# Estimating Production Functions of Multiproduct Firms*

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**Abstract**

Multiproduct firms constitute a considerable share of firms and account for an even greater share of production. Nevertheless, the vast majority of production function estimates are based on the assumption that firms are single-product manufacturers. This assumption is due to a lack of data on how firms allocate their inputs across their various product lines. I provide a strategy for estimating product-specific input allocations and production functions of multiproduct firms involved in monopolistic competition. The strategy is based on using firms' product prices and output demand in solving for product-level input allocations.

Keywords: Multiproduct firm, production function, productivity

JEL codes: D24, L11, L25

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# 1 Introduction

Firms' production functions are estimated in order to identify important values like marginal costs, returns to scale, and productivity differences between firms. These attributes are essential in evaluating how changes in, for example, competition and regulation affect market outcomes. Production function estimates are typically based on the often implicit assumption that the firm produces all of its output with a firm-level production technology that is independent of the firm's product set and that the firm is equally productive in manufacturing all of its products. If a firm produces a range of products, however, it is possible that the firm uses multiple production technologies, in addition to facing distinct demand for these products. Recognising this is important because a substantial share of firms are multiproduct firms, and an even greater share of products are provided by these firms. For example, in the US manufacturing sector from 1987 to 1997, 39% of firms manufactured more than one product, and these multiproduct firms accounted for 87% of the sector's output (Bernard, Redding, and Schott, 2010). In international trade, multiproduct firms are even more prominent: they accounted for more than 99% of US exports in 2000 (Bernard, Jensen, Redding, and Schott, 2007). Moreover, empirical findings in the international trade literature suggest that multiproduct firms have productivity differences across product lines.[1] The reason for assuming a firm-level production technology that is independent of the product set is simply pragmatic: in a typical data set of a cross-section of firms, the input allocation across the firm's product lines is unobservable.

In this paper, I present a strategy for estimating product-level production functions of firms that are multiproduct producers and face monopolistic competition in the output market. Production functions may vary across products, and there may be productivity differences across products within firms. As usual, inputs are observed only at the firm or establishment level. The challenge is to solve for the product-level inputs, which are functions of, among other factors, the unobservable productivity levels. The key to the estimation strategy is that for firms that are not price-takers in the output markets, the products' output demand may be utilised to estimate the optimal product-level input allocations. The estimation strategy requires data on firms' product prices, which fortunately are available in several data sets used in the literature[2]. My strategy for estimating product-specific production functions[3] builds on the structural production function lit-

[1] Reductions in barriers to international trade are shown to raise firms' productivity and reduce their product scopes, i.e., the number of products produced, presumably because firms drop the least productive products from their selections of exported products (e.g., Baldwin, Caves, and Gu, 2005; Bernard, Redding, and Schott, 2011; Mayer, Melitz, and Ottaviano, 2014).

[2] Other data sets reporting outputs and revenues at the product-firm or product-plant levels are available from, for example, Belgium (*Statistics on the Production of Manufactured Goods* by Statistics Belgium), Colombia (*Annual Manufacturers Survey* by the National Administrative Department of Statistics of Colombia), India (*Prowess Data* by the Centre for Monitoring the Indian Economy), and the United States (*Longitudinal Research Database* by the U.S. Census Bureau).

[3] In this paper I estimate Cobb-Douglas production functions but, as shown in Appendix A, the identification strategy is also compatible with the translog production function.

erature that includes Olley and Pakes (1996), Levinsohn and Petrin (2003), Doraszelski and Jaumandreu (2013), Ackerberg, Caves, and Frazer (2015), and Gandhi, Navarro, and Rivers (2020). I show that, under the two model assumptions presented in this section, product-level inputs can be solved parametrically from the firm's static profit maximisation problem as functions of the observable data and the parameters to be estimated. While within-firm differences in output demand[4] and demand shocks provide a source for identifying variation, supply- and demand-side instruments can be used to identify the production function parameters. The only demand function parameters that need to be estimated are the own-price elasticities of demand. I demonstrate the estimation strategy by estimating production functions for products in the wood industry and find that the production technologies are statistically different across products.

In general, there are several reasons why an applied economist may want to allow for product-specific production technologies instead of a firm-level technology that is independent of the product set the firm produces. First, products may differ in how intensive their production processes are in terms of inputs such as labour. Suppose a firm-level Cobb-Douglas production function is estimated, whereas the production technologies are in fact product-specific. In that case, the estimated output elasticity of, say, labour is a linear combination of all the product-specific output elasticities of labour and of all the product-specific output elasticities of other inputs such as capital and materials (Mundlak, 1963). Second, returns to scale may take place at the product level instead of the firm level and may vary across products. Third, there may be productivity differences across products but within firms. Fourth, there may be economies of scope due to producing multiple products. In other words, when a firm-level production technology is assumed, even if some firms produce a variety of products, the estimated production function is likely to be misspecified. Consequently, the production function parameter estimates may not be consistent. Furthermore, the inferences on returns to scale, productivity, and marginal costs may be false. An additional advantage of physical product-specific production function estimates is that they enable computing products' price-cost markups.

The first assumption I make in order to solve for the product-level input allocation concerns economies of scope and flexible manufacturing. Economies of scope are obtained when simultaneous manufacturing of multiple products results in lower fixed or variable costs of production. They are a potential reason for firms to produce multiple products (e.g., Panzar, 1989). An alternative supply-side explanation for the existence of multiproduct firms is that firms can add new products to their product assortments without making large investments in production technology, even if the product-specific variable costs of the new products are higher. This is referred to as flexible manufacturing, and it is assumed in several theoretical studies concerning multiproduct firms (e.g., Eckel and Neary, 2010; Mayer, Melitz, and Ottaviano, 2014). Flexible manufacturing does not en-

[4] The demand function estimated in this paper is isoelastic, but also other functional forms can be used, as shown in Appendix A.

tail economies of scope in the form of variable costs, but some theoretical models allow for economies of scope in the form of lower fixed costs[5]. Instead, an essential feature of flexible manufacturing is the capability of a multiproduct firm to produce one or a few of its products more efficiently than the rest of its products; these are referred to as its core competency[6] (e.g., Eckel and Neary, 2010).[7] The estimation strategy of this paper is based on the assumption that there are no economies of scope, but firms may have core competencies in producing some of their products, as postulated in the theory of flexible manufacturing.[8] Therefore, the estimation strategy is not suitable for industries where products' production technologies are joint because product lines have joint inputs or by-products of a product line are used as inputs in another product line.[9]

The second assumption I make in order to solve for product-level inputs is that capital and labour have already been chosen in the previous period and can be costlessly reallocated across product lines in response to demand and productivity shocks. This assumption is most likely to hold for data on plants producing products in a single industry, as opposed to conglomerates or multiplant firms observed only at the firm level. The reason for assuming labour to be predetermined is the institutional features of the Finnish labour market. However, the estimation strategy can, under certain conditions, also be applied when the labour input is flexible.

The production functions are estimated as physical production functions; i.e., the output is measured in physical quantities instead of value added or gross output value. By doing so, the omitted price bias discussed by Klette and Griliches (1996) can be avoided. The omitted price bias arises when output is measured in value added or gross output value, while output prices and input choices are correlated due to firms' pricing power. If this correlation and demand shocks are not taken into account, estimates of the production function parameters and returns to scale are likely to be biased (De Loecker, 2011).[10] Another advantage of estimating physical production functions is that physical productivity changes can be disentangled from price changes, which is not possible when estimating revenue production functions with conventional methods (Garcia-Marin and Voigtländer, 2019). In order to estimate physical production functions, the output quan-

[5] For example, Eckel and Neary (2010) and Mayer et al. (2014) assume economies of scope in the form of lower fixed costs, whereas Qiu and Zhou (2013) and Flach and Irlacher (2018) do not.

[6] "Core competency" has an alternative definition in the multiproduct firm literature: it is the most profitable activity of a firm (Bernard, Redding, and Schott, 2011).

[7] There are empirical findings consistent with firms having core competencies in producing some of their products (Arkolakis, Ganapati, and Muendler, 2021; Dhyne, Petrin, Smeets, and Warzynski, 2021).

[8] However, the estimation strategy may be adjusted to allow for the number of products to have an effect on the unobservable product-specific productivity level, as in De Loecker, Goldberg, Khandelwal, and Pavcnik (2016).

[9] In other words, the firm's production correspondence is additively separable in the production functions for each product.

[10] An additional problem arises when firm-level revenue production functions of multiproduct firms are estimated. As the composite output (i.e., the product-level outputs aggregated to the firm level) depends on the products' relative prices, so do the elasticity estimates of the firm-level function, even when controlled for the endogeneity between product prices and input choices (Mundlak, 1963).

tities have to be observable and, for any given product, comparable across manufacturers. This data requirement is a matter of how heterogeneous the products in the industry are and, eventually, how detailed the product classification is.

Another data requirement relates to dealing with product selection for which the production function estimation strategy does not account. It is well known in the literature that firm productivity and capital are correlated among the observations some distance above the market entry threshold, which may give rise to selection bias (e.g., Wedervang, 1965). This kind of selection may take place also at the product-level. To avoid selection bias, one needs a data set with a low probability of firm entry and exit and a low probability of change in the firm's product assortment. To this end, data on large, established firms with little variation in product assortment are desirable.

The present contributes to the literature in which multiproduct firms' product-dependent production technologies are estimated. Early research considers how product-specific production functions can be aggregated to the firm level (e.g., Mundlak, 1963, 1964), and how marginal costs can be obtained by estimating joint cost functions (e.g., Brown, Caves, and Christensen, 1979; Caves, Christensen, and Tretheway, 1980). More recently, product-level production functions have been estimated using the assumption that firm-level inputs are allocated across products in proportion to products' revenue shares (e.g., Foster, Haltiwanger, and Syverson, 2008; Collard-Wexler and De Loecker, 2015) or in proportion to the number of products (De Loecker, 2011).

Some of the most recent methods for estimating multiproduct production technologies neither make any assumptions on input allocation nor estimate the allocation. The technological relationship between individual outputs along the firm's production possibilities frontier has been estimated by Malikov and Lien (2021). A transformation function where the output quantity of a given product is related to the firm-level inputs and the output quantities of the other products the firm produces has been estimated by Dhyne, Petrin, Smeets, and Warzynski (2020, 2021). As the output quantities of the other products are endogenous to the productivity shock of a given product, Dhyne et al. (2020, 2021) instrument them with lagged values of the output quantities or inputs lagged even further back. Their estimation methodology is suitable for estimating production technologies for the joint production of products.

Product-specific production functions have been estimated by De Loecker, Goldberg, Khandelwal, and Pavcnik (2016). The purpose of the estimation is to evaluate how marginal costs and price cost markups respond to trade liberalisation. De Loecker et al. use data on single-product firms and the estimation strategy of Ackerberg et al. (2015) to estimate product-specific production function parameters which are assumed to be the same for single- and multiproduct firms. This enables De Loecker et al. to estimate the parameters without simultaneously solving for the unobservable input allocations. They assume that output quantities are set in the period before production and that flexible inputs are set to minimise costs, conditional on the chosen output quantities and firm-level predetermined inputs. All the predetermined and flexible inputs are allocated to product lines such that the input expenditure shares are constant across the inputs for any given product. Economies of scope may exist, but productivity differences between product lines are assumed away. De Loecker et al. show that both cost efficiency and profitability vary across the different products a firm produces.

Another solution for estimating product-level production functions is provided by Orr (2022). In contrast to De Loecker et al. (2016), who make no assumptions concerning output demand or market structure, and more similar to the present study, Orr uses information from output demand in solving for the unobservable product-level inputs. His methodology focuses on uncovering within-firm productivity dispersion while assuming that production function parameters do not vary across industries. Orr first estimates multiproduct firms' input allocations using estimates of the shape of the firm's demand function. He then estimates product-level production functions using the estimated input allocations. Orr finds that plants in machinery manufacturing have sizeable within-plant variation in efficiency.

The estimation strategy of the present study differs from the strategies discussed above because it enables estimation of non-joint and product-specific production functions with productivity differences across products, while the unobservable product-level inputs are solved accordingly. This requires, in contrast to the methodologies discussed above, simultaneous estimation of product-level input allocations and production functions.

The model and the estimation strategy are presented in Sections 2 and 3. In Section 4 I introduce the data set and provide further details of the estimation procedure. Empirical results are presented in Section 5. In Section 6 I discuss the key identifying assumptions and data requirements of the estimation strategy, how they relate to the current production function literature, and the kinds of industries for which the estimation strategy is suitable. Section 7 concludes.

## 2 The Model

The model consists of product-specific production and demand functions and of assumptions about the timing of production decisions. Firm-level production functions are typically estimated without considering demand for the products. Estimation of product-specific production functions, however, requires solving for the unobservable product-level input allocations, which in turn requires the econometrician to specify the output demand functions[11]. In demonstrating the estimation strategy, I assume Cobb-Douglas production

[11] De Loecker (2011) estimates a demand system together with production functions to control for unobserved product prices and demand shocks. In a single-product setting, Doraszelski and Jaumandreu (2013) allow for imperfect competition in the output market, but they need to specify only the own-price elasticity of demand.

and isoelastic demand functions but, as shown in Appendix A, the estimation strategy is compatible also with other functional forms, including demand functions that allow for substitution and complementarity patterns between other products and output of other producers. The identifying assumptions are discussed in more detail in Section 6.1.

## 2.1 Production

In an industry, a group of firms produces $N$ similar or related products $i$, $i \in \{1, ..., N\}$. Firm $j$ produces at least one of these products, $O_{ijt} = 1$ denotes that firm $j$ produces title $i$ at time $t$ and $O_{ijt} = 0$ otherwise, and firm $j$ does not make products in other industries. The physical output of product $i$ that firm $j$ produces at time $t$ is denoted by $Q_{ijt}$. The firm uses three inputs in making product $i$: materials $M_{ijt}$, labour $L_{ijt}$, and capital $K_{ijt}$.[12] Labour and capital are substitutable across the product lines of the firm.[13] All the factors of production are continuously divisible and exclusive across product lines. This means that they can be flexibly allocated across the different lines and that any given share of a firm-level input is used in only one product line. Outputs or by-products of other product lines are not used as factors of production. All the factors other than $M_{ijt}$, $L_{ijt}$, and $K_{ijt}$ that affect the firm's production volume of product $i$ at time $t$ are comprised in productivity, $\omega_{ijt}$. Productivity may represent (product line-specific) factors such as management and organisation of production, downtime due to, for example, maintenance work, and defect rates in the manufacturing process (Ackerberg et al., 2015). Allowing for productivity differences across product lines within firms is in line with the concept of flexible manufacturing, as discussed in the Introduction. The production technology is product-specific and independent of production of other products; that is, the firm's production correspondence is additively separable in the production functions for each product. This implies that there are no economies of scope in the form of lower variable costs.[14] The production function takes the form of Cobb-Douglas, which is a special case of the translog production function considered in Appendix A. In addition, the output elasticities of materials $\beta_{Mi}$, labour $\beta_{Li}$, and capital $\beta_{Ki}$ are product-specific. The product-specific constant of the production function is denoted by $\beta_{0i}$. Hence, the production function is written as

$$Q_{ijt} = \exp\left(\beta_{0i}\right) M_{ijt}^{\beta_{Mi}} L_{ijt}^{\beta_{Li}} K_{ijt}^{\beta_{Ki}} \exp\left(\omega_{ijt}\right). \tag{1}$$

[12] Materials $M_{ijt}$ may be an aggregate of a variety of materials. Quantities of different types of materials add up to $M_{ijt}$ with weights. The econometrician does not need to observe these weights provided that the model assumptions hold and that the quantities of and expenditures on the different material types are observable, as discussed in Section 4.1

[13] As discussed in Section 2.3, firm-level $L_{jt}$ and $K_{jt}$ are chosen at $t-1$ and they are predetermined at the time of production. However, if the econometrician rather assumes $L_{jt}$ to be flexible, i.e., to be chosen at $t$, and the requirements for making this assumption are fulfilled as discussed in Section 6.1, the assumption of labour substitutability across product lines can be relaxed.

[14] However it is possible to allow for economies of scope in the form of lower variable costs by modelling productivity $\omega_{ijt}$ as a function of the producer's product scope, as De Loecker et al. (2016) do.

Productivity $\omega_{ijt}$ follows an exogenous first-order Markov process. It can be divided into a conditional expectation of productivity, which is a function of the productivity attained at $t-1$, and a deviation from the conditional expectation of productivity; that is, a productivity shock $\xi_{ijt}$:[15]

$$\omega_{ijt} = E\left[\omega_{ijt}|\omega_{ijt-1}\right] + \xi_{ijt}. \tag{2}$$

The productivity shock $\xi_{ijt}$ is mean zero and independent across products, firms, and time.

## 2.2 Demand

Competition in the output market is monopolistic, that is, price changes of other products or by other producers have only a negligible effect on the demand for product $i$ of firm $j$. Firm $j$ faces a downward-sloping and isoelastic demand curve for product $i$ at time $t$. The demand for and price of product $i$ of firm $j$ at time $t$ are denoted by $Q_{ijt}$ and $P_{ijt}$, respectively. The demand function is written as

$$Q_{ijt} = \exp\left(\alpha_{ij}\right) P_{ijt}^{\eta_i} \exp(\varepsilon_{ijt}). \tag{3}$$

The demand for product $i$ of firm $j$ depends on factors unobservable to the econometrician and denoted by $\alpha_{ij}$. These factors vary across products and firms but are constant over time. Any shocks to the product- and firm-specific demand levels are captured by $\varepsilon_{ijt}$. These shocks can be caused by changes in buyers' preferences or income or by the number of buyers in the market, for example. The demand shock $\varepsilon_{ijt}$ is mean zero and independent of any information at time $t-1$. The price elasticity of demand $\eta_i$ is product-specific. As firm $j$ faces its own demand curve for product $i$, it operates like a monopolist in the elastic part of the demand function. Hence, to ensure positive input allocations to product line $i$, $\eta_i$ is assumed to be lower than $-1$. As competition is monopolistic, $P_{ijt}$ is determined as a function of $Q_{ijt}$, $\eta_i$, $\alpha_{ijt}$, and $\varepsilon_{ijt}$, as discussed in Section 2.3.

## 2.3 Timing of production decisions

The firm's decision-maker makes the product selection at $t-1$, before the production period begins. The choice of whether to make product $i$ depends on the profit the firm's decision-maker expects from manufacturing any product $i \in \{1, ..., N\}$ at time $t$. Product choices are not correlated with the productivity shocks $\xi_{ijt}$ or the demand shocks $\varepsilon_{ijt}$ that take place only at time $t$.

The three types of inputs $M_{ijt}$, $L_{ijt}$, and $K_{ijt}$ differ in how they are determined. The

[15] Defining the structure of the productivity process is essential for structural methods of production function estimation, such as Olley and Pakes (1996).

firm-level labour[16] $L_{jt}$ and capital stock $K_{jt}$ are predetermined at the time of production. $L_{jt}$ is chosen in the previous period, at $t-1$, and changing labour from $L_{jt-1}$ to $L_{jt}$ may involve adjustment costs. In addition, $K_{jt}$ is chosen at $t-1$. It is determined in a dynamic process as a function of the previous period's capital stock $K_{jt-1}$ and investment $I_{jt-1}$, as $K_{jt} = f(K_{jt-1}, I_{jt-1})$. Both $L_{jt}$ and $K_{jt}$ depend on the (expected) prices of labour, capital, and materials. $L_{jt}$ and $K_{jt}$ are also subject to independent variation. The product-level inputs $L_{ijt}$ and $K_{ijt}$ are allocated across product lines in the period of production, subject to the firm-level constraints $\sum_{i=1,...,N;O_{ijt}=1} L_{ijt} \leq L_{jt}$ and $\sum_{i=1,...,N;O_{ijt}=1} K_{ijt} \leq K_{jt}$.

The product-level materials $M_{ijt}$ and the firm-level materials $M_{jt}$, on the other hand, are chosen at the time of production. Setting $M_{ijt}$ and $M_{jt}$ does not involve adjustment costs; that is, the choices of $M_{ijt}$ and $M_{jt}$ do not have dynamic implications. Firm $j$ does not have monopsony power in the materials market, and thus the price level of materials, henceforth called simply the price of materials, $P_{jt}^M$, is determined exogenously to $M_{jt}$. This implies that there are no cost economies of scope or scale in the form of lower input prices. Instead, the price level of materials varies across firms due to, for example, differences in transport costs (Grieco, Li, and Zhang, 2016). Even if the materials input consists of different types of materials, $P_{jt}^M$ does not depend on the composition of $M_{jt}$. In addition, $P_{jt}^M$ is not fully determined by $P_{jt-1}^M$.

The timing regarding the firm's static profit maximisation problem is as follows. At time $t$, the firm has active production lines for titles $i$, $i \in \{1, ..., N\}$, with $O_{ijt} = 1$ denoting an active product line and $O_{ijt} = 0$ otherwise. The firm employs a predetermined capital stock $K_{jt}$ and labour $L_{jt}$. The productivity shocks $\xi_{ijt}$ and the demand shocks $\varepsilon_{ijt}$ are realised and become observable to the firm. The firm also observes the price of materials $P_{jt}^M$, and chooses the quantities of product-level materials $M_{ijt}$. The firm also decides how to allocate its labour $L_{jt}$ and capital stock $K_{jt}$ among the different product lines in which the firm is active; that is, it sets $L_{ijt}$ and $K_{ijt}$. Most of the timing assumptions of this model are similar to the assumptions previously made in the production function literature. The assumptions are compared to those in the previous literature in Section 6.1.

## 2.4 The firm's optimisation problem

Solving the product-level inputs and identifying the production function parameters requires solving the firm's static profit maximisation problem, which consists of setting $M_{ijt}$, $L_{ijt}$, and $K_{ijt}$ for the product lines in which the firm is active in order to maximise the

[16] $L_{jt}$ is often assumed to be a flexible input. I assume $L_{jt}$ to be predetermined because it is more realistic for the Finnish labour market, as discussed in Section 6.1. However, the empirical model can be adjusted to allow for flexible labour input if the wage (level) is observable and firms are price-takers in the labour market.

sum of product-level profits[17]. Denoting the firm's static profit by $\Pi_{jt}$, the static profit maximisation problem is

$$\begin{aligned}\max_{M_{ijt},L_{ijt},K_{ijt}} \Pi_{jt} &= \sum_{i=1|O_{ijt}=1}^{N} \left(P_{ijt}Q_{ijt} - P_{jt}^{M} M_{ijt}\right) \\ \text{s.t.} \sum_{i=1|O_{ijt}=1}^{N} L_{ijt} &\leq L_{jt} \text{ and} \sum_{i=1|O_{ijt}=1}^{N} K_{ijt} \leq K_{jt}. \end{aligned} \tag{4}$$

Substituting in the inverse demand, $P_{ijt} = \left(Q_{ijt}\left(\exp(\alpha_{ijt}+\varepsilon_{ijt})\right)^{-1}\right)^{\frac{1}{\eta_{ijt}}}$, and the production function, the static profit maximisation problem becomes

$$\begin{aligned}\max_{M_{ijt},L_{ijt},K_{ijt}} \Pi_{jt} &= \sum_{i=1|O_{ijt}=1}^{N} \left( (\exp(\alpha_{ij}+\varepsilon_{ijt}))^{-\frac{1}{\eta_i}} \left(\exp\left(\beta_{0i}\right) M_{ijt}^{\beta_{Mi}} L_{ijt}^{\beta_{Li}} K_{ijt}^{\beta_{Ki}} \exp\left(\omega_{ijt}\right)\right)^{\frac{1}{\eta_i}+1} - P_{jt}^{M} M_{ijt}\right) \\ \text{s.t.} \sum_{i=1|O_{ijt}=1}^{N} L_{ijt} &\leq L_{jt} \text{ and} \sum_{i=1|O_{ijt}=1}^{N} K_{ijt} \leq K_{jt}. \end{aligned} \tag{5}$$

The optimisation problem yields a Lagrangian equation with two constraints. The constraints account for not exceeding $L_{jt}$ and $K_{jt}$ in setting $L_{ijt}$ and $K_{ijt}$ for each active product line. More precisely, given that the firm maximises profit, $L_{jt}$ and $K_{jt}$ are always fully utilised, and the constraints are binding as $\sum_{i=1,...,N;O_{ijt}=1} L_{ijt} = L_{jt}$ and $\sum_{i=1,...,N;O_{ijt}=1} K_{ijt} = K_{jt}$. The Lagrangian is

$$\begin{aligned} Lagr &= \sum_{i=1|O_{ijt}=1}^{N} \left( (\exp(\alpha_{ij}+\varepsilon_{ijt}))^{-\frac{1}{\eta_i}} \left(\exp\left(\beta_{0i}\right) M_{ijt}^{\beta_{Mi}} L_{ijt}^{\beta_{Li}} K_{ijt}^{\beta_{Ki}} \exp\left(\omega_{ijt}\right)\right)^{\frac{1}{\eta_i}+1} - P_{jt}^{M} M_{ijt}\right) \\ &\quad +\lambda_{Ljt}\left(L_{jt} - \sum_{i=1|O_{ijt}=1}^{N} L_{ijt}\right) + \lambda_{Kjt}\left(K_{jt} - \sum_{i=1|O_{ijt}=1}^{N} K_{ijt}\right). \end{aligned} \tag{6}$$

The first-order conditions of static profit maximisation for firm $j$, $\forall j \in \{1,...,J\}$ at time $t$, $\forall t \in \{1,...,T\}$ are

$$\begin{aligned} \frac{\partial Lagr}{\partial M_{ijt}} &= \left(\frac{1}{\eta_i}+1\right) (\exp(\alpha_{ij}+\varepsilon_{ijt}))^{-\frac{1}{\eta_i}} \left(\exp\left(\beta_{0i}\right) M_{ijt}^{\beta_{Mi}} L_{ijt}^{\beta_{Li}} K_{ijt}^{\beta_{Ki}} \exp\left(\omega_{ijt}\right)\right)^{\frac{1}{\eta_i}+1} \frac{\beta_{Mi}}{M_{ijt}} - P_{jt}^{M} \\ &= 0 \;\forall\; i \in \{1,...,N\} \mid O_{ijt}=1, \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\partial Lagr}{\partial L_{ijt}} &= \left(\frac{1}{\eta_i}+1\right) (\exp(\alpha_{ij}+\varepsilon_{ijt}))^{-\frac{1}{\eta_i}} \left(\exp\left(\beta_{0i}\right) M_{ijt}^{\beta_{Mi}} L_{ijt}^{\beta_{Li}} K_{ijt}^{\beta_{Ki}} \exp\left(\omega_{ijt}\right)\right)^{\frac{1}{\eta_i}+1} \frac{\beta_{Li}}{L_{ijt}} - \lambda_{Ljt} \\ &= 0 \;\forall\; i \in \{1,...,N\} \mid O_{ijt}=1, \end{aligned} \tag{8}$$

[17]Firm-level labour and capital are predetermined so their costs are not included in the static profit function.

$$\frac{\partial Lagr}{\partial K_{ijt}} = \left(\frac{1}{\eta_i}+1\right)(\exp(\alpha_{ij}+\varepsilon_{ijt}))^{-\frac{1}{\eta_i}}\left(\exp(\beta_{0i})\,M_{ijt}^{\beta_{Mi}}L_{ijt}^{\beta_{Li}}K_{ijt}^{\beta_{Ki}}\exp(\omega_{ijt})\right)^{\frac{1}{\eta_i}+1}\frac{\beta_{Ki}}{K_{ijt}}-\lambda_{Kjt}$$
$$= 0\ \forall\ i\in\{1,...,N\}\mid O_{ijt}=1, \quad (9)$$

$$\frac{\partial Lagr}{\partial \lambda_{Ljt}} = L_{jt}-\sum_{i=1|O_{ijt}=1}^{N} L_{ijt}=0, \quad (10)$$

$$\frac{\partial Lagr}{\partial \lambda_{Kjt}} = K_{jt}-\sum_{i=1|O_{ijt}=1}^{N} K_{ijt}=0. \quad (11)$$

## 3 Identification and Estimation Strategy

Firm-level Cobb-Douglas production functions have been estimated in numerous studies. Estimation of the product-specific production functions in the present study differs from the estimation of firm-level functions in one important aspect: the product-level inputs are unobservable to the econometrician. This implies that all the elements in the production function are unobservable: the input quantities, the output elasticities of the inputs, and productivity. In other words, not only are the inputs endogenous to the unobservable productivity, which is a standard problem in production function estimation, but they are also unobservable. Clearly, these two problems are closely related. The observables to the econometrician are the product-level output quantities and prices $Q_{ijt}$ and $P_{ijt}$, the firm-level predetermined inputs $L_{jt}$ and $K_{jt}$, and the firm-specific price of the materials input $P_{jt}^{M}$. All these observables are measured without error.

My identification strategy builds on the structural production function literature and especially two remarks. The first remark is that the product-level inputs can be solved parametrically from the firm's static profit maximisation problem as functions of the observable data and the parameters to be estimated, as in equations (12)–(14) below. This is possible because the unobservable productivity terms can be substituted away by using the parametric form of the production function and the observable output quantities. The second remark is that when competition in the output market is imperfect, firms make their production decisions not only as a function of supply-side factors but also as a function of the demand for the products. Intuitively, the higher the demand for a given product, the more inputs the firm is willing to allocate to the product line. Product-firm-specific differences in demand generate within-firm variation in the profitability of manufacturing various products, which combine with supply- and demand-side instruments to allow the identification of product-level input allocations and the production function parameters.

The estimation strategy does not consider product or firm selection. It is therefore suitable for a setting where the likelihood of firm entry and exit is low, and changes in firms' product assortments are infrequent. Firm and product selection are discussed in

Section 6.1.

The production and demand functions are estimated by the generalised method of moments (GMM). In Section 3.1 I first show how the product-level input allocations and the productivity shock can be written in terms of the observable data and parameters to be estimated.[18] I then discuss how the production functions can be identified using instrumental variables familiar from estimation of (firm-level) production functions. Alternatively, the production functions can be identified using optimal instruments, as shown in Appendix A, which is my choice in the empirical application of this paper. In Section 3.2 I present an overidentifying moment condition for the output elasticities of flexible inputs such as materials[19], which is based on the firm's input demand function, as opposed to the production function. Finally, in Section 3.3 I discuss how the own-price elasticities of demand are identified.[20] They are the only parameters of the demand function that need to be estimated. The model can be estimated in one step, as in the empirical application of this paper. Alternatively, the own-price elasticities may be estimated first and then be used in estimating the product-level inputs and production functions.

## 3.1 Identification of the production function

The production functions are identified by, first, using the assumption of static profit maximisation and the parametric form of the model to write for the unobservable input allocations. Second, the productivity shocks are written using the Markovian structure of the productivity process. Third, the identifying moments are written using instruments familiar from the literature.

I show first how to solve for the product-level inputs as a function of observable data and the parameters to be identified. The optimal input choices are determined by the first-order conditions of the firm's static profit maximisation problem[21] in equations (7)–(11). As equations (7)–(9) show, the input choices depend on the productivity terms $\omega_{ijt}$, which are unobservable to the econometrician. However, the unobservable $\omega_{ijt}$ can be substituted away from equations (7)–(9) with the production function in equation (1) inverted for $\omega_{ijt}$; that is, the right hand side of $\omega_{ijt} = \log\left(Q_{ijt}\left(\exp(\beta_{0i})M_{ijt}^{\beta_{Mi}}L_{ijt}^{\beta_{Li}}K_{ijt}^{\beta_{Ki}}\right)^{-1}\right)$. In addition, the right hand side of the inverse demand function $P_{ijt} = \exp\left(\alpha_{ij}+\varepsilon_{ijt}\right)^{-\frac{1}{\eta_i}} Q^{\frac{1}{\eta_i}}$ that appears in equations (7)–(9) can be substituted away with the left hand side of the inverse

[18] I provide solutions for the product-level input allocations for alternative functional forms of production and demand in Appendix A.

[19] The identification strategy does not require firm-level flexible inputs to be observable. But as data sets like the one used in this paper often include firm-level expenditures on flexible inputs, they can, together with the firm-level price of materials, be used in overidentifying moment conditions such as this one.

[20] I discuss identification of the own-price elasticity of demand of alternative demand functions in Appendix A.

[21] Doraszelski and Jaumandreu (2013) also define the first-order conditions for static profit maximisation. They solve for the productivity term by parametrically inverting the intermediate input demand function thus obtained.

demand function; that is, the observable $P_{ijt}$. Consequently, the only demand function elements that remain in equations (7)–(9) are the own-price elasticity $\eta_i$ and the observable $P_{ijt}$. The Lagrangian multiplier $\lambda_{Ljt}$ in equation (8) is eliminated by substitution among equations (8) and (10). Similarly, $\lambda_{Kjt}$ in equation (9) is eliminated by substitution among equations (9) and (11). After these substitutions, the first order conditions in equations (7)–(11) can be rewritten as follows

$$M_{ijt} = \left(\frac{1}{\eta_i}+1\right) P_{ijt}Q_{ijt}\frac{\beta_{Mi}}{P^M_{jt}} \text{ where } i \in \{1,...,N\} \mid O_{ijt}=1, \tag{12}$$

$$L_{ijt} = \frac{\left(\frac{1}{\eta_i}+1\right) P_{ijt}Q_{ijt}\beta_{Li}L_{jt}}{\sum_{i=1,...,N;O_{ijt}=1}\left(\frac{1}{\eta_i}+1\right) P_{ijt}Q_{ijt}\beta_{Li}} \text{ where } i \in \{1,...,N\} \mid O_{ijt}=1, \tag{13}$$

$$K_{ijt} = \frac{\left(\frac{1}{\eta_i}+1\right) P_{ijt}Q_{ijt}\beta_{Ki}K_{jt}}{\sum_{i=1,...,N;O_{ijt}=1}\left(\frac{1}{\eta_i}+1\right) P_{ijt}Q_{ijt}\beta_{Ki}} \text{ where } i \in \{1,...,N\} \mid O_{ijt}=1. \tag{14}$$

These solutions[22] for product-level inputs are expressed only in terms of the observable data ($P_{ijt}$, $Q_{ijt}$, $P^M_{jt}$, $L_{jt}$, $K_{jt}$) and parameters to be estimated ($\eta_i$, $\beta_{Mi}$, $\beta_{Li}$, $\beta_{Ki}$).

After solving for the product-level inputs, the productivity shock $\xi_{ijt}$ can be written, given the productivity process as defined in equation (2), as

$$\xi_{ijt} = \omega_{ijt} - E\left[\omega_{ijt}|\omega_{ijt-1}\right], \tag{15}$$

where $\omega_{ijt}$ is inverted from the production function in equation (1); that is, $\omega_{ijt} = \log\left(\frac{Q_{ijt}}{\exp(\beta_{0i})M_{ijt}^{\beta_{Mi}}L_{ijt}^{\beta_{Li}}K_{ijt}^{\beta_{Ki}}}\right)$, $M_{ijt}$, $L_{ijt}$, and $K_{ijt}$ are as defined in equations (12)–(14), and the lagged $\omega_{ijt-1}$, $M_{ijt-1}$, $L_{ijt-1}$, and $K_{ijt-1}$ are defined accordingly. In this way, $\xi_{ijt}$ is written as a function of observables ($P_{ijt}$, $Q_{ijt}$, $P^M_{jt}$, $L_{jt}$, $K_{jt}$, $P_{ijt-1}$, $Q_{ijt-1}$, $P^M_{jt-1}$, $L_{jt-1}$, $K_{jt-1}$) and the parameters ($\eta_i$, $\beta_{Mi}$, $\beta_{Li}$, $\beta_{Ki}$, $\beta_0$, and the parameters in $E\left[\omega_{ijt}|\omega_{ijt-1}\right]$) to be estimated.

The output elasticities of inputs $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$ can be identified by using instruments familiar from the structural production function literature.[23,24] The exogenous $P^M_{jt}$, the predetermined $L_{jt}$ and $K_{jt}$, and the lagged $P_{ijt-1}$, $P^M_{jt-1}$, $L_{jt-1}$, and $K_{jt-1}$ are all orthogonal to $\xi_{ijt}$. Product-level materials $M_{ijt}$ are chosen as a function of $P^M_{jt}$. The higher $P^M_{jt}$ is, the lower is $M_{ijt}$; that is, the two variables are correlated. The product-level

[22]The demand function, as specified in equation (3), does not allow for substitution and complementarity between different products and products of different producers. However, as shown in Appendix A, equations (12)–(14) are also the solutions for the product-level inputs when the demand function is modified to allow for substitution and complementarity.

[23]For a discussion on identification of gross output production functions, and especially on avoiding collinearity problems, see Gandhi et al. (2020).

[24]The discussion on identification of $\beta_{Li}$ and $\beta_{Ki}$ is based on the assumption that the probability of change in the firm's product assortment is low and therefore product selection does not require particular attention. The potential firm or product selection bias is discussed in Section 6.1.

labour $L_{ijt}$ is chosen in static profit maximisation such that $L_{jt} - \sum_{i=1,...,N;O_{ijt}=1} L_{ijt} = 0$. Hence, the higher the firm-level $L_{jt}$, the higher the product-level $L_{ijt}$, as equation (13) shows. Similarly, the firm-level $K_{jt}$ and the product-level $K_{ijt}$ are correlated because the product-level $K_{ijt}$ in equation (14) is set in static profit maximisation such that $K_{jt} - \sum_{i=1,...,N;O_{ijt}=1} K_{ijt} = 0$. In the same way, the lagged $P^M_{jt-1}$, $L_{jt-1}$, and $K_{jt-1}$ determine the lagged $M_{ijt-1}$, $L_{ijt-1}$, and $K_{ijt-1}$,[25] respectively. In addition to the supply-side factors, the product-level inputs are chosen as functions of the product-specific demand. Furthermore, the demand side is a source of product-specific, within-firm identifying variation. The lagged output price $P_{ijt-1}$ increases in the product-firm-specific demand level $\alpha_{ij}$ and is thus correlated with the profit maximising $M_{ijt}$, $L_{ijt}$, and $K_{ijt}$.[26]

It is important to note that in the solution for $E\left[\omega_{ijt}|\omega_{ijt-1}\right]$, $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$ interact with the parameters governing $E\left[\omega_{ijt}|\omega_{ijt-1}\right]$. As a consequence, the production function parameters can be separately identified (except for $\beta_0$) only if the current supply-side variables - $P^M_{jt}$, $L_{jt}$, and $K_{jt}$ - vary independently from the lagged supply-side variables ($P^M_{jt-1}$, $L_{jt-1}$, and $K_{jt-1}$).[27] In other words, $P^M_{jt}$ has to vary over time such that $P^M_{jt}$ is not functionally dependent on $P^M_{jt-1}$. In addition, $L_{jt}$ and $K_{jt}$ have to vary over time such that the lagged supply-side variables, including $P^M_{jt-1}$, do not fully determine $L_{jt}$ and $K_{jt}$. These assumptions are stated in Section 2.3. In addition, the firm chooses $L_{jt}$ and $K_{jt}$ based on the same information set, but $L_{jt}$ and $K_{jt}$ are subject to sufficient independent variation.[28]

To summarise, $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$ can be identified with instruments that are familiar from the estimation of firm-level production functions. The difference in the moment conditions is that the productivity shocks are defined at the product-firm level. The moment conditions[29] are

$$\begin{aligned} E\left[\xi_{ijt}|P^M_{jt}\right] &= E\left[\xi_{ijt}|L_{jt}\right] = E\left[\xi_{ijt}|K_{jt}\right] = E\left[\xi_{ijt}|P^M_{jt-1}\right] \\ &= E\left[\xi_{ijt}|L_{jt-1}\right] = E\left[\xi_{ijt}|K_{jt-1}\right] = E\left[\xi_{ijt}|P_{ijt-1}\right] = 0 \\ \text{where } i &\in \{1, ..., N\}. \end{aligned} \tag{16}$$

As a consequence of the mean independence conditions in equation (16), $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$ can be identified using optimal instruments. They are discussed and derived in Appendix B. The GMM objective function formed using the moments in equation (16) is not convex

[25] In addition $M_{ijt-1}$, $L_{ijt-1}$, and $K_{ijt-1}$ are solved for because they are needed for solving $\omega_{ijt-1}$, which is the explanatory variable in $E[\omega_{ijt}|\omega_{ijt-1}]$.

[26] Lagged output prices have previously been used as instruments by Doraszelski and Jaumandreu (2013).

[27] Potential functional dependence problems in production function estimation are discussed in Doraszelski and Jaumandreu (2013), Ackerberg et al. (2015), and Gandhi et al. (2020). Gandhi et al. (2020) use a support condition on the regressors adapted from Newey, Powell, and Vella (1999) to avoid the functional dependence problem.

[28] As $P^M_{jt}$ is observed after $K_{jt}$ and $L_{jt}$ have been set, it is a source of variation for $M_{ijt}$ that does not determine $K_{jt}$ and $L_{jt}$.

[29] These moment conditions do not identify the price elasticity of demand, $\eta_i$. The moment conditions that identify $\eta_i$ are discussed and presented in Section 3.3.

because $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$ enter the moments non-linearly. The optimisation algorithm is discussed in Section 4.5.

In addition to the output elasticities $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$, the production function has parameters that govern the first-order Markov process of productivity, $E\left[\omega_{ijt}|\omega_{ijt-1}\right]$, as well as a constant, $\beta_{0i}$. As discussed in Section 4.3, $E\left[\omega_{ijt}|\omega_{ijt-1}\right]$ is approximated by a polynomial. Because the variables in the polynomial are exogenous, they are used as instruments for themselves.

## 3.2 Overidentifying restriction from firm-level demand for the flexible input

In addition to estimating the production functions, I estimate the firm-level demand function for the flexible input $M_{jt}$. The firm-level $M_{jt}$ comprises the unobservable product-level materials choices $M_{ijt}$ in equation (12). Observing the firm-level $M_{jt}$ is not necessary for identifying the production functions, but if $M_{jt}$ is observable,[30] it can be used to write an overidentifying restriction for the output elasticity of the flexible input, $\beta_{Mi}$. In practice, firm-level expenditures on materials are often observable in financial statement data. Given the firm-level price of materials, $P_{jt}^{M}$, a firm-level measure of materials input quantity $M_{jt}$ can be obtained, as discussed in Section 4.1.

If the econometrician is confident that $M_{jt}$ is measured with zero measurement error, the restriction of $M_{jt} = \sum_{i=1,...,N;O_{ijt}=1} M_{ijt}$ can be imposed. I require the firm-level $M_{jt}$ to be measured with a multiplicative mean zero measurement error $\epsilon_{Mjt}$. As discussed in Section 4.1, the source of measurement error in $M_{jt}$ is not $P_{jt}^{M}$ but the variable of firm-level expenditures on materials in the financial statement data. The firm-level demand for the flexible input is written as

$$M_{jt} = \sum_{i=1,...,N;O_{ijt}=1} M_{ijt}\left(1+\epsilon_{Mjt}\right), \tag{17}$$

where $M_{ijt}$ is defined as in equation (12). The firm-level input demand function can be identified using the product-level sales $P_{ijt}Q_{ijt}$ as the instruments. $P_{ijt}Q_{ijt}$ is correlated with the underlying input choice $M_{ijt}$ for two reasons. First, the higher the product-firm-specific demand level $\alpha_{ij}+\varepsilon_{ijt}$ and hence $P_{ijt}$, the higher the profit maximising $M_{ijt}$. Second, the higher the productivity and the resultant output $Q_{ijt}$, the higher the input choice $M_{ijt}$. Since $P_{ijt}Q_{ijt}$ is uncorrelated with the measurement error $\epsilon_{Mjt}$, $P_{ijt}Q_{ijt}$ is a valid instrument for identifying $\beta_{Mi}$. One more moment condition for identifying $\beta_{Mi}$ is therefore

$$E\left[\epsilon_{Mjt}|P_{ijt}Q_{ijt}\right] = 0 \text{ where } i \in \{1,...,N\}. \tag{18}$$

[30] Observing the firm-level materials input $M_{jt}$ does not eliminate the need for the observable materials price $P_{jt}^{M}$.

As the above moment condition is an overidentifying one, it can also be left out of estimation. But if $M_{jt}$ is observable, allowing for non-zero measurement error and including the overidentifying assumption in estimation uses the available information to its fullest extent.

### 3.3 Identification of the price elasticity of demand

There are two parameters in the demand function: $\alpha_{ij}$ and $\eta_i$. The product-firm-specific $\alpha_{ij}$ is a constant. Identification of $\eta_i$ requires an instrument because the price is endogenous.[31] The materials price $P_{jt}^M$, the firm-level labour $L_{jt}$, and the firm-level capital stock $K_{jt}$ correlate with the product price, while they are uncorrelated with the demand shock $\varepsilon_{ijt}$. Hence, $\eta_i$ is identified with the following moment conditions:

$$E\left[\varepsilon_{ijt}|P_{jt}^M\right] = E\left[\varepsilon_{ijt}|L_{jt}\right] = E\left[\varepsilon_{ijt}|K_{jt}\right] = 0 \text{ where } i \in \{1,...,N\}. \tag{19}$$

If $T_{ij}$ is small, however, $\alpha_{ij}$ is an incidental parameter. As discussed in Section 3.1, identification of the production function does not require $\alpha_{ij}$ to be identified, but the potential incidental parameter problem may lead to an inconsistent estimator of $\eta_i$. To avoid this, identification of $\eta_i$ requires an instrument that is uncorrelated with $\alpha_{ij}$. As $P_{jt}^M$ is determined exogenously, it is not correlated with $\alpha_{ij}$ and is therefore also a valid instrument when $\alpha_{ij}$ is an incidental parameter.[32]

## 4 Data and Empirical Implementation

### 4.1 Data

I use the Longitudinal Database on Plants in Finnish Manufacturing (LDPM) and the Industrial Output data from Statistics Finland for the years 2004–2011 (updated in 2012). The two data sets include plants that belong to manufacturing firms with at least 20 employees and a subset of plants of firms with less than 20 employees. The reporting units are mainly plants. The only exceptions are in the Industrial Output data, where a few plants that belong to the same firm report jointly. For these reporting units, I aggregate the observations in the LDPM accordingly.

I estimate product-specific production functions of plants declaring "Manufacture of wood and of products of wood and cork, except furniture; manufacture of articles of straw and plaiting materials", as Division 16 of Eurostat's PRODCOM (Production

[31] For a discussion of the instruments used in demand estimation, see, for example, Ackerberg, Benkard, Berry, and Pakes (2007).

[32] The discussion of the identification of $\eta_i$ is based on the assumption that the probability of change in the firm's product assortment is low, and therefore product selection does not require particular attention. Firm and product selection are discussed in Section 6.1.

communautaire) classification is described. This industry is a good example of an industry in which firms manufacture several different products and presumably employ multiple technologies. The products are classified according to eight-digit PRODCOM codes that are supplemented by national 10-digit subclasses. The titles are provided in Table 1. From the production perspective, products within the fairly narrowly defined titles appear to be comparable in physical quantities. At the same time, product differentiation is likely for several products, and while there are a large number of producers in the industry, competition in the output market is likely to be monopolistic. The output is sold to several other industries, including construction, trade, manufacture of paper and paper products, and generation of (bio)energy.[33] For each product a plant produces in a given year, I observe the output measured in a physical unit, $Q_{ijt}$, as well as the sales revenue. These two factors yield the average price of the product in a given year, $P_{ijt}$.

Similarly for the intermediate products and materials, I observe physical quantities and expenditures by the PRODCOM titles. These data are used to obtain the "price" of materials, $P_{jt}^M$, computed as the Elteto-Koves-Szulc (EKS) multilateral price index (e.g., Hill, 2004; Neary, 2004). For firm $j$, it can be expressed as follows (suppressing subscripts $t$ for time):

$$P_j^M = \prod_{j'=1}^{J} \left( \frac{P_F\left(\mathbf{q}^{j'}, \mathbf{q}^{j}, \mathbf{p}^{j'}, \mathbf{p}^{j}\right)}{P_F\left(\mathbf{q}^{j'}, \mathbf{q}^{b}, \mathbf{p}^{j'}, \mathbf{p}^{b}\right)} \right)^{\frac{1}{J}}, \tag{20}$$

where $\mathbf{q}^{j'}$ and $\mathbf{p}^{j'}$ are the quantity and price vectors of firm $j'$, and $P_F\left(\mathbf{q}^{j'}, \mathbf{q}^{j}, \mathbf{p}^{j'}, \mathbf{p}^{j}\right)$ is the bilateral Fisher price index between firm $j$ and firm $j'$, $j' = 1, ..., J$ ($J$ is the number of firms), which is given by

$$P_F\left(\mathbf{q}^{j'}, \mathbf{q}^{j}, \mathbf{p}^{j'}, \mathbf{p}^{j}\right) = \left( \frac{\mathbf{q}^{j'} * \mathbf{p}^{j}}{\mathbf{q}^{j'} * \mathbf{p}^{j'}} \times \frac{\mathbf{q}^{j} * \mathbf{p}^{j}}{\mathbf{q}^{j} * \mathbf{p}^{j'}} \right)^{\frac{1}{2}},$$

where $\mathbf{q}^{j'} * \mathbf{p}^{j'} = \sum_{n=1,...,N} \mathbf{q}_i^{j'} \mathbf{p}_i^{j'}$ ($N$ is the number of intermediate product and material titles); similarly for $P_F\left(\mathbf{q}^{j'}, \mathbf{q}^{b}, \mathbf{p}^{j'}, \mathbf{p}^{b}\right)$, where $b$ stands for the base firm chosen. The EKS multilateral index satisfies the circularity (transitivity) requirement, which implies that the same index is obtained irrespective of whether firms are compared with each other directly or through their relationships with other firms (Hill, 2004; Neary, 2004). The EKS multilateral index is thus well suited for my purpose of comparing firms when no representative firm exists and bundles of products differ between firms.

[33] The output sold to the paper and paper products industry and the energy sector are by-products of wood manufacturing. The data used in this paper have seven products that can be considered by-products: 16.10.23.03 Coniferous wood in chips or particles; 16.10.23.05 Non-coniferous wood in chips or particles; 16.10.41.00.10 Sawdust; 16.10.41.00.20 Woodchips; 16.10.41.00.40 Lathes, borders, etc.; 16.10.41.00.60 Bark; 16.10.41.00.80 Other wood waste (excluding sawdust, woodchips, bark, lathes, borders, pellets, briquettes etc.). The data show that about two thirds of the wood manufacturing plants have either 0 % or 100 % sales in these by-products, which suggests that shared inputs in the production of "main products" and by-products, which are assumed away in the estimation strategy, are not important.

Table 1. Product titles

| PRODCOM | Title |
|---|---|
| 16.10.10.33 | Coniferous wood; sawn or chipped lengthwise, sliced or peeled, of a thickness $>$ 6 mm, end-jointed, sanded or planed |
| 16.10.10.33.10 | Spruce wood (Picea abies Karst.), sanded or planed, end-jointed, sawn or chipped lengthwise, sliced or peeled, of a thickness $>$ 6 mm |
| 16.10.10.33.20 | Pine wood (Pinus sylvestris L.), sanded or planed, end-jointed, sawn or chipped lengthwise, sliced or peeled, of a thickness $>$ 6 mm |
| 16.10.10.35 | Spruce wood (Picea abies Karst.), fir wood (Abies alba Mill.) |
| 16.10.10.37 | Pine wood (Pinus sylvestris L.) |
| 16.10.10.50 | Wood, sawn or chipped lengthwise, sliced or peeled, of a thickness $>$ 6 mm (excluding coniferous and tropical woods and oak blocks, strips and friezes) |
| 16.10.21.10 | Coniferous wood continuously shaped (including strips and friezes for parquet flooring, not assembled) |
| 16.10.23.03 | Coniferous wood in chips or particles |
| 16.10.23.05 | Non-coniferous wood in chips or particles |
| 16.10.41.00.10 | Sawdust |
| 16.10.41.00.20 | Woodchips |
| 16.10.41.00.40 | Lathes, borders, etc. |
| 16.10.41.00.60 | Bark |
| 16.10.41.00.80 | Other wood waste (excluding sawdust, woodchips, bark, lathes, borders, pellets, briquettes, etc.) |
| 16.21.11.00 | Plywood, veneered panels and similar laminated wood, of bamboo |
| 16.21.12.14 | Plywood consisting solely of sheets of wood (excluding of bamboo), each ply not exceeding 6 mm thickness, with at least one outer ply of non-coniferous wood (excluding tropical wood) |
| 16.21.12.17 | Plywood consisting solely of sheets of wood (excluding of bamboo), each ply not exceeding 6 mm thickness (excluding products with at least one outer ply of tropical wood or non-coniferous wood) |
| 16.21.13.13 | Particle board, of wood |
| 16.21.21.18.30 | Veneer for plywood, cross-banded plywood and other wood, of coniferous wood, sawn lengthwise, sliced or peeled, of a thickness $<=$ 6 mm (excluding end-jointed, planed, sanded and board for manufacturing pencils) |
| 16.21.21.18.80 | Veneer for plywood, cross-banded plywood and other wood, of hardwood, sawn lengthwise, sliced or peeled, of a thickness $<=$ 6 mm (excluding end-jointed, planed, sanded and board for manufacturing pencils) |
| 16.21.22.00 | Densified wood, in blocks, plates, strips or profile shapes |

| | |
|---|---|
| 16.22.10.60 | Parquet panels of wood (excluding those for mosaic floors) |
| 16.23.11.10 | Windows, French-windows and their frames, of wood |
| 16.23.11.50 | Doors and their frames and thresholds, of wood |
| 16.23.19.00.12 | Carpenter's produce for walls, of wood |
| 16.23.19.00.16 | Carpenter's produce for stairs, of wood |
| 16.23.19.00.26 | Components for sauna, of wood |
| 16.23.19.00.32 | Panel elements (also glulam and cellular panels), of wood |
| 16.23.19.00.36 | Ceiling elements, of wood |
| 16.23.19.00.42 | Glulam beams and columns |
| 16.23.19.00.46 | Vertical and horizontal beams (excluding glulam beams and columns) |
| 16.23.19.00.52 | Log frames for buildings of wood |
| 16.23.19.00.90 | Other carpenter's produce, of wood (excluding doors, windows, produce for floors, walls, stairs and sauna, panel and ceiling elements, beams, columns and log frames) |
| 16.23.20.00.20 | Residential buildings of wood, for permanent habitation |
| 16.23.20.00.40 | Residential buildings of wood, for recreational use |
| 16.23.20.00.60 | Saunas of wood (outdoor saunas, assembled or prefabricated) |
| 16.23.20.00.90 | Buildings of wood (assembled or prefabricated) (excluding residential buildings and saunas) |
| 16.24.11.35 | Box pallets and load boards of wood (excluding flat pallets) |
| 16.24.13.20 | Cases, boxes, crates, drums and similar packings of wood (excluding cable drums) |
| 16.24.13.50 | Cable-drums of wood |
| 16.29.14.90 | Other articles of wood (excluding pallet collars) |

In addition to the physical quantities of and expenditures on the intermediate products and materials by PRODCOM title, I observe plant-level intermediate product and material purchase values in the Longitudinal Database on Plants in Finnish Manufacturing, which I denote by $M_{jt}^{LDPM}$. To obtain a plant-level measure of intermediate product and material use that does not comprise input price differences across plants, $M_{jt}$, I deflate $M_{jt}^{LDPM}$ by the price index for intermediate products and materials, $P_{jt}^{M}$. While $P_{jt}^{M}$ is assumed to be measured without error, $M_{jt}^{LDPM}$ and hence $M_{jt}$ are allowed to be measured with error, as discussed in Section 3.2.

The labour input is measured in labour costs, which comprise salary and social payments. The capital stock is estimated using the perpetual inventory method, $K_{jt} = (1-\delta) K_{jt-1} + I_{jt-1}$, where $\delta = 0.1$ and $I_{jt}$ is investment.[34]

The estimation strategy poses certain requirements on the estimation sample. First, observations with missing variables cannot be used in estimation. Second, each product has to be observed in at least four pairs of observations, with each observation pair being from two consecutive years in a given plant.[35] The four observation pairs may be from, for example, five consecutive years in a single plant, or two consecutive years in four plants. This is because, for each product, there are four parameters to be estimated by solving the non-linear optimisation problem,[36] and because estimating the first-order Markov process of productivity evolution requires sequences of at least two observations. To be included in the estimation sample, a plant-level observation has to contain at least one product that the plant also produced in the previous period or in the following period. Observations that do not fulfil the aforementioned criteria are dropped from the sample.

Recall that measurement error in output is assumed to be zero. Unfortunately, there is no other output variable that could be used to verify the accuracy of the product-level sales revenue variables. The only other output variable available is the plant-level gross output value reported in the LDPM. Gross output value is defined as the sum of sales revenue, deliveries to other plants of the firm, changes in inventories, production for own use, and other business revenue, deducting capital gains and acquisition of merchandise. Not surprisingly, gross output value is not equal to the sum of product-level sales revenues from production in all of the plants. There are several possible explanations for this. Plants may produce output that is not included in the sales revenue from production (due deliveries to other plants of the firm, positive changes in inventories, production for own use), or the sales revenue data may include output produced in some previous year (due to negative changes in inventories). Moreover, because capital gains and acquisition of merchandise

[34] I have set $\delta = 0.1$ following the examples of, among others, Levinsohn and Petrin (2003), who estimate production functions of Chilean manufacturing firms, and Doraszelski and Jaumandreu (2013), who estimate production functions of Spanish manufacturing firms.

[35] The requirement of at least four observation pairs can cut down the data set considerably if there are many products that only very few plants or firms produce. For most industries, however, the aforementioned requirement is not likely to affect the sample considerably.

[36] The rest of the parameters are concentrated out of the non-linear optimisation problem, as discussed in Section 4.3.

are deducted from gross output value, it is not possible to make strong inferences about potential measurement error in output. Unfortunately, the various components of gross output value are not reported in the LDPM, and hence I cannot identify why gross output value may differ from sales revenue. However, to reduce the likelihood of using observations with major measurement error in output, I use only those observations for which the ratio of the sum of sales revenue to gross output value is at least 0.6 but not more than 1.4.

In practice, the aforementioned requirements on data restrict the estimation sample as follows. I start with 3660 product-plant-year observations (exactly 1200 plant-year observations) and 66 products once the observations with any missing variables have been excluded. I first drop products with fewer than four observations pairs (step 1). I then drop i) those plant-year observations for which at least one product has been dropped in step 1, as well as ii) plant-year observations for which the ratio of the sum of sales revenue (reported in the Industrial Output data) to gross output value (reported in the LDPM) is less than 0.6 or more than 1.4 (step 2). I repeat these two steps sequentially until no observations drop out. Repeating steps 1 and 2 is required because executing step 2 may result in some of the remaining products having less than four observation pairs. Similarly, dropping products brings about plant-year observations for which step 2 has to be carried out again. After repeating these two steps, there are 2936 product-plant-year observations (904 plant-year observations) and 42 products left. This estimation sample comes from 190 plants during an eight-year time period.

In this estimation sample, 673 out of 904 plant-year observations have a preceding observation. These 673 plant-year observations have altogether 2280 product-plant-year observations, of which 2053 are of products that the plant also produced in the previous year. This means that the product entry rate, excluding plants entering the data, is 10%. The product exit rate, excluding plants exiting the data, is 7%. Completely changing the product selection is very uncommon: there is only one plant-year observation with a totally different product selection in year $t$ than in $t-1$, with only one product produced in each year. Plants' product assortments range from one to 17 products, with 659 out of 904 plant-year observations containing at least two product-level observations. Product assortments vary across plants such that there are no typical product combinations. The correlation between making two products is low for most product pairs: the absolute value of the correlation coefficient is lower than 0.05 (0.1) [0.2] for 63% (81%) [91%] of all product pairs.

The estimation sample is summarised in Table 2. The descriptive statistics show that the plants vary in size, as measured by their sales and by their input use (capital, labour, and materials, reported in hundreds of thousands of euros in Table 2). For example, the 5th percentile plant has sales of $240,000$ euros in a year, while the 95th percentile plant has sales of 8 million euros. The price of materials, which is measured by the EKS multilateral price index, also varies across plants. For the 5th percentile plant, the materials price index is 0.42, while for the 95th percentile plant the index is 2.38. A median plant produces three products, and the 95th percentile plant produces seven products. In Table 2, the product-specific output prices are characterised as relative prices, where the relative price is the ratio of the price to the product-year-specific mean. The relative output prices also have a rather wide distribution, with the 5th percentile relative price being 0.21 and the 95th percentile being 1.73.

Table 2. Descriptive statistics

| | Percentiles | | | | | Mean | Std. dev. |
|---|---|---|---|---|---|---|---|
| | 5th | 25th | 50th | 75th | 95th | | |
| Sales* | 2.42 | 6.87 | 14.48 | 33.05 | 80.02 | 27.12 | 46.21 |
| Capital* | 0.30 | 1.25 | 3.71 | 11.55 | 27.30 | 8.83 | 16.04 |
| Labour* | 0.65 | 1.22 | 2.38 | 4.03 | 10.04 | 4.18 | 9.16 |
| Materials* | 1.74 | 5.09 | 11.56 | 31.57 | 70.40 | 23.88 | 38.69 |
| Price of materials** | 0.42 | 0.64 | 0.86 | 1.22 | 2.38 | 1.04 | 0.63 |
| Number of products | 1 | 1 | 3 | 4 | 7 | 3.25 | 2.35 |
| Observations | 904 | | | | | | |
| Relative price*** | 0.21 | 0.83 | 0.95 | 1.05 | 1.73 | 1 | 0.97 |
| Observations | 2936 | | | | | | |

* Measured in 100 000 euros

** Measured by the EKS multilateral price index

*** Relative to the product-year-specific mean

Output prices vary not only between plants but also to a large extent within plants. Table 3 summarises the mean differences between product prices at the plant level. In a

Table 3. Within-plant price differences

| | Mean differences between relative prices, ordered from highest to lowest (Standard deviation in brackets) | | | | |
|---|---|---|---|---|---|
| Number of products | 1st and 2nd | 2nd and 3rd | 3rd and 4th | 4th and 5th | Other |
| 2 | 0.69 | | | | |
| | (1.07) | | | | |
| 3 | 0.35 | 0.25 | | | |
| | (0.78) | (0.27) | | | |
| 4 | 0.45 | 0.15 | 0.27 | | |
| | (2.78) | (0.25) | (0.29) | | |
| 5 | 0.74 | 0.15 | 0.12 | 0.34 | |
| | (3.21) | (0.28) | (0.17) | (0.31) | |
| $\geq 6$ | 0.23 | 0.10 | 0.07 | 0.07 | 0.14 |
| | (0.26) | (0.13) | (0.09) | (0.12) | (0.20) |

two-product plant, the mean difference between the relative prices is 0.69. In a three-product plant, the mean difference is 0.35 between the highest and the second highest and 0.27 between the second highest and the lowest price. Plants with larger scopes also have within-plant price differences, as shown in Table 3. This suggests that demand varies across the different products that a plant produces.

### 4.2 Choice of instruments

Instead of using the moment conditions with traditional instruments in equation (16), I use the moment conditions with optimal instruments[37] as defined in Appendix B (equations (A.19)–(A.21)), as well as the overidentifying moment condition in equation (18). The demand equations are identified using the moment conditions in equation (19). As four moment conditions are sufficient for exact identification of the model, there are three overidentifying restrictions in this set of moments. Some of the eight- or 10-digit products have at least four but fewer than seven observation pairs; with these cases, I cannot use all seven moment conditions. Instead of dropping observations of the product entirely, I drop some of the overidentifying moments for these products. For product $i$ with only four observations pairs, I adopt moments $E\left[\xi_{ijt}|z_{Mijt}\right]=0$, $E\left[\xi_{ijt}|z_{Lijt}\right]=0$, $E\left[\xi_{ijt}|z_{Kijt}\right]=0$, and $E\left[\varepsilon_{ijt}|P_{jt}^{M}\right]=0$. Moment $E\left[\epsilon_{Mjt}|P_{ijt}Q_{ijt}\right]=0$ ($E\left[\varepsilon_{ijt}|L_{jt}\right]=0$) [$E\left[\varepsilon_{ijt}|K_{jt}\right]=0$] is used when there are at least five (six) [seven] observation pairs. This gives a total of 272 moments.

### 4.3 Estimation of the constants and the productivity process

The production and demand functions involve parameters that can be concentrated out of the non-linear optimisation problem that is solved for estimating $\beta_{Mi}$, $\beta_{Li}$, $\beta_{Ki}$, and $\eta_i$. In the production function, these parameters are the product-specific constant $\beta_{0i}$ and the parameters governing the productivity process $E\left[\omega_{ijt}|\omega_{ijt-1}\right]$, for fixed $\beta_{Mi}$, $\beta_{Li}$, $\beta_{Ki}$, and $\eta_i$. The productivity process is estimated nonparametrically by approximating it with a second-order polynomial of the lagged productivity term $\omega_{ijt-1}\left(\beta_{Mi},\beta_{Li},\beta_{Ki},\eta_i\right)$, denoted by $g\left(\omega_{ijt-1}\right)$. Estimates of the parameters in $g\left(\omega_{ijt-1}\right)$, denoted by $\boldsymbol{\gamma}_i$, and $\beta_{0i}$, are obtained by regressing the productivity level implied by a given set of parameter values, $\omega_{ijt}\left(\beta_{Mi},\beta_{Li},\beta_{Ki},\eta_i\right)$, on the second-order polynomial terms of the implied lagged productivity, $\omega_{ijt-1}\left(\beta_{Mi},\beta_{Li},\beta_{Ki},\eta_i\right)$.

The demand function has one parameter, the product-firm-specific constant $\alpha_{ij}$, that can be concentrated out of the non-linear optimisation problem. For a given $\eta_i$, the estimation equation is $q_{ijt}-\eta_i p_{ijt}=\alpha_{ij}+\varepsilon_{ijt}$, which can be estimated by ordinary least squares (OLS).

[37] The objective function appears smoother and the estimates are less responsive to the starting values when the optimal instruments are used instead of the traditional instruments. This is because the functional forms are exploited to a fuller extent.

### 4.4 Parameter restrictions

Every product $i$ is related to four parameters that are estimated by solving the non-linear optimisation problem: the price elasticity $\eta_i$ and the output elasticities $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$. If I defined the parameters at the eight- or 10-digit level, I would need to estimate $42 \times 4 = 168$ parameters by solving the non-linear optimisation problem. At least in my setting, this is too many. Instead, I define these parameters at the three-digit level,[38] which yields two product categories: "Sawmilling and planing of wood" (PRODCOM code 161) and "Products of wood, cork, straw and plaiting materials" (PRODCOM code 162). This specification implies estimating $2 \times 4 = 8$ parameters by solving the non-linear optimisation problem. The parameters governing the productivity process $g(\omega_{ijt-1})$ are also specified at the three-digit level. The production function constant $\beta_{0i}$ is specific to the product as defined at the eight- or 10-digit level.[39] The demand constant $\alpha_{ij}$ is specific to the eight- or 10-digit product and the plant.

There are 15 products in PRODCOM category 161 and 27 titles in PRODCOM category 162. A plant produces on average 2.17 titles in category 161 and 1.08 titles in category 162. Of the plants in the sample, 56% produce at least one product in category 161, and 61% of the plants produce at least one product in category 162. Of the plants that produce any product in category 161, 86% produce at least two titles in that category. Similarly, 43% of the plants that produce any product in category 162 produce more than one title in that category.

### 4.5 Estimation algorithm

The parameters are estimated by iterated GMM. The estimation algorithm for computing the moments is as follows. At the beginning of each outer GMM iteration, I compute the optimal instruments given the starting values for $\eta_i$, $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$. First, I compute the productivity level implied by the starting values for $\eta_i$, $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$, which I call $h_{ijt}$ instead of $\omega_{ijt}$ (step 1). Second, I estimate the parameters of the implied productivity process $g(h_{ijt-1})$ by OLS, where $h_{ijt}$ is the dependent variable, and the explanatory variables are polynomial terms of $h_{ijt-1}$, as in the Markov process for productivity (step 2). Finally, I compute the optimal instruments as functions of the data, the starting values for $\eta_i$, $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$, and the implied estimates of the parameters in $g(h_{ijt-1})$. On the second and any subsequent outer GMM iteration, the starting values for $\eta_i$, $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$ are the parameter estimates obtained in the previous outer GMM iteration.

For each inner GMM iteration, I compute the moments, given some values for $\eta_i$, $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$, and the optimal instruments computed at the beginning of the outer GMM iteration. First, I compute $h_{ijt}$ as in step 1 of the outer GMM iteration. Second,

[38] The implications of this restriction are discussed in Section 6.1.

[39] Different products' physical output productivities cannot be meaningfully compared unless the different products' productivity distributions are normalised in some way.

I estimate the parameters of $g(h_{ijt-1})$ as in step 2 of the outer GMM iteration. Third, I compute the productivity shock $\xi_{ijt}$ and the measurement error in the firm-level flexible input $\epsilon_{Mjt}$, as described in equations (15) and (17), respectively. The demand shock $\varepsilon_{ijt}$ is obtained from equation (3), as the product-firm-specific constant $\alpha_{ij}$ is estimated by OLS given the output and price data and some value for $\eta_i$. Finally, I compute the moments.

As multiple parameters are estimated by solving the non-linear optimisation problem, I need to make sure that the estimation routine reaches the global minimum among the local minima of the GMM objective function. I experiment with various minimisation algorithms; the Gauss-Newton algorithm turns out to be the least sensitive to starting values. I also run the estimation routine with a large set of alternative starting values.[40] The GMM estimator uses the efficient weighting matrix. Since the parameter estimates are obtained in one step, the asymptotic standard errors are computed using standard GMM formulas and numerical derivatives.

## 5 Results

The estimation results are presented in Table 4. The production and demand functions are estimated for the products in "Sawmilling and planing of wood" (PRODCOM 161) and "Products of wood, cork, straw and plaiting materials" (PRODCOM 162). All the output elasticities of inputs and price elasticities of demand are statistically significant.[41] In addition, the estimates of the output elasticities are statistically different for the technologies in the two product groups. The output elasticity of materials $\beta_{Mi}$ is considerably higher for PRODCOM 162 than for PRODCOM 161 (0.73 and 0.37, respectively). The output elasticity of labour $\beta_{Li}$ is again considerably lower for PRODCOM 162 than for PRODCOM 161 (0.13 and 0.36, respectively). The order of these estimates is not what we may expect as the output titles in PRODCOM 162 are more processed than the output titles in PRODCOM 161. However, a plausible explanation for these estimates relates to the composition of the material inputs used for PRODCOM 161 and PRODCOM 162. The materials used in manufacturing PRODOM 161 are raw in the sense that they have barely been processed before being transported to the plant. At least some of the materials and intermediate products used in manufacturing PRODCOM 162, on the other hand, are products of other industries. In other words, some labour and capital inputs have already been used in manufacturing the material inputs for PRODCOM 162, raising its value. As a consequence, $\beta_{Mi}$ for PRODCOM 162 is also rather high. The output elasticity of capital is of the same magnitude for PRODCOM 161 and PRODCOM 162 ($\beta_{Ki}$ is 0.20 and 0.18, respectively). Returns to scale are different across the products in PRODCOM 161 and PRODCOM 162: the technology for the products in PRODCOM 161 is subject

[40] The starting values for $\beta_{Mi}$, $\beta_{Li}$, and $\beta_{Ki}$ range between 0.15 and 0.5, and the starting values for $\eta_i$ between $-8$ and $-1.5$.

[41] The 42 constants $\beta_{0i}$ and the parameters governing the productivity process $g(\omega_{ijt-1})$ are not reported.

to decreasing returns to scale ($\beta_{Mi} + \beta_{Li} + \beta_{Ki} = 0.93 < 1$), while the technology for the titles in PRODCOM 162 has increasing returns to scale ($\beta_{Mi} + \beta_{Li} + \beta_{Ki} = 1.04 > 1$). In short, the various products in the product groups "Sawmilling and planing of wood" and "Products of wood, cork, straw and plaiting materials", which many multiproduct firms produce simultaneously, are not manufactured with a single production technology.

Table 4. Parameter estimates
PRODCOM 161 Sawmilling and planing of wood
PRODCOM 162 Products of wood, cork, straw and plaiting materials

| | Parameter estimate (standard error) | |
|---|---|---|
| | PRODCOM 161 | PRODCOM 162 |
| Materials | 0.37 | 0.73 |
| | (0.008) | (0.002) |
| Labour | 0.36 | 0.13 |
| | (0.011) | (0.003) |
| Capital | 0.20 | 0.18 |
| | (0.008) | (0.003) |
| Price elasticity of demand | −1.29 | −1.13 |
| | (0.020) | (0.004) |
| Prob[Chi-sq.(264)>J] | 0.4632 | |
| Number of obs. | 2053 | |

The demand for the titles in PRODCOM 161 is more price elastic than the demand for the titles in PRODCOM 162, as $\eta_i$ for the titles within PRODCOM 161 is $-1.29$, while $\eta_i$ for the titles within PRODCOM 162 is $-1.13$. This is intuitive because products of wood, cork, straw, and plaiting materials are likely to be more differentiated than the output of sawmilling and planing of wood. Hansen's J-test does not reject the null hypothesis of valid overidentification restrictions (Prob[Chi-sq.(264[42])>J] is 0.4632).

# 6 Discussion

In Section 6.1 I discuss the assumptions and data requirements underlying the identification and compare them to the assumptions and requirements of other structural production function papers. In Section 6.2 I discuss the extent to which the empirical strategy accommodates economies of scope and differentiated products.

[42] The number of moment conditions is 272.

### 6.1 Identification

Most of the assumptions underlying the identification strategy are familiar from the structural production function literature (e.g., Olley and Pakes, 1996; Levinsohn and Petrin, 2003; Doraszelski and Jaumandreu, 2013; Ackerberg et al., 2015; Gandhi et al., 2020). I also make some novel assumptions. One of the key assumptions in all the estimation strategies concerns the timing of the input choices with respect to the evolution of the productivity process. As in other structural production function models, at least one flexible (product-level) input is required to solve the firm's static profit maximisation problem. In this paper, firm-level labour is considered to be a predetermined input. The reason for this is the environment in which the data have been generated: employment protection legislation plays a significant role in Finland.[43] However, the empirical model may be adjusted to allow for flexible labour input if the wage (level) is observable and the assumptions concerning materials input and the exogeneous price of materials are also plausible for labour and wage. In that case, instead of $L_{jt}$ and $L_{jt-1}$, wage and lagged wage are valid instruments for identifying $\beta_{Li}$.

I further specify that, while the firm-level $L_{jt}$ and $K_{jt}$ are predetermined, the product-level allocations of labour and capital, $L_{ijt}$ and $K_{ijt}$, are flexible. This assumption not only facilitates solving for the product-level $L_{ijt}$ and $K_{ijt}$ but also allows firms to reallocate labour and capital in response to demand and productivity shocks. Whether the assumption of zero adjustment costs holds depends on the set of technologies the firm uses and whether the firm is a single-plant firm or observed at the plant level. While most multiproduct firms manufacture products in a single industry, a considerable share of capital and labour is likely to be applicable to several product lines, even if predetermined at the plant level. On the other hand, the adjustments producers are willing to make in response to demand and productivity shocks are likely to be moderate in magnitude. Hence, the assumption of costless reallocation is likely to be realistic to the extent that single-plant firms, or multiplant firms observed at the plant-level, active in a single industry are willing to make reallocations due to demand and productivity shocks.

One more difference in the assumptions on timing and residuals between this and other structural estimation strategies is that I assume away productivity shocks once the product-level inputs have been set. I also assume away measurement error in output $Q_{ijt}$. I make these assumptions in order to solve for the unobservable input allocations while controlling for the unobservable productivity $\omega_{ijt}$.

In addition to the timing assumptions, the proxy methods are based on the assumption

[43] The OECD indicators of employment protection (OECD, 2013) measure the strictness of legislation on individual and collective dismissals and the strictness of hiring employees on temporary contracts. The measures are based on information about statutory and case law, collective bargaining agreements, and advice from officials from OECD member countries and country experts. According to these indicators, the Finnish labour market was at the OECD average in terms of the strictness of employment protection during the 2004–2011 period. Based on this measure, predetermined labour input is a realistic assumption.

that productivity $\omega_{ijt}$ is the only scalar unobservable that affects the proxy variable. Unobservable inter-firm variation in, say, the input price, is assumed away. I also need to make the scalar unobservable assumption. But in contrast to the other estimation strategies, the materials price[44] $P_{jt}^{M}$ is required for identifying $\beta_{Mi}$. In this model the scalar unobservability assumption implies that $P_{jt}^{M}$ does not depend on the quantity of materials, $M_{ijt}$. This assumption is plausible for industries in which firms do not have monopsony power in the raw materials markets; that is, for firms that are not large relative to the materials markets.[45]

In contrast to most of the other structural methods, this paper requires the econometrician to observe the output prices and specify the output demand function. Moreover, the structural method is applicable only for firms that are not price-takers in the output market. Because differences in the product-firm-specific demand level, $\alpha_{ij}$, and product-firm-time-specific demand shocks, $\varepsilon_{ijt}$, generate within-firm variation in the profitability of manufacturing various products, the product-level input allocations and production function parameters can be identified with valid instruments. However, the econometrician has to identify only the own-price elasticities of the products that the firm produces, even when substitution and complementarity between different products and products of different producers exist, as shown in Appendix A. The demand function estimated in the empirical application is isoelastic, but the product-level inputs are solved also for linear and exponential demand functions in Appendix A.

I make the assumption of monopolistic competition; that is, the assumption that price changes in substitutes and complementary products have only a negligible effect on the demand for a given product of a specific producer. This assumption is likely to be most appropriate in an industry where a large number of firms produces differentiated products. Assuming isoelastic demand, as in this paper, implies that for a given product, price-cost markups are constant across producers. Alternative functional forms of demand, such as the exponential and linear demand considered in Appendix A, break the assumption of constant price-cost markups for a given product. Identification of the isoelastic demand function of the empirical application is based on two assumptions. First, the level of demand for a given product of a given firm is constant over time, except for mean independent demand shocks. If appropriate, potential industry-level changes in demand can be accounted for by estimating product-time-specific fixed effects, as shown in Appendix A. Second, changes in input prices and predetermined input stocks shift the supply curve,

[44] In the wood products industry considered in the present study, input prices vary across plants because of transportation costs that constitute a considerable share of the cost of wood purchases.

[45] This assumption is plausible, at least, in the wood products industry. First, the most important material for wood product plants or firms is raw wood. The majority of raw wood is purchased by the paper and paper manufacturing industry. Thus, demand shocks for wood products are not likely to have an important effect on wood prices. Second, the Natural Resources Institute Finland (Luke) publishes monthly statistics on wood prices. These statistics are based on more than 90% of the volume purchased by the forest industry. This suggests that buyers of raw wood are well informed about the current price levels of various raw wood types, leaving little room for negotiation.

while the demand curve, including the demand shock $\varepsilon_{ijt}$, is not affected. Using material prices and predetermined input stocks as instruments is a standard practice.

The estimation strategy takes into account neither firm nor product selection that takes place due to the expected profitability of manufacturing any particular product in the industry. The demand-side factor that affects the profitability of manufacturing product $i$ is the product-firm specific level of demand $\alpha_{ij}$. The supply-side factors are, for example, the expected product-specific productivity level the firm would have for the product, as well as the stock of capital and labour the firm employs at $t-1$. As is well attested in the literature, firm selection leads to firm productivity and capital being negatively correlated among the observations that are some distance above the threshold of market entry and exit (e.g., Wedervang, 1965). If this is not taken into account, the estimated output elasticity of capital is biased downwards. In general, large, established firms have a low likelihood of exit, so selection is expected to be less of a problem when the data are on large firms.[46]

Product selection regarding supply-side factors may be a concern for the following reason. A large stock of firm-level capital or labour may enable the firm to profitably open, or not close, a product line even if its productivity is low. As equations (13) and (14) show, the product-level $K_{ijt}$ and $L_{ijt}$ increase in the firm-level $K_{jt}$ and $L_{jt}$, respectively. Hence, it is possible that the product-level $K_{ijt}$ or $L_{ijt}$ is negatively correlated with the product-specific $\omega_{ijt}$. However, firms are likely to add (drop) products to (from) their product selection in descending (ascending) order of $\omega_{ijt}$. This implies that the more multiproduct firms there are, or the more products firms have in their assortments,[47] the lower the correlation between $\omega_{ijt}$ and $K_{ijt}$ or $L_{ijt}$ is likely to be.[48] Infrequent changes in firms' product assortments is another indication of the low probability of product-level entry and exit in the data.[49]

Product selection also requires consideration of output demand. Recall from Section 3.3 that $\alpha_{ij}$ is an incidental parameter, and therefore consistent estimation of $\eta_i$ requires an instrument that is uncorrelated with $\alpha_{ij}$. Materials price $P^M_{jt}$ is assumed to be exogenous and hence uncorrelated with $\alpha_{ij}$. However, if $P^M_{jt}$ is correlated over time, it is possible

[46] According to the two data sets I use, the annual exit rate of plants in the wood products industry is 5.6%. Unfortunately, the data set is not informative about whether the exits from the data are real plant closures or whether the plants have not replied to the survey. Most of the plants are large, though, which suggests that the probability of plant exit is low.

[47] I use data on 904 plant-year observations in which 659 plants produce at least two products; the average scope of all plants is 3.25 products.

[48] De Loecker et. al (2016) estimate product-specific production functions using a panel of single-product firms, some of which may be multiproduct firms in other periods. They estimate the production functions correcting for the selection to become a multiproduct firm using the insights of Olley and Pakes (1996) and compare the results to those estimated without correcting for such selection. They conclude that "the use of the unbalanced panel of single-product firms (which includes firms that are always single-product and firms that ultimately transition to a multi-product status) likely alleviates most of the concerns about the selection bias" (De Loecker et al., 2016, p. 483).

[49] I use data where product entry rate, excluding plants entering the data, is 10%, and product exit rate, excluding plants exiting the data, is 7%.

that $P_{jt}^{M}$ is positively correlated with $\alpha_{ij}$ among the observations close to the product-level entry-exit threshold. In addition, for this reason the estimation strategy is suitable for a setting where the probability of change in the firm's product assortment is low. Alternatively, the model may be extended to control for, first, firm selection and, second, selection into various product lines by computing propensity scores for entry, as in Olley and Pakes (1996).

The production function parameters are estimated by solving a non-linear optimisation problem. This implies that the parameters may not be estimated as specific to the most detailed product classification available, as discussed in Section 4.4. In the empirical example in the present study, the production functions are estimated at the eight- or 10-digit level of the PRODCOM classification, but the output elasticities of inputs are estimated as specific to the three-digit classification. The implication of this restriction is that the elasticity estimates are weighted averages of the true product-specific elasticities within the three-digit product class. Consequently, the estimated product-level inputs and the implied productivity levels are also determined by, among other factors, weighted averages of the true product-specific elasticities within the three-digit product class. The extent to which the restriction of constant elasticities within a product class may bias the production function estimates depends on the true product-specific elasticities. Presumably, the more detailed the product class definitions, the more similar the true elasticities within a given class are likely to be and the smaller any potential estimation biases. It is important to note, however, that the elasticity estimates obtained under these parameter restrictions are likely to be different than the estimates that would be obtained by estimating production functions for outputs and inputs aggregated to the three-digit level. An essential difference between these two cases is that when estimating production functions for aggregated outputs and inputs, returns to scale are specified to take place at the aggregate level, which may be a misspecification and one more source of estimation bias. For example, in the case of aggregated output and inputs, elasticity estimates are biased even if the true production functions are identical across product lines but do not have constant returns to scale.

The estimation strategy is parametric, which requires the econometrician to carefully consider the choices of functional form. As shown in Appendix A, the identification strategy is not restricted to the Cobb-Douglas production function and the isoelastic demand function estimated in the empirical application. The solutions for product-level input allocations when production technology takes the form of a translog, another common functional form in production function estimation, and demand is linear, log-linear, or exponential, are provided in Appendix A. The requirement of the functional form is that is has to allow for closed-form solutions for product-level inputs. An example of a production technology that does not allow for this is the constant elasticity of substitution production function.

### 6.2 Industry characteristics

The model assumptions and data requirements underlying the estimation strategy are more suitable for some industries than others. Two model assumptions are about economies of scope and product characteristics.

#### 6.2.1 Economies of scope and flexible manufacturing

Economies of scope may arise due to lower average fixed or average variable costs of production. Average fixed costs decrease when the producer's fixed costs are spread over a greater number of products. For example, opening a new product line does not necessarily require larger premises and thus higher rental costs. The estimation strategy of the present study assumes away fixed costs, as usual in production function estimation.

Economies of scope due to lower variable costs arise if, for example, the production process of a product yields by-products that can be used as inputs for other products. Alternatively, there can be some units of inputs that are joint between product lines, leading to a decrease in average variable costs. These kinds of inputs cannot be solved for when only firm-level inputs are observable. The estimation strategy therefore assumes away the aforementioned types of economies of scope and is not suitable for industries where such economies of scope are plausible. However, not allowing for economies of scope in the form of lower variable costs is in line with the theory of flexible manufacturing (e.g., Eckel and Neary, 2010). Instead, the estimation strategy allows firms to have productivity differences between product lines and hence core competencies in making some products, which is an essential feature of flexible manufacturing, as discussed in the Introduction. Whether the assumption of flexible manufacturing holds depends on the particular industry. For example, joint inputs in petroleum refining lead to lower variable costs and economies of scope, so the estimation strategy is not appropriate. The estimation strategy is, however, compatible with modelling productivity as a function of the producer's product scope and thus allowing for (dis)economies of scope in the form of higher (lower) product-specific total factor productivity and hence (dis)economies of scope due to variable costs, as in De Loecker et al. (2016).

#### 6.2.2 Product heterogeneity

The estimation strategy is suitable for industries where competition is imperfect, where product differentiation is likely. Hence, the product classification has to be detailed enough for the physical output quantities of a given product to be comparable across manufacturers.[50] At the same time, the data have to include a sufficient number of observations

[50] In the empirical example used in the present study, unobservable quality differences or different units of output are not admissible for a product defined at the eight- or 10-digit level of the PRODCOM classification.

of the product for identification to be possible. If the physical quantities of a product cannot be compared due to unobservable quality variation or different units of output, the estimation strategy is not suitable for that industry.

A large share of manufacturing industries make products that are comparable in physical units even across manufacturers. Examples include food products, textiles, paper and paper products, printing and reproduction of recorded media, chemicals and chemical products, basic pharmaceutical products and pharmaceutical preparations, rubber and plastic products, other non-metallic mineral products, basic metals, and fabricated metal products. Industries that do not make products comparable in physical units are likely to manufacture products that involve advanced technology. Certain transport equipment such as ships and boats, or air and spacecraft, are good examples. If an econometrician wished to estimate production functions for cars, for example, (s)he should consider whether subcompact cars such as the Citroën C1 and Renault Twingo, or entry-level luxury cars such as the Audi A4 and Mercedes-Benz C-Class, are comparable in physical units, and if not, whether the product classes can be defined more narrowly without making the number of observations in product classes too small.

In short, product differentiation is not a challenge to identification as long as quality differences are observable, or if differentiation is horizontal and does not require different production technologies. Ultimately, this requirement is a matter of whether the output data are sufficient regarding the output characteristics. Many national statistics authorities provide data sets that are likely to be sufficiently rich for several manufacturing industries. There are also several manufacturing industries, such as automakers, whose output is not reported in sufficient detail in these data sets and would have to be complemented with additional output data such as car type and characteristics.

## 7 Conclusion

This paper contributes to the substantial empirical literature on production function estimation, which underlies an even larger body of applied economic research. The standard assumption made in production function estimation is that firms produce all of their output with a firm-level production technology that is independent of the firm's product set and that the firm is equally productive in manufacturing all of its products. However, an empirical fact is that a remarkable share of firms are multiproduct firms, and it is possible that they use several, product-specific production technologies. The empirical literature, with a few exceptions, disregards this fact in production function estimation because data sets do not report how firms allocate their inputs to the various product lines. Building on previous structural estimation strategies, I provide a strategy for estimating product-specific production functions of firms that are not price-takers in the output market. The empirical model does not require data on input allocations to various product lines but does require data on firms' output prices. The optimal input allocations are solved parametrically by, first, specifying the output demand and the firm's static profit maximisation problem. The unobservable productivity terms are then substituted for by using the definition of productivity and the observable output quantities. While differences in output demand and product-firm-time-specific demand shocks generate within-firm variation in the profitability of manufacturing various products, supply- and demand-side instruments are used to identify the product-level input allocations and the production function parameters. The empirical model also accommodates productivity differences between a firm's product lines, which the empirical literature suggests is important in determining firms' production and product market entry decisions and hence market outcomes.

I demonstrate the estimation strategy by estimating product-specific production functions for products in the industry "Manufacture of wood and of products of wood and cork, except furniture; manufacture of articles of straw and plaiting materials". I find that the technologies used in "Sawmilling and planing of wood" (PRODCOM 161) and "Products of wood, cork, straw and plaiting materials" (PRODCOM 162) are statistically different from each other. These findings show that, at least in this industry, the single-product technology assumption should not be imposed on multiproduct firms.

Several questions remain for future research. First, there are industries where production technologies are joint because product lines have joint inputs or because by-products of a product line are used as inputs in another product line. Applied econometricians would appreciate identification strategies that accommodate these kinds of economies of scope. Second, while it is important to consider endogenous product selection when estimating product-specific production functions, those estimates, likewise, are useful for studying how firms select their products.

## 8 Data Availability Statement

The data are confidential micro data from Statistics Finland. The data are compiled of the following two data sets:

1. Industrial Output for the years 2004 – 2011, cited as:

   Statistics Finland, 2012. "Industrial Output". https://stat.fi/en/statistics/documentation/tti

2. Longitudinal Database on Plants in Finnish Manufacturing (LDPM) for the years 2004 – 2011, cited as:

   Statistics Finland, 2012. "Longitudinal Database on Plants in Finnish Manufacturing (LDPM)".

These data sets are provided by the Research Services department of Statistics Finland. Statistical legislation and data protection and confidentiality practices specified in legislation are applied in releasing the data. Release of data is subject to a user licence. Instructions for applying for a user licence are provided here:

https://www.tilastokeskus.fi/tup/mikroaineistot/hakumenettely_en.html

The replication package can be accessed here: http://doi.org/10.5281/zenodo.7474464

## References


Ackerberg, D., C. L. Benkard, S. Berry, and A. Pakes (2007): "Chapter 63 Econometric Tools for Analyzing Market Outcomes," vol. 6 of *Handbook of Econometrics*, pp. 4171–4276. Elsevier.

Ackerberg, D., K. Caves, and G. Frazer (2015): "Identification Properties of Recent Production Function Estimators," *Econometrica*, 83(6), 2411–2451.

Arkolakis, C., S. Ganapati, and M.-A. Muendler (2021): "The Extensive Margin of Exporting Products: A Firm-Level Analysis," *American Economic Journal: Macroeconomics*, 13(4), 182–245.

Baldwin, J. R., R. Caves, and W. Gu (2005): "Responses to Trade Liberalization: Changes in Product Diversification in Foreign- and Domestic-controlled Plants," Economic Analysis (EA) Research Paper Series 31, Statistics Canada, Analytical Studies Branch.

Bernard, A. B., J. B. Jensen, S. J. Redding, and P. K. Schott (2007): "Firms in International Trade," *Journal of Economic Perspectives*, 21(3), 105–130.

Bernard, A. B., S. J. Redding, and P. K. Schott (2010): "Multiple-Product Firms and Product Switching," *American Economic Review*, 100(1), 70–97.

——— (2011): "Multiproduct Firms and Trade Liberalization," *Quarterly Journal of Economics*, 126(3), 1271–1318.

Brown, R. S., D. W. Caves, and L. R. Christensen (1979): "Modelling the Structure of Cost and Production for Multiproduct Firms," *Southern Economic Journal*, 46(1), 256–273.

Caves, D. W., L. R. Christensen, and M. W. Tretheway (1980): "Flexible Cost Functions for Multiproduct Firms," *Review of Economics and Statistics*, 62(3), 477–481.

Collard-Wexler, A., and J. D. Loecker (2015): "Reallocation and Technology: Evidence from the US Steel Industry," *American Economic Review*, 105(1), 131–171.

De Loecker, J. (2011): "Product Differentiation, Multiproduct Firms, and Estimating the Impact of Trade Liberalization on Productivity," *Econometrica*, 79(5), 1407–1451.

De Loecker, J., P. K. Goldberg, A. K. Khandelwal, and N. Pavcnik (2016): "Prices, Markups, and Trade Reform," *Econometrica*, 84(2), 445–510.

Dhyne, E., A. Petrin, V. Smeets, and F. Warzynski (2020): "Theory for Extending Single-Product Production Function Estimation to Multi-Product Settings," Working paper.

——— (2021): "Multi Product Firms, Import Competition, and the Evolution of Firm-product Technical Efficiencies," Working paper.

Doraszelski, U., and J. Jaumandreu (2013): "R&D and Productivity: Estimating Endogenous Productivity," *Review of Economic Studies*, 80(4), 1338–1383.

Eckel, C., and J. P. Neary (2010): "Multi-Product Firms and Flexible Manufacturing in the Global Economy," *Review of Economic Studies*, 77(1), 188–217.

Flach, L., and M. Irlacher (2018): "Product versus Process: Innovation Strategies of Multiproduct Firms," *American Economic Journal: Microeconomics*, 10(1), 236–277.

Foster, L., J. Haltiwanger, and C. Syverson (2008): "Reallocation, Firm Turnover, and Efficiency: Selection on Productivity or Profitability?," *American Economic Review*, 98(1), 394–425.

Gandhi, A., S. Navarro, and D. Rivers (2020): "On the Identification of Gross Output Production Functions," *Journal of Political Economy*, 128(8), 2973–3016.

Garcia-Marin, A., and N. Voigtlander (2019): "Exporting and Plant-Level Efficiency Gains: It's in the Measure," *Journal of Political Economy*, 127(4), 1777–1825.

Grieco, P. L. E., S. Li, and H. Zhang (2016): "Production Function Estimation with Unobserved Input Price Dispersion," *International Economic Review*, 57(2), 665–690.

Hill, R. J. (2004): "Constructing Price Indexes across Space and Time: The Case of the European Union," *American Economic Review*, 94(5), 1379–1410.

Klette, T. J., and Z. Griliches (1996): "The Inconsistency of Common Scale Estimators When Output Prices Are Unobserved and Endogenous," *Journal of Applied Econometrics*, 11(4), 343–361.

Levinsohn, J., and A. Petrin (2003): "Estimating Production Functions Using Inputs to Control for Unobservables," *Review of Economic Studies*, 70(2), 317–341.

Malikov, E., and G. Lien (2021): "Proxy Variable Estimation of Multiproduct Production Functions," *American Journal of Agricultural Economics*, 103(5), 1878–1902.

Mayer, T., M. Melitz, and G. I. P. Ottaviano (2014): "Market Size, Competition, and the Product Mix of Exporters," *American Economic Review*, 104(2), 495–536.

Mundlak, Y. (1963): "Specification and Estimation of Multiproduct Production Functions," *Journal of Farm Economics*, 45(2), 433–443.

——— (1964): "Transcendental Multiproduct Production Functions," *International Economic Review*, 5(3), 273–284.

Neary, P. J. (2004): "Rationalizing the Penn World Table: True Multilateral Indices for International Comparisons of Real Income," *American Economic Review*, 94(5), 1411–1428.

Newey, W. K., J. L. Powell, and F. Vella (1999): "Nonparametric Estimation of Triangular Simultaneous Equations Models," *Econometrica*, 67(3), 565–603.

OECD (2013): "OECD Indicators of Employment Protection," https://www.oecd.org/els/emp/oecdindicatorsofemploymentprotection.htm.

Olley, G. S., and A. Pakes (1996): "The Dynamics of Productivity in the Telecommunications Equipment Industry," *Econometrica*, 64(6), 1263–1297.

Orr, S. (2022): "Within-Firm Productivity Dispersion: Estimates and Implications," *Journal of Political Economy*, Online publication before print, https://www.journals.uchicago.edu/doi/abs/10.1086/720465?journalCode=jpe.

Panzar, J. C. (1989): "Chapter 1 Technological Determinants of Firm and Industry Structure," vol. 1 of *Handbook of Industrial Organization*, pp. 3–59 (1st. ed.). Elsevier.

Qiu, L. D., and W. Zhou (2013): "Multiproduct Firms and Scope Adjustment in Globalization," *Journal of International Economics*, 91(1), 142–153.

Statistics Finland (2012a): "Industrial Output," https://stat.fi/en/statistics/documentation/tti.

——— (2012b): "Longitudinal Database on Plants in Finnish Manufacturing (LDPM)," .

Wedervang, F. (1965): *Development of a Population of Industrial Firms: The Structure of Manufacturing Industries in Norway, 1930-1948*. Scandinavian University Books (Universitetsforlaget).