}^\theta$ and nominal wage $W_{h,t}^\theta$ to solve

$$\max_{K_t^\theta, I_t^\theta, u_t^\theta, B_t^\theta, C_t^\theta, N_{h,t}^\theta, W_{h,t}^\theta} \left[\ln(C_t^\theta - b\overline{C}_{t-1}^\theta) - \frac{(N_{h,t}^\theta)^{1+\eta}}{1+\eta} + \beta \mathbb{E}_t^\theta \mathcal{V}(\mathcal{S}_{t+1}^\theta) \right] \tag{4.30}$$

subject to the budget constraint

$$P_t^\theta C_t^\theta + P_t^\theta I_t^\theta + P_t^{B,\theta} B_t^\theta + (\varphi_w/2)\left(W_{h,t}^\theta / W_{h,t-1}^\theta - \gamma\Pi\right)^2 W_t^\theta$$
$$= B_{t-1}^\theta + P_t^\theta R_t^{k,\theta} u_t^\theta K_{t-1}^\theta + W_{h,t}^\theta N_{h,t}^\theta + \int_0^1 D_{i,t}^\theta di - P_t^\theta a(u_t^\theta) K_{t-1}^\theta.$$

where P_t^θ is the price level, $R_t^{k,\theta}$ is the capital rental rate, and $\int_0^1 D_{i,t}^\theta di$ is the combined current nominal profits from intermediate firms, given below in the firms' profit maximization problem. $P_t^{B,\theta}$ is the price of bond that pays 1 unit of consumption at $t+1$ so $P_t^{B,\theta} = 1/R_t^\theta$, where R_t^θ is the gross nominal interest rate. We allow for a capital utilization rate u_t^θ choice, subject to a resource cost specified as $a(u_t^\theta) = R^k(1+\tau)^{-1}\left((u_t^\theta)^{1+\tau} - 1\right)$. We explain the continuation value $\mathcal{V}(.)$ in detail below.

Each household is monopolistically competitive in its labor supply. A perfectly competitive labor packer combines household labor and sells the composite labor N_t^θ to intermediate firms, described below, using the CES technology $N_t^\theta = \left[\int_0^1 (N_{h,t}^\theta)^{\frac{1}{\lambda_n}} dh\right]^{\lambda_n}$, where λ_n controls the steady-state wage markup. The packer's cost minimization leads to a standard demand curve taken by the household as an additional constraint in solving equation (4.30), namely $N_{h,t}^\theta = N_t^\theta \left(W_{h,t}^\theta / W_t^\theta\right)^{-\lambda_n/(\lambda_n - 1)}$, where W_t^θ is the aggregate wage level.

As we detail below, our approach handles large state space models, which allows us to incorporate DE into a NK model with *nominal and real frictions* that are typical of such quantitative business cycle models (see e.g., Christiano et al. (2005) and Smets and Wouters (2007)). In particular, the budget constraint above describes how nominal wages are subject to an adjustment cost (as in Kim (2000)), governed by the parameter φ_w, where γ is the rate of deterministic technological progress and Π is the steady-state inflation rate. On the preference side, note that in equation (4.30) we allow for habit formation, where $\overline{C}_{t-1}^\theta$ is the average aggregate consumption in the previous period and b is the external habit parameter.

Finally, the optimization in equation (4.30) is further subject to the physical capital law of motion, which features a standard quadratic investment adjustment cost

$$K_t^\theta = (1-\delta) K_{t-1}^\theta + \left\{1 - (\kappa/2)\left((I_t^\theta / I_{t-1}^\theta) - \gamma\right)^2\right\} I_t^\theta,$$

where δ is the depreciation rate and κ is the adjustment cost parameter.

As explained in Section 3, in the naïveté approach, in evaluating the continuation value $\mathcal{V}(.)$ in equation (4.30), the household assumes that her and other agents' future conditional preferences and resulting conditionally optimal actions will be taken under perfect memory (or RE), given values of the states entering next period, collected in the vector $\mathcal{S}_{t+1}^\theta$. To construct that continuation value we thus set up a 'shadow' economy (indexed by RE) where the household problem is solved under perfect memory, conditional on inherited states:

$$\mathcal{V}(\mathcal{S}_t^\theta) = \max_{K_t^{RE}, I_t^{RE}, u_t^{RE}, B_t^{RE}, C_t^{RE}, N_{h,t}^{RE}, W_{h,t}^{RE}} \left[\ln(C_t^{RE} - b\overline{C}_{t-1}^\theta) - \frac{(N_{h,t}^{RE})^{1+\eta}}{1+\eta} + \beta \mathbb{E}_t \mathcal{V}(\mathcal{S}_{t+1}^{RE}) \right],$$

subject to the budget constraint

$$P_t^{RE} C_t^{RE} + P_t^{RE} I_t^{RE} + P_t^{B,RE} B_t^{RE} + (\varphi_w/2) \left(W_{h,t}^{RE} / W_{h,t-1}^\theta - \gamma \Pi \right)^2 W_t^{RE}$$
$$= B_{t-1}^\theta + P_t^{RE} R_t^{k,RE} u_t^{RE} K_{t-1}^\theta + W_{h,t}^{RE} N_{h,t}^{RE} + \int_0^1 D_{i,t}^{RE} di - P_t^{RE} a(u_t^{RE}) K_{t-1}^\theta.$$

The law of motion for capital is given by

$$K_t^{RE} = (1-\delta) K_{t-1}^\theta + \left\{ 1 - (\kappa/2) \left(I_t^{RE} / I_{t-1}^\theta - \gamma \right)^2 \right\} I_t^{RE},$$

while the labor demand curve is simply $N_{h,t}^{RE} = N_t^{RE} \left(W_{h,t}^{RE} / W_t^{RE} \right)^{-\lambda_n/(\lambda_n - 1)}$.

Firms. The final output is produced by a perfectly competitive representative firm that combines a continuum of intermediate goods $Y_{i,t}^\theta$ using the CES technology:

$$Y_t^\theta = \left[\int_0^1 (Y_{i,t}^\theta)^{\frac{1}{\lambda_f}} di \right]^{\lambda_f},$$

where λ_f controls the steady-state markup. Intermediate goods firms' production function is $Y_{i,t}^\theta = (u_{i,t}^\theta K_{i,t}^\theta)^\alpha (\gamma^t N_{i,t}^\theta)^{1-\alpha}$, where $K_{i,t}^\theta$ and $N_{i,t}^\theta$ are the capital and labor employed by firm i. From the cost minimization problem, the real marginal cost is given by

$$MC_t^\theta = \frac{(R_t^{k,\theta})^\alpha (W_t^\theta / P_t^\theta)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha} u_{i,t}^\theta (\gamma^t)^{1-\alpha}}.$$

Intermediate firms face an adjustment cost à-la Rotemberg (1982) in changing their nominal price. Their problem is to choose $P_{i,t}^\theta$ to maximize

$$(C_t^\theta - bC_{t-1}^\theta)^{-1} D_{i,t}^\theta / P_t^\theta + \beta \mathbb{E}_t^\theta \mathcal{V}_f(P_{i,t}^\theta), \tag{4.31}$$

where $D_{i,t}^\theta = \left(P_{i,t}^\theta Y_{i,t}^\theta - P_t^\theta MC_t^\theta Y_{i,t}^\theta - (\varphi_p/2) \left(P_{i,t}^\theta / P_{i,t-1}^\theta - \Pi \right)^2 P_t^\theta Y_t^\theta \right)$ and φ_p is the price adjustment cost parameter. $\mathcal{V}_f(.)$ is the continuation value

$$\mathcal{V}_f(P_{i,t-1}^\theta) = \max_{P_{i,t}^{RE}} \left[(C_t^{RE} - bC_{t-1}^\theta)^{-1} D_{i,t}^{RE} / P_t^\theta + \beta \mathbb{E}_t \mathcal{V}_f(P_{i,t}^{RE}) \right],$$

where $D^{RE}_{i,t}=\left(P^{RE}_{i,t}Y^{RE}_{i,t}-P^{RE}_{t}MC^{RE}_{t}Y^{RE}_{i,t}-(\varphi_p/2)\left(P^{RE}_{i,t}/P^{\theta}_{i,t-1}-\Pi\right)^2 P^{RE}_{t}Y^{RE}_{t}\right)$.
Thus, in equation (4.31), firms' instantaneous payoff is given by current real profits and the continuation value is given by the discounted sum of real profits $V_f(P^{\theta}_{i,t})$. Under naïveté, in computing continuation value, agents assume that firms inherit the chosen price $P^{\theta}_{i,t}$ (which is relevant for the adjustment cost) but future prices are set according to RE.

Market clearing and monetary policy. The resource constraint is given by

$$C^{\theta}_t+I^{\theta}_t+(\varphi_p/2)\left(\Pi^{\theta}_t-\Pi\right)^2 Y^{\theta}_t+(\varphi_w/2)\left(\Pi^{\theta}_{w,t}-\gamma\Pi\right)^2 W^{\theta}_t/P^{\theta}_t+a(u^{\theta}_t)K^{\theta}_{t-1}=Y^{\theta}_t,$$

where $\Pi^{\theta}_{w,t}\equiv W^{\theta}_t/W^{\theta}_{t-1}$ is nominal wage inflation. The central bank follows a Taylor rule:

$$R_t/R^{\theta}=\left(R^{\theta}_{t-1}/R\right)^{\rho_R}\left\{\left(\widetilde{\Pi}^{\theta}_t/\Pi\right)^{\phi_\pi}\left(Y^{G,\theta}_t/(\gamma Y^{G,\theta}_{t-1})\right)^{\phi_Y}\right\}^{1-\rho_R}\exp(\varepsilon_t),\qquad \varepsilon_t\sim N(0,\sigma^2_R),$$

where $\widetilde{\Pi}^{\theta}_t\equiv\left[\Pi^{\theta}_t\Pi^{\theta}_{t-1}\Pi^{\theta}_{t-2}\Pi^{\theta}_{t-3}\right]^{1/4}$ is annual inflation, ε_t is the iid monetary policy shock, and GDP is defined as $Y^{G,\theta}_t\equiv C^{\theta}_t+I^{\theta}_t$. We provide the equilibrium conditions in Online Appendix C.

4.2. Solution method

Our solution method exploits the fact that naive DE agents expect future actions to be taken under the RE policy function. Below we outline our solution method. We provide additional details and formulas in Online Appendix D.

1. The first step of the solution algorithm consists of obtaining the shadow RE law of motion used by agents to form DE. We start from a linear RE system

$$\mathbf{\Gamma}_0\mathbf{x}^{RE}_t=\mathbf{\Gamma}_1\mathbf{x}^{RE}_{t-1}+\mathbf{\Psi}\varepsilon_t+\mathbf{\Pi}\eta^{RE}_t, \tag{4.32}$$

where $\mathbf{x}^{RE}_t$, ε_t and η^{RE}_t are vectors of endogenous variables, shocks, and expectation errors, respectively. This RE system is simply the RE version of the economy, with linear equilibrium conditions where DE ($\mathbb{E}^{\theta}_t$) is replaced with RE ($\mathbb{E}_t$).
A recursive law of motion can be obtained, using for example Sims (2000),

$$\mathbf{x}^{RE}_t=\mathbf{T}^{RE}\mathbf{x}^{RE}_{t-1}+\mathbf{R}^{RE}\varepsilon_t. \tag{4.33}$$

2. Consider a linear DE system

$$\mathbf{\Gamma}^{\theta}_0\mathbf{x}^{\theta}_t=\mathbf{\Gamma}^{\theta}_2\mathbb{E}^{\theta}_t[\mathbf{y}^{RE}_{t+1}]+\mathbf{\Gamma}^{\theta}_1\mathbf{x}^{\theta}_{t-1}+\mathbf{\Psi}^{\theta}\varepsilon_t, \tag{4.34}$$

where we provide expressions for $\mathbf{\Gamma}^{\theta}_0$, $\mathbf{\Gamma}^{\theta}_2$, $\mathbf{\Gamma}^{\theta}_1$ and $\mathbf{\Psi}^{\theta}$ in the Online Appendix. Relative to the RE system (4.32), which implicitly defines expectations in $\mathbf{x}^{RE}_t$ by using expectation errors η^{RE}_t, the DE system (4.34) explicitly accommodates DE ($\mathbb{E}^{\theta}_t[\mathbf{y}^{RE}_{t+1}]$).

We can substitute the $\mathbb{E}_t^\theta[\mathbf{y}_{t+1}^{RE}]$ in the DE system (4.34) as

$$\mathbb{E}_t^\theta[\mathbf{y}_{t+1}^{RE}] = \mathbb{E}_t[\mathbf{y}_{t+1}^{RE}] + \theta(\mathbb{E}_t[\mathbf{y}_{t+1}^{RE}] - \mathbb{E}_t^r[\mathbf{y}_{t+1}^{RE}]), \tag{4.35}$$

where $\mathbb{E}_t^r[\mathbf{y}_{t+1}^{RE}]$ denotes the comparison group, or the *reference* distribution, characterizing the representativeness heuristic. Our method allows for a general form of memory recall and thus of this comparison group. In particular, as we further explain below, we model this reference distribution in a flexible, yet parsimonious manner, as a weighted average of lagged RE expectations:

$$\mathbb{E}_t^r[\mathbf{y}_{t+1}^{RE}] = \sum_{j=1}^{J} \alpha_j \mathbb{E}_{t-j}[\mathbf{y}_{t+1}^{RE}], \tag{4.36}$$

where $\{\alpha_j\}_{j=1}^J$ are weight parameters on lagged expectations such that $\sum_{j=1}^J \alpha_j = 1$. Let $\mathbf{y}_t^{RE} = \mathbf{M}\mathbf{x}_t^{RE}$, where $\mathbf{M}$ is a selection matrix that selects variables from a vector $\mathbf{x}_t^{RE}$. Given the DE beliefs characterized by (4.35) and (4.36), the system (4.34) then becomes

$$\mathbf{\Gamma}_0^\theta \mathbf{x}_t^\theta = \mathbf{\Gamma}_2^\theta \left[(1+\theta)\mathbf{M}\mathbf{T}^{RE}\mathbf{x}_t^\theta - \sum_{j=1}^{J} \theta\alpha_j \mathbf{M} \left(\mathbf{T}^{RE}\right)^{j+1} \mathbf{x}_{t-j}^\theta \right] + \mathbf{\Gamma}_1^\theta \mathbf{x}_{t-1}^\theta + \mathbf{\Psi}^\theta \varepsilon_t, \tag{4.37}$$

where $\mathbf{T}^{RE}$ is the auto-regressive component of the RE solution. The expression (4.37) also clarifies that agents form DE based on state variables inherited from the DE economy, but assuming that in the future the economy follows the RE law of motion.

3. Inverting matrices and rewriting (4.37) more compactly gives the DE law of motion:

$$\mathbf{z}_t^\theta = \mathbf{T}^\theta \mathbf{z}_{t-1}^\theta + \mathbf{R}^\theta \varepsilon_t, \tag{4.38}$$

where we provide expressions for $\mathbf{T}^\theta$ and $\mathbf{R}^\theta$ in Online Appendix D and $\mathbf{z}_t^\theta$ is a vector that includes not only $\mathbf{x}_t^\theta$ but also its lags. Finally, we check that all variables over which we take DE present residual uncertainty using the formula we provide in Online Appendix D.

The key advantages of our solution method are thus its *portability and tractability*. A researcher interested in solving a model under DE would implement the following steps. First, derive a set of linearized equilibrium conditions under DE. Second, solve the corresponding shadow RE model (4.32). Third, use a few lines of code to obtain the solution under DE (4.38) by combining the DE equilibrium conditions (4.34) with the RE solution (4.33).

4.3. Estimation

Our aim is to demonstrate that DE matter in practical and policy-relevant settings. We choose the estimation method that aligns with this goal. The starting point of our analysis is a local projection estimation of impulse responses to a monetary policy shock using U.S. quarterly macroeconomic data over the sample period 1969Q1–2006Q4.[13] Specifically, we estimate the following regressions:

$$x_{t+h} = c^h + \tau_1^h t + \tau_2^h t^2 + \sum_{l=1}^{L} A_l^h x_{t-l} + \sum_{i=0}^{I} B_i^h e_{t-i} + u_{t+h}, \quad h = 0,\ldots,H$$

where x_t is the variable of interest and e_t is the Romer and Romer (2004) monetary policy shock, extended by Coibion et al. (2017). The coefficients of interest are $\{B_0^h\}_{h=0}^H$. We set $L=4$ and $I=0$ and compute the impulse response for $H=32$ horizons.

We estimate the model parameters using the Bayesian version of the impulse-response-matching method, developed by Christiano et al. (2010). In this method, the likelihood depends on how closely the model matches the empirical response to a shock. The likelihood is then combined with priors on the model parameters.[14] In our empirical analysis below, we target the impulse responses of four variables: log real per capita consumption, log per capita hours worked, log GDP deflator inflation, and the log Federal Funds rate (FFR). We then also use the implied responses of four other variables, namely log real per capita investment, log real per capita GDP, SPF expected inflation, and SPF GDP growth expectations, as 'untargeted' moments that serve as external validation.[15]

We fix several parameters before the estimation. The deterministic growth rate γ and the steady-state inflation rate Π are set to 1.004 and 1.01, respectively, which imply a steady-state annual output growth rate of 1.6% and the annualized inflation rate of 4%. The capital share α, the discount factor β and the depreciation rate δ are set to 0.3, 0.99 and 0.025, respectively. We set λ_f and λ_n to 1.1, which imply steady-state price and wage markups of 10%. For parameters that are common in the New Keynesian literature, we center our priors around conventional values.

For the diagnostic parameter θ, we choose a conservative prior that puts significant weight on the RE case ($\theta=0$) but also encompasses the estimates found in Bordalo et al. (2018) and Bordalo et al. (2019) ($\theta \approx 1$). Specifically, we chose a Normal distribution with mean 0 and standard deviation 0.2, but we truncate this prior above $\theta \geq 0$ so as to be consistent with the theoretical restriction that the diagnosticity parameter has to be (weakly) positive. Note that the prior mode of this truncated Normal distribution is 0.[16] As explained above (see equation (4.36)), we allow for flexible reference expectations in memory recall and thus the comparison group is a weighted average of lagged expectations. To estimate the weights $\{\alpha_j\}_{j=1}^J$ on past memory, we consider a parsimonious parameterization. We set $J=32$ and estimate the mean μ and the standard deviation σ of a Beta distribution. We then rescale and discretize the implied $Beta(\mu,\sigma^2)$ distribution to span the discrete interval [0,32] and obtain the weights $\widetilde{\alpha}_j$. We then apply the transformation $\alpha_j = \widetilde{\alpha}_j / (\sum_{j=1}^J \widetilde{\alpha}_j)$ so that $\{\alpha_j\}_{j=1}^J$ sum to one. We report the priors

13. We do not include the period after 2007Q1 to avoid complications arising from the zero lower bound.

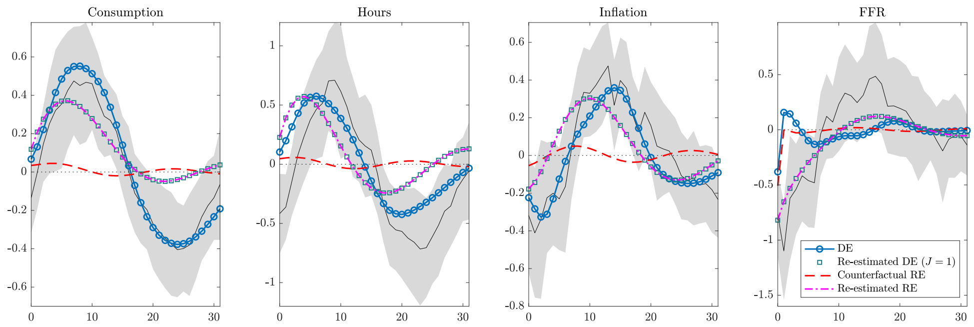
14. We provide a detailed description of the estimation method in the Online Appendix E.

15. To obtain real per capita GDP we divide real GDP by total population. Real per capita consumption is measured by the sum of personal consumption expenditure on nondurables and services divided by total population. Real per capita investment is the sum of gross private domestic investment and personal consumption expenditure on durables divided by total population. Per capita hours worked is the total hours in nonfarm business sector divided by total population.

16. Online Appendix F shows that results are similar if we center the prior for θ around 1.

FIGURE 2

Impulse responses to a monetary policy shock: Fit for targeted responses



Notes: This figure reports impulse responses for targeted variables. The black lines are the mean responses from the local projection and the shaded areas are the 90% confidence bands. The blue circled lines are IRFs from the baseline model with DE, allowing for distant memory. The green squares are IRFs from the DE model imposing that recall is based only on recent memory. In this case, $J=1$ by assumption. The red dashed lines are IRFs from the counterfactual RE model where we set $\theta=0$ while holding fixed other estimated parameters. The magenta dashed lines are IRFs from the re-estimated RE model. The consumption and hours responses are in percentage deviations from steady states while inflation and the FFR are in annual percentage points.

and all estimated parameters in Table F1 in the Online Appendix, while below we focus on the key parameters that control the effects of DE.

4.4. Results

Figure 2 presents the local projection impulse responses (black solid lines) to a one-standard-deviation expansionary monetary policy shock along with the 90% confidence bands. In response to a reduction in the FFR, real variables such as hours and consumption all increase in a hump-shaped manner, peaking around 10 quarters after the initial shock. These variables then undershoot below the steady states and reach their trough around 5 to 6 years after the shock, followed by a gradual recovery.[17] Inflation builds up slower and tends to peak at the end of the boom, followed by a slow return to the steady state.

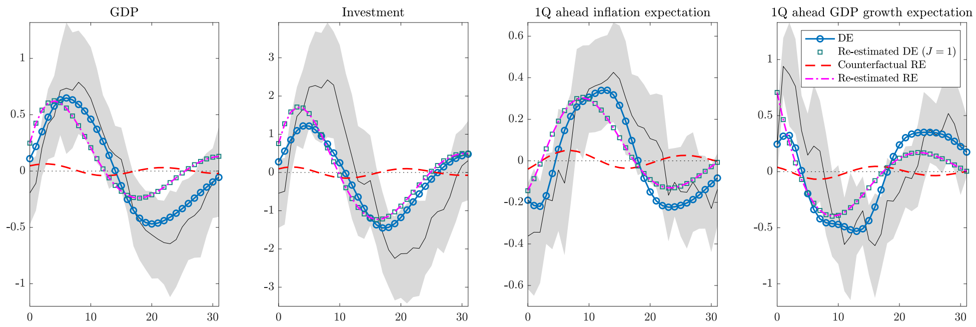
The DE New Keynesian model (blue lines with circles) reproduces the empirical impulse response functions (IRF) well, successfully generating the boom-bust cycle observed in the data. We use three alternative models to argue that the DE distortion *and* distant memory provide the key mechanism for this successful fit.

Alternative models. First, we evaluate a *counterfactual* RE model, where we set the diagnostic parameter $\theta=0$, while holding fixed all other estimated parameters. This counterfactual model (red dashed lines) generates transitory and negligible responses, indicating that much of our success is due to the DE mechanism. Second, we re-estimate the model under RE, i.e., imposing the constraint that $\theta=0$. The re-estimated RE model

17. McKay and Wieland (2021) find a similar boom-bust pattern in their estimated responses to a monetary policy shock.

FIGURE 3

Impulse responses to a monetary policy shock: Fit for untargeted responses



Notes: This figure reports impulse responses for untargeted variables. The black lines are the mean responses from the local projection and the shaded areas are the 90% confidence bands. The blue circled lines are IRFs from the baseline model with DE, allowing for distant memory. The green squares are IRFs from the DE model imposing that recall is based only on recent memory. In this case, $J=1$ by assumption. The red dashed lines are IRFs from the counterfactual RE model where we set $\theta=0$ while holding fixed other estimated parameters. The magenta dashed lines are IRFs from the re-estimated RE model. The responses of GDP and investment are in percentage deviations from the steady states while the inflation and output growth expectations are in annual percentage points.

(magenta dashed lines) fails in delivering the empirical boom-bust dynamics and the amplitude of the IRFs. As a result, the marginal likelihood, a Bayesian measure of fit that penalizes models with more parameters, is $(-345-(-369)=)$ 24 log points higher in the DE model.

In the third and final exercise, we re-estimate the DE model (i.e., we allow for $\theta \geq 0$) but *impose* the constraint that $J=1$. Strikingly, we then estimate a value of $\theta=0$. As a result, in this alternative model that constrains memory recall to be based entirely on the immediate past, the IRFs (green squares) coincide with those of the re-estimated RE model.[18] This exercise showcases how in our model distant memory and DE distortion θ are *complements*, since they interact and jointly magnify the role of DE.

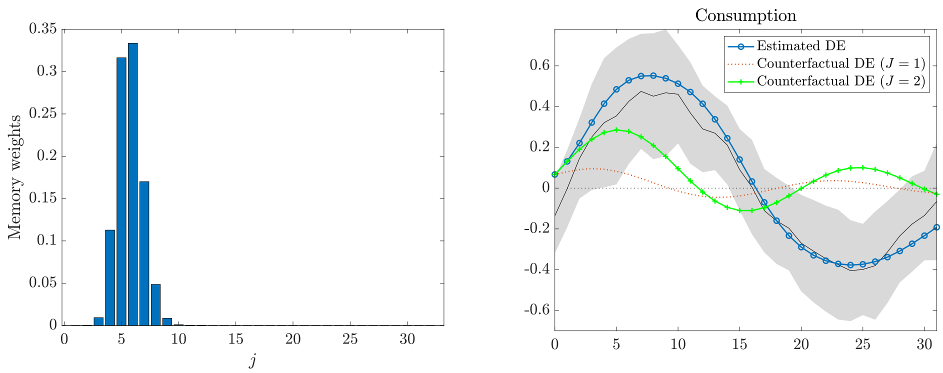
Untargeted moments. The model also matches remarkably well the IRFs that were not targeted in the estimation. The first two panels of Figure 3 report the responses of GDP and investment to the monetary policy shock. The model delivers a good fit. The right two panels of Figure 3 report the impulse response of expected inflation and expected output growth.[19] The model generates expectations that are very much in line with those observed in the data, even though we did not target those expectations in

18. It is then not surprising that the marginal likelihood of the re-estimated RE model beats that of the re-estimated DE model with $J=1$ by $(-369-(-371)=)$ 2 log points, because the DE model with $J=1$ has an additional parameter θ that is not estimated to be significant.

19. We measure inflation and output growth expectations using the median of the SPF survey responses of one-quarter-ahead inflation and output growth expectations, respectively. We assume that the model implied inflation and output growth expectations coincide with what a DE agent in the model would predict ($\mathbb{E}_t^{\theta}[\widehat{\pi}_{t+1}^{RE}]$ and $\mathbb{E}_t^{\theta}[\Delta\widehat{Y}_{t+1}^{G,RE}]$).

FIGURE 4
Estimated selective memory

(a) Estimated memory weights α_j (b) Empirical role of distant memory



Notes: The left panel reports the estimated memory weights α_j. The right panel reports the consumption IRF in the estimated DE model (blue circled line) and the counterfactual model where only recent memory matters (orange dotted line) and when only two-periods-ago memory matters (green line with plus signs).

our estimation exercise.[20] Figure 3 also shows that the re-estimated RE version does a worse job in accounting for these untargeted moments. Formally, we find that the root-mean-square error (RMSE) for the DE model is 0.52, while for the re-estimated model RE it is 0.64. Crucially, focusing only on the untargeted survey moments (inflation and output growth expectations), the RMSE for the DE model is 0.25, while for the re-estimated RE model it is 16% larger, at 0.29.[21]

DE parameters. We estimate $\theta = 1.97$ for the parameter controlling the severity of the DE distortion. This value is in the same order of magnitude of previous estimates (for example, Bordalo et al. (2018), Bordalo et al. (2019), d'Arienzo (2020), L'Huillier et al. (2021), which tend to estimate $\theta \approx 1$), even if larger. However, we note that the existing estimates are based primarily on models where imperfect memory is assumed to be driven only by the immediate past, an assumption that we show fundamentally changes inference in our structural model (per our discussion of the DE version imposing $J = 1$).

The mean and standard deviation of the Beta distribution that controls the weights $\alpha_j' s$ attached to each of the $J = 32$ lagged expectations entering the comparison group are 0.17 and 0.03, respectively. As shown in the left panel of Figure 4, these estimates imply that the weights are centered on the expectations formed six quarters ago, with positive weights assigned to expectations formed between three and ten quarters ago.

20. In the Online Appendix F, we estimate the model targeting inflation and output growth expectations and show that it can generate boom-bust cycles in macro variables.

21. We compute $RMSE = \sqrt{\sum_{i=1}^{N} \sum_{t=1}^{T} (IRF^i_{data,t} - IRF^i_{model,t})^2 / T}$, where $IRF^i_{data,t}$ and $IRF^i_{model,t}$ indicate the local projection IRF and model IRF, respectively, for (i) GDP, investment and expected inflation and output growth (all untargeted moments) or for (ii) expected inflation and output growth (untargeted survey moments only).

To further examine the importance of distant memory and complement our previous discussion on alternative models, the right panel of Figure 4 shows how the impulse response for consumption changes as we vary the lag for the reference distribution. We consider the counterfactual case in which only recent memory matters ($J=1$) or when only two-periods-ago expectations matter ($J=2$). Other parameters are fixed at the benchmark estimates. Reducing the lag impacts the frequency and the amplitude of the boom-bust cycles. As we discuss below, when J increases the effects of past misperceptions accumulate, leading to larger fluctuations.

4.5. Mechanism

We have emphasized throughout this paper that the interaction of endogenous states and distant memory affects equilibrium actions and DE beliefs. In this class of NK models, these effects occur both on the real and nominal side of the economy, which in equilibrium are jointly determined. To describe the overall mechanism in this model, we first focus on consumption dynamics by leveraging the qualitative insights on repeated boom-busts of the PIH model of Subsection 3.3. In contrast to the PIH model, where the real interest rate was constant, a consumption boom-bust is now accompanied by a corresponding movement in the perceived real interest rate. Thus, our second line of argument is to describe the nominal side, and in particular the novel and critical role played by perceptions over inflation.

4.5.1. Capital surprise as an endogenous informational state We have described in detail in the PIH model how the interaction of endogenous predictability and distant memory delivers a novel, informational state, capturing surprises over the endogenous state of capital. A similar force characterizes this richer NK model. In particular, resembling Proposition 6, the surprise $\mathbb{N}^k_{J,t-1} \equiv \widehat{k}^{\theta}_{t-1} - \mathbb{E}^r_t[\widehat{k}^{RE}_{t-1}]$ emerges as an endogenous informational state. Here $\mathbb{E}^r_t[\widehat{k}^{RE}_{t-1}]$ is the reference expectation for $\widehat{k}_{t-1}$, where we use lowercase letters with hats to denote variables in log-deviations from the steady states.[22]

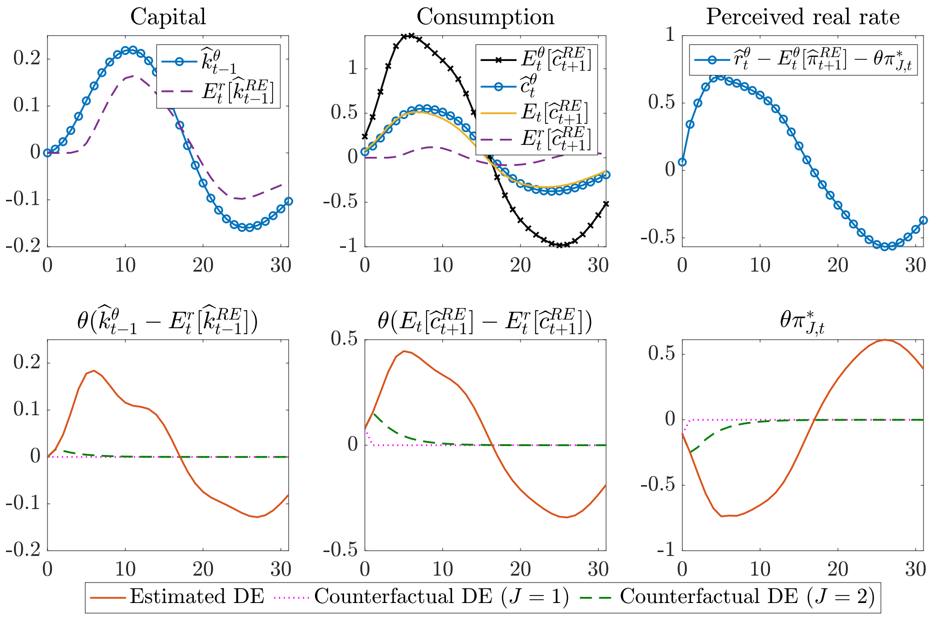
Figure 5 shows the path of equilibrium capital $\widehat{k}^{\theta}_{t-1}$, its reference expectation $\mathbb{E}^r_t[\widehat{k}^{RE}_{t-1}]$ (in top left panel), and the resulting equilibrium surprise $\mathbb{N}^k_{J,t-1}$ (solid line in bottom left panel). An expansionary monetary policy shock stimulates consumption and investment so capital $\widehat{k}^{\theta}_{t-1}$ increases. The reference distribution $\mathbb{E}^r_t[\widehat{k}^{RE}_{t-1}]$ moves slowly, so the agent is positively surprised by the resources available. Due to these positive surprises, consumption and investment further rise, which in turn leads to more capital stock and further positive capital surprises. This virtuous feedback loop continues until the reference expectation $\mathbb{E}^r_t[\widehat{k}^{RE}_{t-1}]$ of capital begins to catch up to the realized $\widehat{k}^{\theta}_{t-1}$. The agent is then less optimistic about the future and begins cutting back on consumption and investment.

Eventually, the economy enters a bust phase when the agent becomes *disappointed* in the level of capital relative to the reference distribution formed at the height of the boom. The capital surprise $\mathbb{N}^k_{J,t-1}$ thus turns from positive to negative, leading now the

22. By equation (4.36), this reference expectation is a weighted average of projections over capital at $t-1$, conditional on $t-j$ information. When $j=1$, this projection is simply the observed realized state $\widehat{k}^{\theta}_{t-1}$, while for $j>1$ the projection makes $t-j$ conditional forecasts over the uncertain $\widehat{k}_{t-1}$ using the RE law of motion.

FIGURE 5

Capital, consumption, and perceived real rate



Notes: The figure reports the response to the monetary policy shock for capital, consumption, and the perceived real rate (top panels). We also report responses of the capital and consumption surprises and the perceived innovation in cumulative inflation, scaled by θ (bottom panels).

agent to over-correct. Consumption is now reduced below the steady-state level, pushing down the level of aggregate demand and capital accumulation. A bust period arises, where the feedback between perceived pessimism leads to further economic declines and disappointment in the agent's perceptions of her resources (relative to her more optimistic forecast formed several periods ago). Thus, similar to the illustrations in the PIH model, in this quantitative model an endogenous boom-bust in the informational state $\mathbb{N}_{J,t-1}^k$ emerges, triggered by the sole realization of an iid shock (here the monetary policy shock).

As emphasized in Section 3, when memory recall is based only on immediate past (i.e., $J=1$), there is no perceived surprise over the endogenous state $\widehat{k}_{t-1}^{\theta}$, since its realization entirely informs the comparison group relevant for the DE beliefs. Indeed, in a counterfactual case where $J=1$, the reference expectation $\mathbb{E}_t^r[\widehat{k}_{t-1}^{RE}]=\widehat{k}_{t-1}^{\theta}$ and thus the perceived surprise $\mathbb{N}_{J,t-1}^k=0$ at all times, as shown by the flat magenta dotted line in the bottom left panel of Figure 5. The same panel also shows that as soon as $J=2$ the surprise $\mathbb{N}_{J,t-1}^k$ is activated (the green dashed line). However, in the counterfactual DE model where $J=2$, the duration and magnitude of the surprise are both very small. In contrast, consistent with the discussion around the PIH model (see also Figure 1), the more distant memory estimated in our model increases both the duration and the magnitude of the boom-bust cycles in the surprise $\mathbb{N}_{J,t-1}^k$.

4.5.2. Joint real and nominal dynamics We now analyze the key mechanism behind the joint determination of the real and the nominal side, and in particular the novel and critical role played by inflation dynamics. For this purpose, we focus on the Euler equation for bonds, momentarily ignoring habit formation:

$$\frac{\left(C_t^\theta\right)^{-1}}{P_t^\theta}=\beta R_t^\theta \mathbb{E}_t^\theta\left[\frac{(C_{t+1}^{RE})^{-1}}{P_{t+1}^{RE}}\right],$$

that, expressing in terms of deviations from the steady state and rearranging, becomes

$$\begin{aligned}\mathbb{E}_t^\theta[\widehat{c}_{t+1}^{RE}]-\widehat{c}_t^\theta &= \widehat{r}_t^\theta-\mathbb{E}_t^\theta[\widehat{\pi}_{t+1}^{RE}]-\ \theta\underbrace{\left[p_t^\theta-\mathbb{E}_t^r\left[p_t^{RE}\right]\right]}_{\text{Surprise in price level}}\\ &=\underbrace{\widehat{r}_t^\theta-\mathbb{E}_t^\theta[\widehat{\pi}_{t+1}^{RE}]-\theta\pi_{J,t}^*}_{\text{Perceived real rate}},\end{aligned} \tag{4.39}$$

where

$$\pi_{J,t}^*\equiv\underbrace{\textstyle\sum_{j=1}^J\alpha_j\left(\widehat{\pi}_{t-j+1,t}^\theta-\mathbb{E}_{t-j}[\widehat{\pi}_{t-j+1,t}^{RE}]\right)}_{\text{Perceived innovation in cumulative inflation}}=p_t^\theta-\mathbb{E}_t^r\left[p_t^{RE}\right],$$

and each term $\widehat{\pi}_{t-J+1,t}=\widehat{\pi}_{t-J}+\widehat{\pi}_{t-J+1}+...\widehat{\pi}_t=p_t-p_{t-J}-\pi$ denotes the *cumulative* inflation between $t-J$ and t. For further reference, we denote that surprise, or *perceived innovation in cumulative inflation*, as the equilibrium object $\pi_{J,t}^*$.[23]

Thus, in the Euler equation (4.39) the perceived consumption growth (on the LHS) equals the perceived real rate (on the RHS), where both equilibrium objects are jointly formed under beliefs driven by DE. We analyze these two objects in turn.

Consider first expected consumption growth under DE, given by:

$$\mathbb{E}_t^\theta[\widehat{c}_{t+1}^{RE}]-\widehat{c}_t^\theta=\mathbb{E}_t[\widehat{c}_{t+1}^{RE}]+\quad\theta\underbrace{\left(\mathbb{E}_t[\widehat{c}_{t+1}^{RE}]-\mathbb{E}_t^r[\widehat{c}_{t+1}^{RE}]\right)}_{\text{Surprise in expected consumption}}\quad-\widehat{c}_t^\theta \tag{4.40}$$

where $\mathbb{E}_t^r[\widehat{c}_{t+1}^{RE}]=\sum_{j=1}^J\alpha_j\mathbb{E}_{t-j}\widehat{c}_{t+1}^{RE}$, by equation (4.36). The top middle panel of Figure 5 plots the elements entering equation (4.40). During the boom (bust) phase $\widehat{c}_t^\theta$ and $\mathbb{E}_t[\widehat{c}_{t+1}^{RE}]$ rise (fall) by a similar amount, while the reference expectation $\mathbb{E}_t^r[\widehat{c}_{t+1}^{RE}]$ moves sluggishly. In turn, the DE beliefs $\mathbb{E}_t^\theta[\widehat{c}_{t+1}^{RE}]$ overreact by a factor of θ to the surprise $(\mathbb{E}_t[\widehat{c}_{t+1}^{RE}]-\mathbb{E}_t^r[\widehat{c}_{t+1}^{RE}])$ in expected consumption. This surprise, plotted in the bottom middle panel of Figure 5, is an endogenous equilibrium object. The top and bottom middle panels thus show that this surprise is a key driver of the expected consumption growth under DE.

23. In the special case of $J=1$, per our earlier analytical results, equilibrium variables under the RE law of motion respond to endogenous states in the same way as they do under the DE law of economy, making the equilibrium perceived innovation in cumulative inflation take the simpler but equivalent form $\pi_{1,t}^*=\widehat{\pi}_t^\theta-\mathbb{E}_{t-1}\widehat{\pi}_t^\theta$. This form recovers the nominal price surprise object that distorts consumption smoothing in the NK model of L'Huillier et al. (2021) who focus their analysis entirely on the $J=1$ case.

As made transparent by our consumption-smoothing model, the surprise in expected consumption is generally a function of the surprise in both (i) the exogenous innovation and (ii) the endogenous states. It is only when $J=1$ that the latter does not matter, since then the time t exogenous shock is the only change in the information set from the immediate past $t-1$ to current t. The magenta dotted line in Figure 5 confirms that in a counterfactual with $J=1$ the surprise in expected consumption moves only at the time of the exogenous shock.

Instead, when memory is based on more distant past, the perceived surprises embedded in the realized path of the endogenous states matter for the DE overreaction to $\mathbb{E}_t[\widehat{c}_{t+1}^{RE}]$ in equation (4.40). In fact, the middle bottom panel of Figure 5 shows that the path of the surprise in expected consumption tracks closely the path of the surprise in capital, $\mathbb{N}_{J,t-1}^k$ (plotted in the bottom left panel). Intuitively, like in the consumption-smoothing model of Section 3, a positive (negative) surprise $\mathbb{N}_{J,t-1}^k$ makes the agent overly optimistic (pessimistic) about future resources.[24] The endogenous boom-bust in the information state $\mathbb{N}_{J,t-1}^k$ is thus reflected in periods of endogenous optimism and pessimism over future consumption.[25] The counterfactual of $J=2$ (green dashed line) indicates again that further memory lags, like in our estimated model, amplify both the duration and magnitude of the boom-bust dynamics.

We now turn to the formation of DE beliefs over nominal prices and their role in affecting the perceived real rate. The top right panel of Figure 5 shows how the boom and bust in expected consumption growth $\mathbb{E}_t^\theta[\widehat{c}_{t+1}^{RE}]-\widehat{c}_t^\theta$ discussed above is mirrored by a corresponding rise and fall in the perceived real interest rate, $\widehat{r}_t^\theta-\mathbb{E}_t^\theta[\widehat{\pi}_{t+1}^{RE}]-\theta\pi_{J,t}^*$. By equation (4.39), DE affect this rate through two channels. The first is DE over future inflation, $\mathbb{E}_t^\theta[\widehat{\pi}_{t+1}^{RE}]$. The second channel is the perceived surprise in the price level, or $\pi_{J,t}^*$. Intuitively, holding constant $\mathbb{E}_t^\theta[\widehat{c}_{t+1}^{RE}]$ and $\left(\widehat{r}_t^\theta-\mathbb{E}_t^\theta[\widehat{\pi}_{t+1}^{RE}]\right)$, a higher innovation $\pi_{J,t}^*$ makes the perceived expected future price relatively high, thus lowering the incentives to postpone consumption.

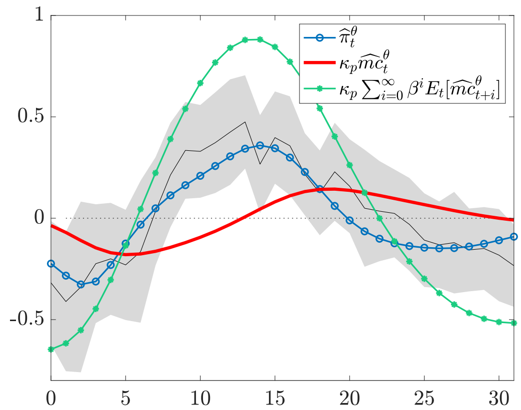
The bottom right panel of Figure 5 shows how this second channel, operating through the surprise $\pi_{J,t}^*$, drives most of the variation in the perceived real rate. To understand the equilibrium path of $\pi_{J,t}^*$, recall from Figure 2 that on impact, due to an increase in utilization, inflation $\widehat{\pi}_t^\theta$ declines, which determines a negative surprise in the price level. As shown in Figure 2, $\widehat{\pi}_t^\theta$ then starts to gradually recover and eventually rises above steady state during the economic boom, as in the data. This rise first leads to a recovery in the surprise $\pi_{J,t}^*$ back to steady state from below. Importantly, as inflation accelerates at the end of the boom, selective memory recall starts to increasingly weigh the high price level states, leading to positive surprises in $\pi_{J,t}^*$, as indicated in Figure 5. This path determines a *reversal* in the perceived innovation in cumulative inflation, which now moves into the positive territory during the bust part of the cycle.

24. While this intuition is similar to the consumption-smoothing model where capital was the only relevant endogenous state, in this rich NK model, due to its nominal and real frictions, the set of relevant endogenous states that affect $\mathbb{E}_t\left[\widehat{c}_{t+1}^{RE}\right]$ is larger than just $\widehat{k}_{t-1}$. However, the close proportionality between the path of surprises in $\mathbb{E}_t\left[\widehat{c}_{t+1}^{RE}\right]$ and $\widehat{k}_{t-1}$ indicates that in equilibrium the former is primarily influenced by the latter.

25. The Online Appendix shows that the perceived increase in consumption more than compensates for the habit stock. In other words, not only agents expect consumption to be higher in the future, but they also expect it to grow with respect to the habit stock, lowering the marginal utility. Furthermore, we show that even without consumption habit our model is able to generate boom-bust cycles.

Figure 6

Inflation, marginal costs, and New Keynesian Phillips Curve



Notes: The figure reports the response to the monetary policy shock for inflation, marginal costs (scaled by κ_p), and a counterfactual measure of inflation built assuming that agents can correctly foresee the future path of marginal costs.

DE beliefs determine misperceptions on the real and nominal side of the economy that are consistent with each other. Indeed, the bottom row of Figure 5 shows how the cycle of optimism/pessimism over future consumption tracks in equilibrium the cycle of negative/positive perceptions of prices. In particular, in the boom phase, the over-reaction to negative surprises in inflation leads to a high perceived real rate that is consistent with a perceived acceleration in consumption arising from over-reactions to capital surprises. As the reference distribution for both capital and prices slowly adjusts to their corresponding realized path, a reversal occurs. The economic boom endogenously creates the conditions for a bust, where negative surprises over capital and perceived deceleration in future consumption are consistent with high perceptions of future price levels and a low perceived real rate.

The same bottom row of Figure 5 shows, systematically across its three panels, the importance of our estimated distant memory process. Distant memory creates larger revisions in expectations, leading to larger surprises, and larger belief distortions. This explains why the parameter J does not only affect the frequency of the boom-bust cycle but also the amplitude. When $J=1$, there are no surprises in the endogenous states, and the over-reaction in expected consumption and price level arises only at the time of the shock. When $J=2$, the dynamics are still small and short-lived. Instead, under a more distant memory, agents' expectations are constantly revised and misperceptions build. In our model, past decisions affect current expectations and generate new distortions that feed into current decisions, creating endogenous waves of optimism and pessimism — a form of Minsky (1977) moments.

The success of the DE model also stems from its ability to accurately match inflation dynamics. As in the data, inflation movements seem relatively small with respect to fluctuations in real activity. To study this disconnect, consider the relation between inflation and marginal costs (see Online Appendix G for the derivation):

$$\widehat{\pi}_t^{\theta} = \kappa_p \sum_{i=1}^{\infty} \beta^i \mathbb{E}_t^{\theta} [\widehat{mc}_{t+i}^{RE}] + \kappa_p \widehat{mc}_t^{\theta}.$$

The above expression makes clear that inflation depends on the DE of future marginal costs for a given starting value of current marginal costs. To understand the effect of the distorted beliefs about future marginal costs, consider the following counterfactual measure of inflation:

$$\widehat{\pi}_t^{CF} = \kappa_p \sum_{i=0}^{\infty} \beta^i \mathbb{E}_t [\widehat{mc}_{t+i}^{\theta}].$$

The expression above captures the path of inflation that an econometrician who can accurately predict the *realized* path of marginal costs would compute.[26]

Figure 6 reports the path for marginal costs (scaled by κ_p), inflation, and for the counterfactual measure of inflation. On impact, DE inflation drops because an increase in utilization lowers marginal costs. DE inflation keeps declining for a couple of quarters, as in the data, because agents keep revising their expectations about future marginal costs. The counterfactual measure of inflation also drops on impact, but it immediately starts increasing because there are no further revisions in expectations after the first period. As marginal costs start increasing, DE inflation starts recovering, and so does the counterfactual measure of inflation.

Importantly, the counterfactual measure of inflation shows much larger fluctuations than actual inflation. This is because the actual path of marginal costs is more persistent than what is perceived by the DE agents. Thus, for a given movement in marginal costs, DE lead to *underreaction* of inflation because agents expect a relatively fast return of marginal costs to the steady state. An external observer who were endowed with the path of real activity and marginal costs would conclude that the Phillips curve is quite flat. Indeed, the estimated value for the slope of Phillips curve, κ_p, is significantly larger for the DE model, at 0.0502, compared to 0.0337 in the re-estimated RE model. While the RE model needs to appeal to a flat Phillips curve to account for the joint dynamics in real activity and inflation, the DE model is able to reconcile them based on the distorted expected path for marginal costs.

We conclude this subsection by discussing one additional point. In this economy there is positive co-movement between the key real aggregate variables (consumption, investment and hours). The economic channel is typical to the New Keynesian models. Intuitively, following the expansionary monetary policy shock, the demand for goods (consumption and investment) is stimulated. In this demand-driven economy, equilibrium is largely restored through a higher capacity utilization, which not only directly increases the supply of goods, but also leads to a higher labor productivity and thus stimulates firms' labor demand.

5. Conclusions

In this paper, we build on the DE paradigm proposed by Bordalo et al. (2018) to analyze the qualitative and quantitative implications of the joint determination of DE

26. Notice that this measure of inflation does not coincide with the shadow RE inflation, because the expected path of marginal costs is based on the DE economy.

beliefs and optimal actions in the presence of (i) endogenous states and (ii) distant memory recall. In the first part of the paper, we use a three-period consumption-savings model as a laboratory to provide behavioral micro-foundations for our analysis that we argue are psychologically and model-coherent. We then extend the model to the infinite horizon to show that under distant memory the interaction between actions and DE beliefs naturally generate repeated boom-bust cycles in response to a single initial shock. In the second part of the paper, we develop a portable solution method that can be used to solve rich general equilibrium models featuring DE. We incorporate DE into a quantitative New Keynesian model of the type widely used for policy analysis. We uncover a critical and novel role played by endogenous states and distant memory recall, which allows the DE model to replicate the empirical boom-bust cycle dynamics in response to a monetary policy shock.

There are two main avenues for future research. First, deriving and studying optimal monetary policy under different behavioral assumptions regarding agents' expectations would have important policy implications and further expand the practical relevance of DE. Second, it will be interesting to allow for non-linearities, such as changes in policy makers' behavior, stochastic volatility, and occasionally binding constraints. The methods developed in this paper can be extended to accommodate these cases by leveraging the *conditional* log-normality of the equilibrium distributions, as in the work of Dew-Becker (2014) and Bianchi et al. (2022). These extensions will allow researchers to incorporate asset pricing, breaks in the transmission mechanisms of the shocks, and changes in volatility of the macroeconomy in general equilibrium models featuring DE.

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