Adverse Selection and Liquidity Distortion*

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Abstract

This paper develops a tractable model with two-dimensional asymmetric information in asset markets: sellers are privately informed about their asset quality and distress positions. Illiquidity arises endogenously and manifests itself through two distinct market outcomes. The first outcome features limited market participation, resulting in a dry-up in trading volume. The second outcome involves a large volume at a depressed price. Only in the latter outcome do distressed sellers engage in fire-sales, quickly unwinding their positions at a steep price discount. The paper further establishes that this equilibrium can arise only when buyers expect that sellers with a higher need for immediacy will on average have higher-quality assets. Hence, both the information structure and the distribution of sellers’ distress are crucial for the existence of fire-sales.

Key words: Liquidity; Fire Sales; Market dry-up; Decentralized trading.

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1 Introduction

This paper proposes an information-based theory to explain a variety of illiquidity patterns in asset markets. It considers two-dimensional asymmetric information: sellers are privately informed about the quality of their assets as well as their distress positions. While, starting from Akerlof (1970), it is well known that adverse selection can lead to illiquidity in financial markets, the main contribution of this framework is to show how various information structures and market conditions can lead to different types of equilibria, generating two distinct notions of illiquidity.

The first outcome results in a dry-up in trading volume, which implies that distressed sellers have to hold on to their assets longer. By contrast, the second outcome implies a high trading volume at a depressed price, in which distressed sellers sell quickly at a steep discount (i.e., a "fire sale").

To capture explicitly both aspects of market liquidity—price and trading rate—as well as traders' different incentives, I allow sellers and buyers to sort themselves optimally into different submarkets. Each submarket is characterized by a pair of price and trading rate, which endogenously depends on the composition of sellers and the resulting buyer-seller ratio in that market. Without informational friction, each asset with a positive gain from trade will be sold at the price that reflects the asset fundamental and at the optimal trading rate (i.e., at an optimal level of participation). I therefore define liquidity distortion as the deviation from the first-best outcome of the selling price and the trading rate.

I establish that the market outcome crucially depends on buyers' perceptions of sellers' trading motives. To be precise, the value of immediacy to a seller can be understood as his marginal rate of substitution between prices and trading rates, which depends on both the asset quality (common value component) and distress position (private value component). Recognizing sellers' incentives, buyers, on the other hand, only value the asset quality. The interaction of sellers' incentives and buyers' willingness to pay thus leads to different market segmentation and liquidity distortion.

Specifically, while the trading rate can naturally be used as a mechanism to separate sellers with different needs for immediacy, I show that a fully separating mechanism can emerge if and only if the underlying distribution suggests that the type of seller who is willing to wait longer has a higher expected asset quality on average, which I refer to as the "monotonicity condition".

Such a condition holds automatically for the special case in which there is only private information on asset quality, a category into which existing adverse selection models with one-

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1 Specifically, the distress position of a seller is modeled as his holding cost.
2 Such a phenomenon has been referred to as a buyers' strike or market freeze. See, for example, Diamond and Rajan (2011) and Tirole (2010).
3 This setup is known as competitive search.
dimensional asymmetric information fall. In that case, the theory predicts a separating equilibrium that exhibits downward-distorted trading rates in which sellers who are more willing to wait longer trade at a higher price but more slowly. In other words, this equilibrium predicts low trading volume, limited market participation, and trading delay for distressed sellers.

With two-dimensional asymmetric information, the monotonicity condition may be violated, resulting in different market phenomena. In particular, when the impact of a distress position dominates the effect of asset quality, buyers know that the average asset bought from sellers who are more willing to wait can actually be of lower quality. As a result, buyers are no longer willing to pay a higher price for the asset, undermining the full separation mechanism.

In this case, I focus on one type of semi-pooling equilibrium that yields several novel predictions and rationalizes fire-sale phenomena. First, it involves upward-distorted trading rates and a discontinuity of price with respect to types, implying a high volume of trades at a common depressed price. A distressed seller with a high-quality asset finds it optimal to enter a pooling market, in which he can unload his assets quickly but at a price discount. Second, in contrast to standard lemon models that impose a pooling equilibrium, this type of semi-pooling equilibrium predicts that the larger the market share of highly distressed sellers, the steeper the required discounts.

Note that, unlike most existing models of fire sales or buyers’ strikes that exogenously assume buyers are constrained and sellers are forced to sell, all buyers in my framework are unconstrained but may choose not to participate; moreover, sellers always have the choice to retain their assets or to sell them quickly at lower prices. My model thus provides a micro-foundation for these phenomena.

Furthermore, the theory sheds lights on when and why fire sales actually arise. Indeed, as Shleifer and Vishny (2010) discussed, empirical works have documented different trading outcomes across markets, and no existing theory is able to reconcile these two patterns. My paper therefore contributes to the literature by establishing a fire-sale outcome is only possible when the monotonicity condition is violated. Specifically, this happens when there is two-dimensional private information and when the market condition is such that sellers who value immediacy more tend to have higher-quality assets. The fire-sale outcome is thus more likely when there is a large fraction of distressed sellers in the market and these sellers have relatively high-quality assets.

4Models with one-dimensional private information that leads to trading delay for sellers with high-quality assets include Janssen and Roy (2002), Inderst and Müller (2002), and Camargo and Lester (2014), etc. The same intuition also applies to models in which quantity is used as the instrument to separate types (for example, DeMarzo and Duffie (1999)).

5In those frameworks (such as Akerlof (1970) and Eisfeldt (2004)), an increase in the sellers’ distress level should increase both volume and price because sellers with high-quality assets are more eager to trade as they become more distressed, which improves the average quality of the pool. As discussed in Uhlig (2010), such a prediction is actually counterfactual.
higher-quality assets. In Section 4, I further show how these two equilibria lead to distinct empirical predictions, connecting them to different empirical evidence.

**Relation to Literature** My work is closest to that of Guerrieri et al. (2010), who apply the notion of a competitive search equilibrium to a static environment with adverse selection and uninformed principals who are allowed to post contracts. This paper contributes to the literature by considering asymmetric information on both private and common values. Furthermore, mine is the first paper to show that it is the effect of multidimensional private information that leads to semi-pooling equilibria. These findings in regard to semi-pooling equilibria constitute a novel contribution to the literature on competitive search (e.g., Moen (1997), Burdett et al. (2001), Mortensen and Wright (2002), and Eeckhout and Kircher (2010)), where a fully separating equilibrium is always obtained.

The paper also makes a methodological contribution by developing a characterization method for continuous types with mechanism design approach. This approach not only simplifies the equilibrium characterization to solving a differential equation, but also facilitates extending the analysis to a more general environment. Specifically, the constructing algorithm in Guerrieri et al. (2010) is designed for the monotonicity case and thus for a fully separating equilibrium only. My approach can be applied to any underlying distribution (including the ones that leads to non-monotonicity) and for semi-pooling equilibria.

Building on Guerrieri et al. (2010), the contemporaneous work of Guerrieri and Shimer (2014a) also emphasizes the idea that liquidity works as a screening mechanism; in that work, they consider private information on the common value only and therefore obtain a fully separating equilibrium. The notion of a fire sale in Guerrieri and Shimer (2014a) refers to the drop in the trading price when the equilibrium resale value decreases. In contrast, the term fire sale in my paper refers to the case in which a relatively distressed seller enters a pooling market, in which he can sell his asset more quickly (because of an upward-distorted buyer-seller ratio) but takes an undervalued price. These features are unique to the constructed semi-pooling equilibria and also imply distinct market activities.

The two-dimensional information problem has been studied in different settings with screening contracts, in which different instruments (such as quantity) are used to screen types. In a two-dimensional information setting with only one instrument, we should expect that prices can reveal at most the type that summarizes two-dimensional preference heterogeneity (see, for example, Bhattacharya and Spiegel (1991) and Biais et al. (2000)). My framework adds insights to this line of literature by showing that, in decentralized markets, a fully separating mecha-

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6 As discussed in Guerrieri et al. (2010), this equilibrium concept is similar to the refined equilibrium concept developed in Gale (1992), and Gale (1996).

7 A follow-up paper by Guerrieri and Shimer (2014b) studies multidimensional private information in an environment where the monotonicity condition holds but with a weaker off-path restriction.
nism in respect to the effective type cannot be sustained when the monotonicity condition fails. In this case, a semi-pooling equilibrium with features of a fire sale exists.\(^8\) It is worth noting that although there are well-known pooling results (such as ironing) in the optimal screening contract, the underlying economic force is very different from the competitive framework here. The construction of the semi-pooling equilibrium in my model is driven by the buyer’s free-entry condition instead of by the principal’s profit maximization. Such a constraint is not present in the optimal screening problem.

This paper is also related to the literature based on the random-matching framework, including the over-the-counter literature (such as Duffie et al. (2005), Duffie et al. (2007), Weill (2008), and Lagos and Rocheteau (2009)) and the monetary search literature (such as Kiyotaki and Wright (1993), Trejos and Wright (1995), and Shi (1995)).\(^9\) In particular, recent works by Chiu and Koeppl (2016) and Camargo and Lester (2014) consider one dimensional private information and use their frameworks to study government intervention.\(^10\)

The rest of the paper is organized as follows. Section 2 presents the model and defines the equilibrium concept. Section 3 establishes my approach to characterizing equilibria and the main results. Section 4 establishes the empirical implications for different informational structures. Section 5 studies efficiency and policy implications. All omitted proofs can be found in the appendix.

\section{Model}

\textit{Players.} There is a continuum of sellers and buyers. Both sellers and buyers are risk-neutral and infinitely lived, with a common discount rate \(r\). Time is continuous, and the time horizon is infinite. Each seller owns a single asset, which is nondivisible. The assets vary in their dividend flows at each instant, indexed by \(s \in S = [s_L, s_H] \subset \mathbb{R}_+\). Sellers’ types are two-dimensional and unobservable: (1) the quality of asset they own \(s\) and (2) their distress positions, which are modeled as holding costs \(c > 0\).\(^{11}\) The support of \(c\) is some arbitrary set \(C\) which can assume

\(^8\)Such force is not present in Bhattacharya and Spiegel (1991) (which studies one single informed trader in a Walrasian framework), nor in Biais et al. (2000) (as it analyzes only the case in which the monotonicity condition holds).

\(^9\)Williamson and Wright (2008) provides a detailed survey of this line of literature.

\(^10\)Camargo and Lester (2014) characterizes a set of equilibria in which sellers with high-quality assets exit the market at a lower rate since they are more likely to reject the offer in a random search framework. They show that the policy might slow down market recovery, depending on the fraction of lemons in the market. Chiu and Koeppl (2016), on the other hand, assume a pooling equilibrium and focus on the timing of the intervention.

\(^11\)As explained in Duffie et al. (2007), we could imagine this holding cost to be a shadow price for ownership due to, for example, (1) low liquidity, that is, a need for cash; (2) high financing cost; (3) an adverse correlation of asset returns with endowments; or (4) a relatively low personal value from using the asset, as in the case of certain durable consumption goods, such as homes.
either discrete or continuous values. I assume that the largest value of holding costs, denoted by $c_H$, satisfies $s_L - c_H > 0$. Let $G(s, c): S \times C \rightarrow [0, 1]$ denote the joint cumulative distribution of asset quality and holding cost. While holding the asset, the flow payoff to a seller $(s, c)$ is given by $s - c$.

The other side of the market consists of homogeneous buyers, who can only buy one unit of the asset and do not need to pay any holding costs but simply enjoy the dividend flow of an asset. Thus, the gain from trade in this environment is the holding cost. A buyer can choose to enter the market by paying a flow cost $k > 0$, which provides him an opportunity to meet a seller. I further assume that the measure of buyers is strictly larger than the measure of sellers. Hence, the measure of buyers who decide to enter the market is determined by the free-entry condition.\footnote{This structure is designed to emphasize the idea that there is plenty of capital available in the market; therefore, any reduction in liquidity, if it arises, is endogenous and driven by informational frictions.}

To simplify the exposition, I assume that when an agent trades, he is removed from the market and replaced by a new agent of identical type. Nevertheless, as shown in Appendix A.1, main results in this stationary model hold in a dynamic environment with endogenous evolution of an aggregate distribution.\footnote{One convenient feature of this setup is that one can construct stationary equilibria while the aggregate distribution evolves over time. This property makes the fully dynamic environment tractable. It is also straightforward to allow for resale. More details are presented in the published working paper version.}

**Bilateral Trades** Buyers can enter the market by posting a trading price $p$ and incurring a flow cost $k$, or they can exit the market. There is a continuum of markets characterized by a price $p \in \mathbb{R}_+$. Both buyers and sellers choose their preferred market, with rational expectations about the buyer-seller ratio associated with each market, denoted by $\theta(p)$. Within each market, traders form bilateral matches subject to a random matching function. A seller who enters the market $(p, \theta(p))$ is matched with a buyer with the Poisson rate $m(\theta(p))$. The matching function $m(\cdot)$ is assumed to be a strictly increasing function of $\theta$, which captures the idea that a higher buyer-seller ratio implies that a seller can find a buyer faster (and therefore implies a higher trading rate for sellers). Furthermore, as is standard, the matching function $m(\cdot)$ is assumed to be twice continuously differentiable and strictly concave. On the other hand, a buyer in market $(p, \theta(p))$ meets a seller at the rate $n(\theta(p))$, where $n(\cdot)$ is assumed to be a strictly decreasing function of $\theta$. That is, a higher buyer-seller ratio makes it harder for a buyer to meet a seller. Trading in pairs requires that $m(\theta) = \theta \cdot n(\theta)$.

When entering the market $(p, \theta(p))$, a seller $(s, c)$ enjoys a flow payoff $s - c$ until he meets a buyer, which happens at Poisson rate $m(\theta)$. The expected utility of such a seller is given by:

$$rV(p, \theta(p), x) = x + m(\theta(p))(p - V(p, \theta(p), x)),$$

where $x$ is defined as $x \equiv s - c \in X \equiv \{x | x = s - c, s \in S \text{ and } c \in C\}$. 


The type $x$ is then the sufficient statistic for sellers’ motivation to trade, which is a combination of the asset quality and distress position. A seller enjoys a higher payoff when holding the asset if the asset quality is higher or if he is less distressed. This two-dimensional problem is made much simpler by the fact that one single variable $x$ entirely captures sellers’ incentives. I therefore directly characterize the equilibrium with respect to the effective type $x$.

Since buyers only care only about asset quality (i.e., the common value component of the seller’s type), a buyer’s expected value for buying the asset from type $x$ is then given by

$$h(x) = \mathbb{E}[s_j | s = x],$$

where $h : X \rightarrow \mathbb{R}_+$ and the function $h(x)$ depends on the underlying joint distribution of $(s, c)$. As I will show later, the property of this function is crucial for the equilibrium outcome.

Given that sellers’ types are unobservable, buyers form rational beliefs about the composition of sellers in each market. Let $(\eta_p)$ denote the conditional cumulative distribution of sellers’ types in each market $p$.

Buyers’ payoffs, denoted by $J(p, \theta(p), \mu(\cdot|p))$, in each market $p$ can be expressed as:

$$r J(p, \theta(p), \mu(\cdot|p)) = -k + \frac{m(\theta(p))}{\theta(p)} \left\{ \int \frac{h(x)}{r} d\mu(x|p) - p - J(p, \mu(\cdot|p), \theta(p)) \right\}.$$

### 2.1 Equilibrium

Let $P$ denote the set of feasible prices: $P = [0, \frac{z u}{r}]$. If a seller chooses not to enter the market, he then receives his autarky value, $\frac{z u}{r}$. Such a choice is denoted by $\varnothing$.

**Definition 1** A stationary equilibrium consists of a set of offer prices $P^* \subseteq P$; a buyer-seller ratio function in each market $p$, $\theta(\cdot) : P \rightarrow [0, \infty]$; and a conditional distribution of sellers in each submarket, $\mu(\cdot|p) : X \rightarrow [0, 1]$ for each market $p \in P$, such that the following conditions hold:

**E1 (Optimality for sellers):** Let

$$V^*(x) = \max_{\bar{p} \in P^* \cup \emptyset} V(\bar{p}, \theta(\bar{p}), x),$$

and for any $p \in P^*$ and $x \in S$, $x \in$ support of $\mu(\cdot|p)$ only if $p \in \arg\max_{\bar{p} \in P^* \cup \emptyset} V(\bar{p}, \theta(\bar{p}), x)$.

**E2 (Optimality for buyers)**

**E2(a) Free-entry condition:** For any $p \in P^*$,

$$J(p, \theta(p), \mu(\cdot|p)) = 0.$$
E2(b) Optimal price-posting: There does not exist any \( p' \notin P^* \) such that

\[
J(p', \theta(p'), \mu(\cdot|p')) > 0,
\]

and \( \theta(p') \) and \( \mu(x|p') \) satisfy the following condition:

\[
\theta(p') = \inf_{x \in X} \bar{\theta}(p', x),
\]

where \( \bar{\theta}(p', x) \equiv \inf\{\tilde{\theta} > 0 : V(p', \tilde{\theta}, x) \geq V^*(x)\} \), and \( x \in \text{support of } \mu(\cdot|p') \) only if \( x \in \arg \inf_{\tilde{x} \in X} \bar{\theta}(p', \tilde{x}) \).

To understand the equilibrium concept, first consider the price that is posted in equilibrium, \( p \in P^* \). The sellers’ optimality condition requires that sellers choose optimally which submarket to enter, taking as given the price \( p \) and the trading rate—which is determined by the buyer-seller ratio \( \theta(p) \)—in each submarket. The buyers’ optimality condition, E2(a), states that a buyer must earn zero profit for any active market \( p \in P^* \), given the expected trading rate \( m(\theta(p)) \) and the expected asset quality \( \mu(\cdot|p) \) in that market.

As standard in the competitive search literature, the equilibrium concept imposes restrictions on beliefs about the buyer-seller ratio for markets that are not open, which is specified in Condition E2(b).16 Specifically, when a buyer contemplates a deviation and offers a new price \( p' \notin P^* \), he takes the sellers’ equilibrium utilities \( V^*(x) \) as given and expects a buyer-ratio larger than zero only if there is a seller type who is willing to come to this market. Moreover, he expects the lowest buyer-ratio for which he can find such a seller type, which means that he expects sellers to queue up for this new price until it is no longer profitable from them to do so.17

One can understand Condition E2(b) by considering the following hypothetical adjustment process. If a buyer expects \( \theta' > \theta(p') \), this implies that some sellers obtain a higher value in this new market than their equilibrium utilities \( V^*(x) \). As a result, they would keep flowing into this market, pulling down the trading rate. Throughout this process, type-\( x \) sellers will stop entering this market until \( \theta' < \bar{\theta}(p', x) \). This process stops at the value of \( \theta(p') = \inf_{x \in X} \bar{\theta}(p', x) \); at this value, the type of seller who is willing to come to this market at the lowest buyer-seller ratio is indifferent, while all other sellers strictly prefer to stay in their equilibrium market. Hence, the belief puts only positive weight on these sellers. In other words, the type of seller who is willing to accept the lowest buyer-ratio at this new price determines both the composition \( \mu(x|p') \) and the buyer-seller ratio \( \theta(p') \).

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16 It also resembles the refined Walrasian general-equilibrium approach developed in Gale (1992). See Guerrieri et al. (2010) for a detailed discussion regarding different refinements developed in the literature.

17 Such a requirement is called the market utility property in the competitive search literature, which is in the spirit of subgame perfection. As discussed in Eeckhout and Kircher (2010), the buyer-seller ratio function \( \theta(p) \) acts similar to hedonic price schedule in Rosen (1974).
2.2 Benchmark

Before characterizing the equilibrium, I first consider an environment where buyers can observe sellers’ type $x$ and where the value to buyers is given by $h(x)$. Such an environment is the canonical competitive search model put forth by Moen (1997).

One can then solve the equilibrium independently for each type. The restriction on beliefs in this environment then suggests that, when opening a new market $p'$ for type $x$, a buyer expects a buyer-seller ratio $\theta(p')$ that makes this seller indifferent. In other words, condition $E2(b)$ is simply reduced to the statement that, there is no market $(p', \theta(p'))$ such that sellers are indifferent while buyers are strictly better off. As a result, the combination of price and trading rate in equilibrium, denoted by $(p^{FB}(x), \theta^{FB}(x))$, must be Pareto efficient. Given that buyers must earn zero, the equilibrium solution must then maximize sellers’ utilities subject to buyers’ free-entry condition $E2(a)$:

$$V^{FB}(x) = \max_{p, \theta} \frac{x + pm(\theta)}{r + m(\theta)}$$
subject: $J(p, \theta, x) = 0$.

The first order condition for the equilibrium buyer-seller ratio yields:

$$\frac{h(x) - x}{k} = \frac{r + m(\theta) - \theta m'(\theta)}{m'(\theta)}.$$  \hfill (2)

In words, the equilibrium trading rate when $x$ is observable depends on the ratio of the gain from trade over the buyers’ searching cost, where $h(x) - x = E[c|x]$. The higher the gain from trade, the higher the trading rate (i.e., a higher level of market participation). In the special case with constant holding cost $c$, sellers’ type is the asset quality $s$ and the first-best trading rate is independent of asset quality. This immediately suggests that first-best allocations cannot be implemented in an environment where asset quality is unobservable: facing the same trading rate, the low-type seller has incentives to pretend to be a higher type in order to obtain a higher price.

3 Characterization

In an equilibrium $\{P^*, \theta(p), \mu(\cdot | p)\}$, the outcome for a type-$x$ seller can be described by the pair of functions $\{p^*(x), \theta(p^*(x))\}$, where $p^*(x)$ denotes the sellers’ optimal market choice $p^*(x) \equiv \arg \max_{\tilde{p} \in P_{\theta(x)}} V(\tilde{p}, \theta(\tilde{p}), x)$. To characterize the equilibrium, I prove that the equilibrium outcome must satisfy a number of properties by setting up a mechanism design problem. One can interpret this problem from the viewpoint of an imaginary market designer, who allocates

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$^{18}$In this counterfactual environment, buyers can only observe $x$, but not asset quality or holding cost separately.
buyers and sellers into different markets by designing the price and the buyer-seller ratio for each market so that both sellers’ and buyers’ optimality conditions are satisfied.

**Sellers’ Optimality** For the sellers’ side, by the revelation principle, an outcome that satisfies the sellers’ optimality condition \( E1 \) can be achieved if and only if it is optimal for the agent to report truthfully under the direct revelation mechanism \( \{ p^\alpha(x), \theta^\alpha(x) \} \), where \( p^\alpha : X \to \mathbb{R}_+ \) and \( \theta^\alpha : X \to \mathbb{R}_+ \). That is, the mechanism, which is a pair of functions \( \{ p^\alpha(\cdot), \theta^\alpha(\cdot) \} \) that assigns the type \( x \) seller into the market \( \{ p^\alpha(x), \theta^\alpha(x) \} \), must satisfy the sellers’ truth-telling condition:

\[
x \in \arg\max_{\tilde{x} \in X \cup \emptyset} V(p^\alpha(\tilde{x}), \theta^\alpha(\tilde{x}), x).
\]

With this interpretation, Lemma 1 establishes the conditions under which the sellers’ optimality condition is satisfied.

**Lemma 1** The pair of functions \( \{ p^\alpha(\cdot), \theta^\alpha(\cdot) \} \) satisfies the sellers’ optimality condition if and only if the following three conditions are satisfied:

\[
\begin{align*}
\theta^\alpha(x) & \text{ is non-increasing;} \\
V^*(x) &= \frac{x + p^\alpha(x) \cdot m(\theta^\alpha(x))}{r + m(\theta^\alpha(x))} = V^*(x_L) + \int_{x_L}^{x} \frac{1}{r + m(\theta^\alpha(\tilde{x}))} d\tilde{x} ; \text{ and} \\
V^*(x) & \geq \frac{x}{r},
\end{align*}
\]

where \( x_L \) denotes the lower bound of \( X \).

The proof follows from standard arguments in the mechanism design literature (e.g., Milgrom and Segal (2002)), where \( \frac{1}{r + m(\theta^\alpha(x))} \) is the partial derivative of \( V(p, \theta, x) \) with respect to \( x \) and so the envelope condition yields condition \( IC \).

Condition \( (M) \) formally establishes that the trading rate must be non-increasing with seller’s type for any incentive-compatible mechanism. The reason is straightforward: the marginal rate of substitution between prices and trading rates of a seller is determined by his flow value of holding an asset. Hence, a seller who enjoys a higher flow value (a higher type \( x \)) is more willing to wait longer in exchange for a higher price.\(^{19}\)

**Buyers’ Optimality** On the buyers’ side, the mechanism must satisfy the buyers’ free-entry condition \( E2(a) \) so that the buyers’ allocation generates the correct buyer-seller ratio in each market. Define \( \theta(p) \equiv \theta^\alpha(x) \) for any \( x : p^\alpha(x) = p \) and \( P^* = \{ p^\alpha(x) : x \in X \} \). The expected asset value to buyers in any market is determined by the conditional distribution of sellers who are assigned to market \( p \). The free-entry condition then requires that the price in

\(^{19}\)Formally, one can see that the single crossing condition holds strictly in this case. That is, for any \( \theta' < \theta, p' > p, \) and \( x' > x, \{ V(p', \theta', x') - V(p, \theta, x') \} - \{ V(p', \theta', x) - V(p, \theta, x) \} > 0. \)
each market equals the expected asset quality minus the buyers’ expected searching cost in that market, which gives:

\[ p = \frac{E[h(x)]p^a(x) = p}{r} - \frac{k\theta(p)}{m(\theta(p))}, \text{ for } \forall p \in P^*. \] (3)

Lemma 1 and the buyers’ free-entry condition (3) establish the properties of any mechanism that is feasible. Each mechanism maps to an equilibrium \( \{P^*, \theta(p), \mu(\cdot|p)\} \) that satisfies \( E1 \) and \( E2(a) \), but not the buyers’ optimal price-posting, \( E2(b) \). I now use condition \( E2(b) \) to further identify a necessary condition for which a feasible mechanism can be implemented as an equilibrium outcome. That is, a buyer will not deviate by posting a price that is not recommended by the market designer.\(^{20}\)

In order to check whether a buyer has an incentive to post a new price \( p' \), one first needs to know which type a buyer will attract, which is established by the following Lemma.

**Lemma 2** Given a feasible mechanism \( \{p^a(\cdot), \theta^a(\cdot)\} \) and given that \( \theta^a(x) > 0 \ \forall x \), for any \( p' \) outside the range of \( p^a(\cdot) \), the unique seller who will come to market \( p' \), denoted by \( T(p') \), is given by:

\[ T(p') = \{x^+ \} \cup \{x^-\}, \]

where \( x^- = \inf \{x \in X | p' < p^a(x)\} \) and \( x^+ = \sup \{x \in X | p' > p^a(x)\} \).

Note that since \( \theta^a(x) \) is weakly decreasing, \( p^a(x) \) must be weakly increasing for any feasible mechanism. According to Lemma 2, if a new price \( p' \) is higher (lower) than any existing price, the unique seller who comes to this market is then the highest (lowest) type. Moreover, when the price schedule jumps upward from \( p_1 \) to \( p_2 \) at a point \( \hat{x} \), for any price \( p' \in (p_1, p_2) \), \( \hat{x} \) is the unique seller who comes to this new market since \( x^- = x^+ = \hat{x} \).

To understand this result, recall that, according to condition \( E2(b) \), a buyer expects the lowest buyer-seller ratio for which he can find a seller type; moreover, the buyer-seller ratio is such that this seller will be indifferent between this new market and his original market. Consider a new price \( p' \) than is higher than any existing price. Since the highest types value immediacy least, if the combination of this new high price and low speed is such that the highest type is indifferent, all other types must find this new combination strictly worse and thus would rather stay in their original markets. This thus explains why the highest type is the unique seller who comes to this market in this case.

There is a similar result when a new price is lower than all existing prices. When the combination of this lowest price and higher speed makes the lowest type indifferent, all other

\(^{20}\)Without condition \( E2(b) \), the entire set of feasible mechanisms characterized above can be supported as equilibria.
sellers, who value immediacy less, will not find this new market attractive. This explains why the lowest type is the only type that will come to this new market. The same logic can be applied to the case when the price schedule jumps.

Since Lemma 2 has identified the seller type that a buyer would attract by opening a new market, one then needs to know only the expected payoff of a buyer who purchases an asset from such a seller. Hence, buyers’ payoff function \( h(x) \), which depends on the joint distribution of asset quality and distress positions, plays an important role.

So far, I have not imposed restrictions on buyers’ payoff functions \( h(x) \). Below, I show that the presence of two-dimensional private information implies that markets have different perceived motives for selling and thus have different equilibrium outcomes.

### 3.1 Monotonicity: A Fully Separating Equilibrium

I first start with the environment where the monotonicity condition is satisfied.

**Assumption A1** (Monotonicity) The function \( h(x) \) is (1a) a continuously differentiable function and (1b) strictly increasing in \( x \) (\( h'(\cdot) > 0 \)).

Such an assumption can be understood as an environment in which the informational motive (on the common value component) dominates. Specifically, the underlying joint distribution of \((s, c)\) is such that sellers with the larger flow value \( x \) (i.e., those who are more willing to wait) have a better asset quality \( s \) on average. The exact condition on the joint distribution is provided in Appendix A.2.

Consider a simple case where \( s \) and \( c \) are independent and \( s \) follows an uniform distribution. In this case, \( E[c|x] \) is then weakly decreasing in \( x \), since a very high (low) willingness to wait implies that the holding cost is coming for the lower (higher) part of the support of \( C \). Given that \( E[s|x] = x + E[c|x] \), the monotonicity condition thus holds when \( E[c|x] \) does not decrease too much relative to the increase in \( x \).

It is worth noting that this assumption holds automatically when there is only one dimensional private information regarding asset quality. That is, conditional on \( c \), asset quality determines the sellers’ type. As a result, those with a higher payoff of holding the asset (higher \( x \)) must have a higher-quality asset (higher \( s \)).

Lemma 3 shows that, whenever this condition holds, the unique equilibrium would be fully separating with respect to \( x \).

**Lemma 3** (Full Separation) Under A1, there exists no submarket where seller types are pooled.

To see this, consider a pooling market for sellers \( x \in [x_1, x_2] \subset X \). If a buyer deviates by posting a new price \( p' \) that is only slightly higher than the original pooling price, he expects to attract the highest type-\( x \) in the original pooling market. That is, according to Lemma 2,
Moreover, since $A_1$ suggests that $h(x)$ is strictly increasing, a buyer can obtain the most valuable asset from the original pool market by simply paying more: $h(x_2) > E[h(x)|x_1 \leq x \leq x_2]$. As a result, this deviation is profitable for buyers.

The argument here also highlights why assumption $A_1$ is crucial. Without it, the type of seller that is most willing to wait may not have the most valuable assets. Furthermore, under $A_1$, this pooling necessarily implies that the asset quality in the market right above $x_2$ must jump upward. Hence, in order to prevent seller-$x_2$ from entering the market above him and to satisfy buyers’ free-entry condition, the price $p^*(x)$ must jump upward and the buyer-seller ratio $\theta^*(x)$ must jump downward at $x_2$. This thus explains why the above deviation is always possible.

Lemma 3 thus allows us to focus on fully separating equilibria. The free-entry condition (3) can then be rewritten as:

$$ p^{\alpha}(x) = \frac{h(x)}{r} - \frac{k\theta^\alpha(x)}{m(\theta^\alpha(x))}. \quad (4) $$

Substituting this payment schedule into $(IC)$, we get:

$$ V^*(x) = x + \frac{\left(\frac{h(x)}{r} - \frac{k\theta^\alpha(x)}{m(\theta^\alpha(x))}\right)m(\theta^\alpha(x))}{r + m(\theta^\alpha(x))} = V^*(x_L) + \int_{x_L}^x \frac{1}{r + m(\theta^\alpha(x))} d\bar{x}. $$

Taking the derivative with respect to $x$ on both sides, one then obtains the following differential equation for $\theta^\alpha(x)$:

$$ \left[ h(x) - x - k \left(\frac{r + m(\theta)}{m'(\theta)}\right) \right] \frac{d\theta}{dx} = -(r + m(\theta)) \cdot \frac{m(\theta)}{m'(\theta)} \cdot \frac{h'(x)}{r}. \quad (5) $$

In summary, in order to satisfy the incentive-compatibility constraints and the free-entry condition, the buyer-seller ratio function $\theta^\alpha(\cdot)$ in any separating equilibrium must satisfy the above differential equation (5), subject to the condition $(M)$. Lastly, the initial condition is pinned down by Lemma 4:

**Lemma 4** (No distortion for the lowest type) Under $A_1$, the lowest type achieves his first-best utilities in any fully separating equilibrium.

Under Assumption $A_1$, buyer’s valuation is higher for a higher-type seller. Hence, a lower-type seller has incentive to pretend to be a higher-type. In other word, the incentive constraint binds downward. The trading rate is thus distorted downward in order to prevent a lower type seller from mimicking. Intuitively, since none has incentives to mimic the lowest type, there is no need to distort the trading rate for the lowest type, which gives the initial condition for Equation (5).

Specifically, suppose that the equilibrium trading rate is also distorted downward for the lowest type (i.e., $\theta^*(x_L) < \theta^{FB}(x_L)$ and $p^*(x_L) > p^{FB}(x_L)$). The lowest type seller’s utility in
Figure 1: If the market \((p^*(x_L), \theta^*(x_L))\) for the lowest type is distorted downward, the new market \((p', \theta(p'))\) that makes the lowest type to be indifferent is then below buyers’ free-entry condition (represented by the dash brown line). Hence, buyers are strictly better off by offering \(p' = p^{FB}(x_L)\).

This case must be lower than his first-best utility. By offering a new price \(p' = p^{FB}(x_L)\) which is lower than all existing prices, buyers expect to attract the lowest type according to Lemma 2. As shown in Figure 1, the buyer-seller ratio \(\theta(p')\) at this new market that makes the lowest seller indifferent must then be strictly smaller than \(\theta^{FB}(x_L)\). As a result, this new market \((p', \theta(p'))\) constitutes a profitable deviation for buyers.

Lemma 4 thus gives the initial condition \(\theta^*(x_L) = \theta^{FB}(x_L)\), which then uniquely pinned down the equilibrium buyer-seller ratio function \(\theta^*(x)\). Figure 2 illustrates the solution of the buyer-seller ratio function for the special case with homogenous holding cost \(c\), in which sellers’ types directly map to asset quality and the first best solution is independent of asset quality.

Given \(\theta^*(x)\), the corresponding price function \(p^*(x)\) is given by the free entry condition (4). We can therefore conclude that the schedule \(\{p^*(x), \theta^*(x)\}\) is the unique mechanism that satisfies both \((E1)\) and \((E2)\), mapping to the least-cost separating equilibrium. Specifically, a downward-distorted buyer-seller ratio implies that fewer buyers enter the market, and high-type sellers trade at a lower speed but at a higher price than do low-type sellers. Due to the distortion, all sellers (expect the lowest type) obtain lower equilibrium utilities than those in the first-best benchmark.

**Proposition 1** Under A1, the unique solution to the mechanism design problem that implements the equilibrium outcome is given by the buyer-seller ratio function \(\theta^* : x \rightarrow \mathbb{R}_+\), and the price function \(p^* : x \rightarrow \mathbb{R}_+\), where \(\theta^*(x)\) is the unique solution to (5) with the initial condition \(\theta^*(x_L) = \theta^{FB}(x_L)\), and where \(p^*(x)\) is given by (4).

\[\text{21} \text{One can see that the standard condition of uniqueness does not hold at } \theta^*(x_L) = \theta^{FB}(x_L). \text{ In fact, there will be two solutions; however, the other solution increases with } s \text{ and therefore violates Condition (M).}\]
Corollary 1 Under A1, (1) the first-best solution \( \{ \theta^{FB}(x), p^{FB}(x) \} \) is not implementable; and (2) the equilibrium buyer-seller ratio \( \theta^*(x) \) is downward-distorted compared to the first-best. That is, \( \theta^*(x) < \theta^{FB}(x) \) for all \( x > x_L \).

3.2 Nonmonotonicity: Semi-pooling Equilibria

I now turn to the case in which the private-value component (i.e., the sellers’ distress positions) dominates the common value (i.e., asset quality) so that the types who are willing to wait longer (i.e., those with the larger flow value \( x \)) do not necessarily have better assets, which critically violates the monotonicity condition (A1).

An Illustrative Example of Nonmonotonicity: Consider the case in which there are two possible holding costs for sellers \( C = \{ c_H, c_L \} \), where \( c_H > c_L > 0 \). For simplicity, assume that \( c \) and \( s \) are independently distributed. Let \( \lambda \) denote the probability that a seller incurs a higher holding cost \( c_H \). The value of \( h(\cdot) \) can then be understood as in Figure 3. Observe that the buyers’ value function \( h(\cdot) \) does not strictly increase with type \( x \). In particular, the expected value of the asset drops in the overlapping region \( [s_L - c_L, s_H - c_H] \), since the asset could be either a low-quality one owned by a seller with a low holding cost or a high-quality one owned by a seller with a high holding cost. If, for example, \( s \) is uniformly distributed over the interval \( [s_L, s_H] \), the expected asset quality in the overlapping region is given by \( h(x) = x + E[c|x] = x + \lambda c_H + (1 - \lambda)c_L \). On the other hand, outside of this region, a very high (low) willingness to wait implies that the holding cost must be low (high).22

I now show that a fully separating equilibrium (with respect to the effective type \( x \)) cannot exist whenever \( h(\cdot) \) has an interior strict local maximum.23 The intuition is as follows. A

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22 Similarly, this situation arises when \( c \) follows a continuous Beta distribution (Beta(0.5, 0.5)). Such a distribution is bimodal, approximating a two-point distribution.

23 That is, \( \exists \hat{x} \in \text{int}(X) \) such that \( h(\hat{x}) > h(x) \) for all \( x \) in a neighborhood of \( \hat{x} \).
fully screening mechanism is a combination of a downward-distorted trading probability and an upward-sloping price scheme. To generate such a price schedule, the market designer must make sure that buyers are willing to pay the price in each market, given the expected asset value in that market.\(^{24}\) However, if the types who are willing to wait longer have assets that are worth less, buyers’ are no longer willing to pay a higher price for those assets, which undermines the full screening mechanism and leads to semi-pooling equilibria.

To facilitate the construction of semi-pooling equilibria, given \(f(p^*(x), \theta^*(x))\), define the function \(\bar{H}(x) \equiv E[h(x)|p^*(x) = p^*(x)]\), which represents how buyers value type \(x\), given the equilibrium sorting of sellers. \(\bar{H}(x)\) coincides with the underlying value \(h(x)\) only if different type-\(x\) sellers go to different markets (a full separation). On the other hand, if there exists a subset of sellers who are pooled in one submarket, the value of type \(x\) in the eyes of the buyers is then the expected value in that submarket, instead of the real value \(h(x)\).

**Proposition 2** For any equilibrium outcome \(\{p^*(x), \theta^*(x)\}\), the corresponding buyers’ function \(\bar{H}(x)\) cannot have an interior strict local maximum.

Proposition 2 thus suggests that if the underlying function \(h(\cdot)\) has an interior strict local maximum, a fully separating equilibrium cannot be sustained.

In general, for any given arbitrary distribution of \((s, c)\), which determines the shape of function \(h(x)\) and the distribution of \(x\), there may exist multiple ways to construct semi-pooling equilibria, and in contrast to the monotonic case, the equilibrium is usually not unique. Nevertheless, one can still apply the developed methods to construct equilibria.

To proceed, I provide detailed construction for two particular types of semi-pooling equilibria and establish the required condition on the underlying distribution for each construction. Then, I conclude with a general discussion on how to apply the developed methods for any arbitrary distribution.

\(^{24}\) Notice that, in contrast to standard mechanism problems, the set of feasible mechanisms is solved subject to the free-entry condition.
3.2.1 Ironing equilibria

The first type of semi-pooling is similar to the standard ironing technique. The idea here is to pool a subset of sellers so that the constructed buyers’ valuation $\tilde{H}(x)$ is weakly increasing and continuous. For example, as illustrated in the left panel of Figure 4, one can pool all sellers within the region $[x_a, x_b]$, where the pair of $(x_a, x_b)$ is picked such that the expected value in the pooling market satisfies $E[h(x) | x_a \leq x \leq x_b] = h(x_a) = h(x_b)$. More generally, the function $h(x)$ may have multiple peaks and corresponding valuation $\tilde{H}(\cdot)$ may have multiple ironing (pooling) intervals. Nevertheless, to apply this method, the necessary condition is that the underlying distribution admits a continuous function $\tilde{H}(\cdot)$ that is weakly increasing.

By construction, $\tilde{H}(x)$ is strictly increasing when types are fully separated. Hence, similar as before, the buyer-seller ratio must be downward distorted. The key difference is that all types in the pooling region $x \in [x_a, x_b]$ are now treated as if they are sellers with value $h(x_a)$ (or, equivalently, $h(x_b)$). Similar to the monotonic case, the equilibrium of this type predicts lower trading volumes. I thus leave the detailed construction and condition for ironing equilibria in Appendix A.3.

3.2.2 Equilibria with Upward-Distorted Trading Rate: Fire-Sale

I now analyze a particular type of semi-pooling equilibrium that predicts upward-distorted trading rate and thus leads to distinct market outcome: a large aggregate trading volume at a depressed price in the pooling market. I refer to this as "fire-sale" equilibria.
The construction involves pooling all sellers below a marginal type $x^*$, as illustrated in the right panel of Figure 4. Moreover, the marginal type $x^*$ must satisfy the following conditions: (1) the monotonicity condition holds above this marginal type; and (2) the average quality below this cutoff type is strictly higher than the quality of his own asset. That is, the following condition holds:

$$q(x^*) \equiv E[h(x)|x \leq x^*] > h(x^*).$$  \hfill (H1)

Whether such a cutoff type exists depends on the underlying distribution. First of all, in order to satisfy the first condition, function $h(\cdot)$ must increase after a certain point. Let $\bar{x}$ denote the greatest local minimum point of function $h(\cdot)$. Moreover, for any $x > \bar{x}$, the ability of condition $(H1)$ to hold then depends on both the shape of $h(x)$ as well as the measure of $x$. Intuitively, condition $(H1)$ holds when underlying distribution suggests that sellers with lower $x$ have relatively high asset quality.\(^{25}\)

Observe that, by construction, the marginal type $x^*$ is now the minimum of $\hat{H}(x)$. Specifically, $\hat{H}(x)$ has a downward jump at $x^*$ for $\hat{H}(x)$, as $\hat{H}(x) = q(x^*) \forall x < x^*$ and $\hat{H}(x) = h(x) \forall x \geq x^*$. That is, from the viewpoint of buyers, the marginal type has the lowest expected asset quality. In particular, condition $(H1)$ suggests that the average asset quality in the pooling market is higher than $h(x^*)$.

The marginal seller thus has incentives to mimic a lower seller type ($x < x^*$) and/or a higher seller type ($x > x^*$). In other words, the incentive constraints bind both directions for the marginal type. As a result, one should expect distortion for both the market above and below this marginal seller. This is in sharp contrast to the case of monotonically increasing $\hat{H}(x)$ and thus only downward incentive constraints bind. Moreover, because the marginal seller is the one with the lowest quality, none has incentives to mimic him; hence, for the same intuition as in Lemma 4, this seller must receive the first-best utility.

The distortion can be implemented as follows: First of all, since $\hat{H}(x)$ is strictly increasing after $x^*$ and the marginal seller $x^*$ is less willing to wait compared with sellers with a higher $x$, all these markets must involve a higher price and lower trading speed. That is, the buyer-seller ratio is again downward distorted for $x > x^*$. On the other hand, since $x^*$ is more willing to wait compared with sellers with a lower-$x$, the distortion in the pooling market must involve a higher trading rate and a lower selling price, as established by Condition $(M)$ in Lemma 1.

The exact combination of price and buyer-seller ratio in the pooling market is thus pinned down so that (1) the marginal type is indifferent between going to the pooling market and his own market where he receives the first-best utility, and (2) the free-entry condition holds, where

\(^{25}\)Specifically, if the underlying distribution admits a seller type, denoted by $\overline{x}$, whose asset quality is the same as the average quality of all sellers below him. That is, $\overline{x}$ is the seller $x$ that satisfies $E[h(\overline{x})|\overline{x} \leq x] = h(x)$ and $x > \overline{x}$. Then, condition $(H1)$ holds for any seller type that is between the local minimum point $\overline{x}$ and the type $\overline{x}$. That is, any $x^* \in (\overline{x}, \overline{x})$ can be used to construct such semi-pooling equilibria.
Figure 5: The marginal type in a fire-sale equilibrium must be indifferent between the pooling market and the first-best market.

the average quality is given by $q(x^*)$. Formally, for any $x > \bar{x}$ that satisfies condition $(H1)$, define a pair of functions $(p_q(x), \theta_q(x))$ that solves the following equations:

$$
\begin{align*}
\theta_q(x) &= \max_\theta \left\{ \theta | V(p, \theta, x) = V^{FB}(x) \text{ and } p = \frac{q(x)}{r} - \frac{k\theta}{m(\theta)} \right\} \\
p_q(x) &= \frac{q(x)}{r} - \frac{k\theta_q(x)}{m(\theta_q(x))}.
\end{align*}
$$

Specifically, the solution for the pooling market, denoted by $(p_q(x^*), \theta_q(x^*))$, is illustrated in Figure 5. It is the intersection point between $V^{FB}(x^*)$ and the buyers’ free-entry condition, giving an upward-distorted trading rate $\theta_q(x^*) > \theta^{FB}(x)$. Note that since the average quality in the pooling market is strictly higher than the marginal type, the free-entry condition for buyers in the pooling market (valued at $q(x^*)$), represented by the dashed line, is thus above the free-entry condition in the market for the marginal type $x^*$, represented by the black line. This explains why condition $(H1)$ is necessary for the presence of an upward-distorted trading rate.

Moreover, the upward-distorted buyer-seller ratio means that buyers must wait for a long time to meet a seller in the pooling market, and hence they must be compensated with a low price given their high search cost, which is represented by $\frac{k\theta_q(x^*)}{m(\theta_q(x^*))}$. Indeed, this is the force that drives down the price in the pooling market, despite the fact that the average asset quality $q(x^*)$ is higher than $h(x^*)$. The combination of high trading rate and low price thus guarantees that only relatively distressed sellers are willing to enter. Proposition 3 summarizes the existence condition and characterization for a fire-sale equilibrium $x^*$.

**Proposition 3** There exists a semi-pooling equilibrium $x^* > \bar{x}$ with an upward-distorted trading
rate, if the following inequalities are satisfied:

\[ q(x^*) > h(x^*) \quad \text{(H1)} \]
\[ V(p_q(x^*), \theta_q(x^*), x_L) \geq V^{FB}(x_L). \quad \text{(H2)} \]

Such an equilibrium is characterized by the following equilibrium buyer-seller ratio and equilibrium price function, respectively:

\[
\theta^*(x) = \begin{cases} 
\theta_q(x^*) & \forall x \in [x_L, x^*) \\
\theta(x; \theta^{FB}(x^*)) & \forall x \geq x^*
\end{cases}
\]
\[
p^*(x) = \begin{cases} 
p_q(x^*) & \forall x \in [x_L, x^*) \\
\frac{h(x)}{\tau} - \frac{m(\theta(x))}{m(\theta^*(x))} & \forall x \geq x^*
\end{cases}
\]

where \( \theta(x; \theta^{FB}(x^*)) \) denotes the solution of (5) with the initial condition \( \theta^{FB}(x^*) \).

It is straightforward to show that this construction satisfies Lemma 1 and the free-entry condition. What is left is to show that condition \( E2(b) \) is also satisfied. In particular, for any \( q(x^*) > h(x^*) \), the price schedule jumps at the marginal type \( x^* \) from the pooling price \( p_q(x^*) \) to the first-best price for the marginal type \( p^{FB}(x^*) \). According to Lemma 2, any deviation \( p' \) between these two prices will attract only the marginal type \( T(p') = x^* \) with quality \( h(x^*) \). Given that the marginal type achieves his first-best utility, any deviation \( p' \) and the corresponding buyer-seller ratio \( \theta(p') \) necessarily imply a distortion, which therefore decreases buyers’ utility.\(^{26}\)

### 3.2.3 Existence of Fire-Sale equilibria

To further highlight the underlying economic force that leads to fire-sales, I now consider the simple example presented in Figure 3, where \( (s, c) \) are independently distributed and \( c \in \{c_L, c_H\} \). Let \( G_s(s) \) denote the marginal distribution of \( s \) and \( \lambda \) denote the measure of sellers with high holding costs.

In this example, the buyers’ value function \( h(\cdot) \) strictly increases in \( x \) after \( \underline{x} = s_H - c_H \), since all sellers above \( \underline{x} \) must have a low holding cost \( c_L : h(x) = x + c_L \). In order to construct a fire-sale equilibrium, one must first find the cutoff type \( x^* > \underline{x} \) so that condition \( (H1) \) holds.

Pick the type right above \( \underline{x} \) to be the cutoff type, \( x^* = \underline{x} + \varepsilon \). By construction, all sellers that are highly distressed (i.e., with holding cost \( c_H \)) enter the pooling market, and thus the average asset quality owned by distressed sellers is the expected mean of the underlying asset quality. On the other hand, sellers with low holding cost enter the pooling market if and only if

\(^{26}\)The argument above further highlights why such a pooling market can be sustained: buyers now attract the type whose asset quality is lower than the average quality in the pooling market. In contrast, under the monotonicity condition, sellers that are attracted by this deviation necessarily have the highest quality asset within the pool.
their asset quality is low enough (i.e., below $x^* + c_L$), and thus the expected asset quality among these sellers yields $E[\tilde{s} | \tilde{s} \leq x^* + c_L]$. The expected asset quality in the pooling market is then the weighted average of asset quality bought from these two groups of sellers. Condition (H1) can thus be rewritten as:

$$E[h(x) | x \leq x^*] = \gamma E[\tilde{s} | \tilde{s} \in S] + (1 - \gamma)E[\tilde{s} | \tilde{s} \leq x^* + c_L]$$

$$> h(x^*) = x^* + c_L,$$

where $\gamma = \lambda + (1 - \lambda)G_s(x^* + c_L)$ denotes the share of sellers with high holding costs in the pooling market.

Clearly, a higher measure of distressed sellers (i.e., a higher $\lambda$) thus improves the average asset quality in the pooling market. That is, it is easier for condition (H1) to be satisfied. By the same logic, when the underlying distribution of assets improves (that is, consider first-order stochastic dominance given a fixed interval) the average asset quality in the pooling market increases, and thus the fire-sale equilibrium is more likely to arise.\(^{27}\)

This simple example illustrates that the fire-sale equilibrium exists only when the effect on the distress level (the private value component) is strong enough so that the asset quality of those who want to sell faster is actually higher than the quality of assets held by some non-distressed sellers. Figure 6 below illustrates the equilibrium price function and buyer-seller ratio in a fire-sale equilibrium. The flat schedule represents the pooling submarket.

### 3.2.4 General Discussion on Equilibrium Construction

Given any arbitrary distribution, there may exist multiple ways to construct semi-pooling equilibria. Nevertheless, Lemma 1 and the free-entry condition characterize the whole set of feasible mechanisms (i.e., conditions $E1$ and $E2(a)$ are satisfied). Hence, one would just need to use Lemma 2 to check the possible deviation (i.e., condition $E2(b)$).

While the exact construction and the shape of $\hat{H}(x)$ may differ, the logic behind the direction of distortion remains the same. Specifically, for any constructed $\hat{H}(x)$, under the region where $\hat{H}(x)$ is increasing (decreasing), IC constraints bind downward (upward) and thus the buyer-seller ratio must be downward (upward) distorted. For regions under which $\hat{H}(x)$ is continuous, one can then apply the differential equation derived in Equation (5).\(^{28}\)

---

\(^{27}\)In Appendix A.4.9, I show that if a fire-sale equilibrium with the cutoff type $x^*$ exists with a $\lambda$ measure of distressed sellers under distribution $G_s(s)$, then the existence of a fire-sale equilibrium with the cutoff type $x^*$ is guaranteed for $\lambda' > \lambda$ and for a distribution $G'_s(s)$ that first-order stochastically dominates $G_s(s)$.

\(^{28}\)Specifically, when $h(x)$ is strictly decreasing in $[x_L, x_H]$, $\theta^*(x)$ is then characterized by Equation (5) with the terminal condition $\theta^*(x_H) = \theta^{FB}(x_H)$. Given that $h'(x) < 0$ and all buyer-seller ratio are upward distorted (i.e. $\theta^*(x) > \theta^{FB}(x)$), one can easily see that Equation (5) suggests that $\frac{d\theta}{dx} < 0$ and thus Lemma 1 holds.
Furthermore, as shown in the construction for fire-sale equilibria, it is possible to combine these two types of distortion. However, according to the monotonicity condition in Lemma 1, upward distortion must come before downward distortion. This also explains why $\hat{H}(x)$ can have an interior local minimum (as in fire-sale equilibria), but not an interior strict local maximum, as established in Proposition 2. The logic thus also suggests that one can also combine both fire-sale and ironing techniques for non-monotonic functions with multiple peaks. However, the fire-sale pooling market must start from the bottom and the ironing interval must come after.

4 Empirical Implications

The developed theory provides a link between the market environment (i.e., information structures and distribution of sellers’ distress positions) and market liquidity (i.e., asset prices and trading volume). Depending on whether the monotonicity condition holds, different types of equilibria arise, thus generating two distinct empirical illiquidity patterns: (1) a dry-up in trading volume and (2) a high trading volume at a steep price discount.

The second illiquid pattern is a unique feature of the semi-pooling equilibria constructed in Section 3.2.2, which I label as a fire sale. Moreover, I now show formally that only this equilibrium can generate the following features often associated with a "fire-sale" in the literature:

(1) An increase in the distress of an individual seller lead to a disproportional price discount. Distressed sellers quickly unwind their positions at a price below the fundamental value.
An increase in market-wide distress leads individual to face a higher distress discount.\footnote{For example, as discussed in Brunnermeier and Pedersen (2008), tightened lending standards in the market (which makes more banks become distressed) may lead to further downward asset prices.}

To formalize these results, I analyze the effect of market-wide distress in a fire sale equilibrium. Specifically, for the sake of illustration, throughout this section I consider the simple example in Section 3.2.3, where the distress level of sellers is binary. A larger measure of highly distressed sellers (denoted by parameter $\lambda = \Pr(c = c_H)$) thus represents an increase in market-wide distress. Nevertheless, as discussed in detail below, the result holds more generally whenever the concerns of sellers’ distress position become more severe.

Moreover, to highlight the importance of the observability of sellers’ distress positions, I compare the result to the model with only 1-D asymmetric information (so that monotonicity condition always holds). I show that model the with 1-D asymmetric information cannot produce the above “fire-sale” features and it has distinct empirical predictions.

In both environments, the cash flow of assets is given by $y + \sigma s$, where $(y, \sigma)$ are the common observable components of the asset’s value and $s$ is the sellers’ private information for this asset, which follows a symmetric distribution with mean zero and with support on $[-1, 1]$. The level of adverse selection is then parameterized by $\sigma$.

Formally, let $\bar{\theta}(s, c)$ and $\bar{p}(s, c)$ denote the trading rate and price of a seller $(s, c)$ in equilibrium. When sellers’ distress levels are observable, which I refer it as 1-D asymmetric information case, one can apply the characterization in Section 3.1 for each distress position separately.\footnote{That is, conditional on $c$, asset quality determines the sellers’ type, thus $\bar{\theta}(s, c) = \theta^*(s - c)$.} With 2-D asymmetric information considered throughout this section, I focus on the fire sale equilibrium; hence, the solution is derived in Section 3.2.3.

4.1 Aggregate Prices and Volumes

The total volume and the average trading price are then given, respectively, by:

$$
\vartheta \equiv \int m(\bar{\theta}(s, c))g(s, c)dsdc,
$$

$$
\varphi \equiv \int \frac{\bar{p}(s, c) \cdot m(\bar{\theta}(s, c))g(s, c)dsdc}{\vartheta},
$$

where $g(s, c) = \frac{\partial^2 G(s, c)}{\partial sdc}$ denotes the joint density. To see clearly the role of asymmetric information compared to other existing channels, it is useful to look at cross-market comparisons.

Specifically, I use the trading volume and the price of a safe asset with no payoff uncertainty (i.e., $\sigma = 0$) as a benchmark, denoted by $\vartheta^0$ and $\varphi^0$, respectively. To fix the idea, one can interpret agency MBS are a safe asset and private-label MBS are the assets that suffer adverse selection,
since the originators are likely to have private information. For the sake of comparison, I assume that everything else is the same for these two assets and buyers have access to both markets.

The predictions across markets distinguish this information based theory from other channels that rely on constrained investors or buyers with downward-sloping demand. One can easily show that without private information, \( \vartheta = \vartheta^0 \) and \( \varphi = \varphi^0 \) for any dispersion and for any distribution of distress costs, respectively. In other words, this channel will affect both markets in the same way, and thus cannot explain why illiquidity arises in one market but not the other. By contrast, the theory predicts that more information-sensitive assets would have distinct prices and volumes compared to safe assets, which is formalized in the Proposition below.

**Proposition 4** In a fire-sale equilibrium, with the level of adverse selection fixed, an increase in the market-wide distress level (\( \lambda \)) leads to a higher trading volume \( \vartheta \) and a lower price \( \varphi \). Further, there exists \( \lambda_g \) such that for any \( \lambda > \lambda_g \), \( \vartheta > \vartheta^0 \). In contrast, for any \( \lambda > 0 \), \( \vartheta < \vartheta^0 \) in the model with 1-D asymmetric info.

Note that there are two conflicting effects on the price in the pooling market. Fixing a marginal type \( x^* \), an increase in the measure of distressed sellers (\( \lambda \)) means that more distressed sellers with higher quality assets enter the pooling market, which improves asset quality in the pool. On the other hand, when buyers’ willingness to pay increases, the marginal seller now has a higher incentive to mimic. Hence, intuitively, the upward distortion that is needed to prevent \( x^* \) from entering the pooling market must increase as well, which then further increases buyers’ expected searching cost, driving down the price.

We can use Figure 5 to see that the buyer-seller ratio effect must dominate in equilibrium. Specifically, a higher quality in the pooling market shifts up the buyers’ free-entry condition (which is represented by the green line). Hence, the corresponding solution \( (p_q(x), \theta_q(x)) \) must move downward along the marginal type’s indifference curve. Therefore, the price in the pooling market goes down despite average seller quality improves. In other word, the buyer-seller ratio effect must dominate.

As a result, the model predicts that the greater the market share of highly distressed sellers (\( \lambda \)), the steeper are the required discounts. One can easily see that, as a result, the aggregate volume rises and the weighted aggregate price declines, because more sellers now sell at a lower distressed price. In general, this result holds for any distribution that leads to an increase in \( E[s|x < x^*] \).

It is worth noting that the prediction on price is distinct from a standard adverse selection model that imposes a pooling equilibrium directly. In those models, there is only quality effect.

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31 Calem et al. (2011), Downing et al. (2009), and Jiang et al. (2010) have found evidence that issuers have used private information about loan quality when choosing which loans to securitize.
Thus, a larger share of distressed sellers necessarily leads to a higher price instead of a lower one. Such a prediction is often deemed to be counterfactual.\footnote{See, for example, Uhlig (2010).}

Proposition 4 also highlights that both the underlying information structure and market conditions are important for the market outcomes. Indeed, when sellers’ distress levels are observable, the volume is always lower for the more information-sensitive assets.\footnote{Under 1-D asymmetric information case, there are two effects on the average price: (1) Assets with lower quality are traded at higher rates. This composition effect pushes down the average price; (2) A downward distorted buyer-seller ratio leads to a higher individual price for sellers $s > s_L$ than the one under complete information. The composition effect dominates when there are more mass on lower-quality assets.} This is because that the model predicts a downward distorted buyer-seller ratio, even though the market becomes more distress. In other words, most trading takes place only in the market for relatively transparent (and hence liquid) assets, which therefore looks like a flight to quality.

### 4.2 Distress Discount

Another aspect to examine regarding whether or not there is a fire sale comprises the price discounts faced by a distressed seller. Hence, I now turn to study the effect of distress on selling prices faced by an individual seller. The distress discounts are defined as a decrease in transaction price due to an increase in distressed level, fixing the asset fundamental:\footnote{When $c$ is continuous, price discount for any asset quality is then given by $\frac{\partial \bar{p}(s,c)}{\partial c}$.}

$$\Delta \bar{p}(s) \equiv \bar{p}(s, c_L) - \bar{p}(s, c_H).$$

**Proposition 5** In a fire-sale equilibrium, distressed sellers with high-quality assets face a larger price discount than in an environment without asymmetric information. Furthermore, such a discount increases with market-wide distress ($\lambda$). In contrast, the distress discount is independent of market-wide distress ($\lambda$) under a 1-D asymmetric information model.

Recall that, in a fire-sale equilibrium, sellers with relatively high quality assets $s > x^* + c_L$ go to the pooling market only if he becomes distressed and enters a separating market otherwise. Given that the price for the separating market is independent of $\lambda$ and the price in the pooling market decreases with $\lambda$, the price discount thus increases with $\lambda$. On the other hand, since all sellers always enter a separating market in the 1-D asymmetric information model, the difference in the price is only a function of holding costs and it is thus independent of $\lambda$.

Note that in a standard channel with downward demand, a more distressed seller would then face a lower price. Hence, it is useful to compare the prediction with the benchmark. The result highlights that the price discount is amplified by sellers’ distress position in a fire sale.
equilibrium. To understand the price discount, one can decompose the distress discount into the following terms for sellers with relatively high asset quality:\(^{35}\)

\[
\hat{p}(s, c_L) - \hat{p}(s, c_H) = \left( \frac{s}{r} - \frac{E[s|x < x^*]}{r} \right) + \left( \frac{k\theta_q(x^*)}{m(\theta_q(x^*))} - \frac{k\theta(s, c_L)}{m(\theta(s, c_L))} \right) > 0. \quad (7)
\]

The first term represents the difference between the average asset quality in the pooling market and the asset quality of a seller. Clearly, this term is zero for any separating equilibrium. As in the standard adverse selection model with one pooling market, sellers with better assets suffer a higher pooling discount. The second term represents the difference in buyers' expected searching costs. A higher buyer-seller ratio in the pooling market implies that it is easier for a seller to find a buyer, and harder for a buyer to find a seller. As a result, the price drops further to compensate buyers.

Given any marginal type \(x^*\), a higher \(\lambda\) leads to a larger buyer-seller ratio, which thus increases the price discount. Note that for the marginal type \(s = x^* + c_L\), the first term, is negative by construction. Hence, for such a marginal type, what drives the price down in this pooling market is indeed the upward-distorted trading rate, not the standard pooling discount. This feature further highlights the additional force in this semi-pooling equilibrium. On the other hand, both forces – namely, the pooling discount and the high transaction cost – exist for sellers with higher asset quality, implying an even sharper discount for those sellers.

### 4.3 Discussion of Predictions and Evidence

Propositions 4 and 5 have established that only the semi-pooling equilibria with upward distorted buyer-seller ratio can generate the features often associated with "fire-sales" in the literature. Hence, the theory provides an answer to why and when sellers actually engage in fire sales, or try to avoid them instead. Indeed, as Shleifer and Vishny (2010) discussed, empirical works have documented different trading outcomes across markets, and no existing theory is able to reconcile these two patterns.\(^{36}\)

Specifically, the model predicts that fire sales can exist only in an environment in which (1) investors observe neither asset quality (the common value) nor sellers' trading motives (the private value), and, more importantly, (2) when distress costs dominate the motives for selling (as discussed in Section A.4.9). Only in this case does the equilibrium suggest a high trading volume at a distressed price. That is, fire sales with large distress discounts actually occur as

\(^{35}\)To be precise, the expression holds for sellers with relatively high asset quality \((s > x^* + c_L)\). Sellers with relatively low asset quality (any type \(s < x^* + c_L\) always go to the pooling market, regardless of their distress position, and therefore always trade at the pooling price.

\(^{36}\)The conventional explanation for fire sales assumes that high-valuation buyers are sidelined and distressed sellers are forced to sell below fundamental value. It is therefore silent about when fire sales actually exist.
One concrete example for this outcome is the housing markets, in which some sell fast at a highly depressed prices, while, at the same time, other sellers trade slower at relatively high prices. In particular, one may interpret the pooling submarket to be the market for foreclosed houses. That is, the group of sellers who choose to default on their mortgages include some who cannot pay (highly distressed) and some who have lower-quality houses (strategic default). Our model thus predicts that, only when the above conditions hold, forced sales would account for a large fraction of total sales, leading to higher price discount.

If the conditions above are not satisfied, the equilibrium always involves downward-distorted trading rates. Sellers choose to hold onto their illiquid assets, so most trading take place only in the market for relatively transparent (and hence liquid) assets. It therefore appears that sellers will tend to avoid selling affected assets at fire-sale prices by mainly selling assets in liquid markets instead, consistent with Schlingemann et al. (2002), Chernenko et al. (2014), and Boyson et al. (2011).

This is true especially in the market for the AAA-rated private MBS. The trading volume and issuance of private-label MBS has declined to negligible levels since the onset of the financial crisis in mid-2007, while the agency MBS market remains liquid. This phenomenon could be interpreted as AAA-MBS becoming more information-sensitive as a result of the drop in housing prices in 2007. On the other hand, given that banks became more distressed during the crisis, it is not clear ex-ante why fire sales do not occur. Through the lens of the model, one can interpret this episode as an environment in which banks’ distress level are observable to investors. In that environment, banks are more likely to divest segments from industries with a more liquid market for corporate assets, even when this means keeping some of their worst performing units. Similarly, in the MBS market, Chernenko et al. (2014) show that distressed mutual funds actively sell agency MBS instead of nontraditional securitization. Boyson et al. (2011) shows that most banks circumvent fire sales by shifting to deposits, issuing equity, and cherry picking.

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37 Pulvino (1998) shows that used planes sold by distressed airlines bring 10 to 20 percent lower prices than planes sold by undistressed airlines. Coval and Stafford (2007) show that widespread selling by financially distressed mutual funds leads to transaction prices below fundamental value. Ellul et al. (2011) found that insurance companies that are relatively more constrained by regulation are more likely to sell downgraded bonds, which are subject to price discount.

38 Garmaise and Moskowitz (2004) documents the existence of asymmetric information in real estate markets.

39 See Campbell et al. (2011), who examined forced sale discounts by using information on individual bankruptcies, controlling observable characteristic of a house.

40 Schlingemann et al. (2002) find that firms are more likely to divest segments from industries with a more liquid market for corporate assets, even when this means keeping some of their worst performing units. Similarly, in the MBS market, Chernenko et al. (2014) show that distressed mutual funds actively sell agency MBS instead of nontraditional securitization. Boyson et al. (2011) shows that most banks circumvent fire sales by shifting to deposits, issuing equity, and cherry picking.

41 This idea derives from the fact the drop in housing prices in 2007 resulted in some AAA-MBS, which were previously treated as safe assets, failing to deliver the high cash flow that was promised. The fact that originators have more information about the credit quality and underlying composition of the securities therefore leads to an increase in adverse selection. See Gorton (2009) for a detailed description about how the chain of interlinked securities was sensitive to house prices and how asymmetric information was created via complexity. His view is also empirically supported by Agarwal et al. (2012).
case, as shown in Propositions 4 and 5, fire sales will not occur even if banks become highly distressed.

In general, our theory provides a new testable prediction regarding the observability of sellers’ distress positions and the existence of fire sales: fire sales are more likely to occur in less-transparent and less-regulated markets in which sellers’ distress levels are not publicly disclosed. Examples include the housing markets, the market for used durable goods (where sellers’ private motives are often unobservable), or when the distress of banks is driven by their counterparty risk exposures to complex financial networks that are difficult for investor to assess.\footnote{Caballero and Simsek (2013), for example, considers the environment when banks are uncertain about the financial network of cross exposures between financial institutions.}

**Remarks on Empirical Measurement** In general, to test the empirical implications, one would need to carefully condition the exact information that is available to market participants. With MBS markets, for example, one should consider all privately MBSs that provide identical information to buyers as one type of assets. To test the predictions established in Proposition 4, the markets under which issuers have more soft information should then be the ones subject to higher levels of adverse selection level (i.e., higher $\sigma$). Moreover, everything else being equal, the model predicts that assets traded with higher price dispersion (conditional on observable characteristics) are more likely to have lower trading volumes.\footnote{Along this line, Jankowitsch et al. (2011) shows that there is indeed a correlation between price dispersion in over-the-counter market and trading volume (as well as common liquidity measures).}

Regarding predictions at the micro level (as in Proposition 5), empirical tests are more challenging given the nature of private information. Nevertheless, although the true quality of an asset is not observable to buyers when it is traded, some existing works use ex-post performance (such as the default rate for MBS) as a proxy of the asset quality. Similarly, when a seller chooses to default on his mortgage, buyers may not be able to differentiate distressed from strategic defaulters. Existing work has made progress in this direction by combining detailed information about employment status, equity, and other assets—see, for example, Gerardi et al. (2015).

## 5 Discussion on Efficiency

The method developed here (i.e., Lemma 1, $E_1$, and the buyers’ free entry condition, $E_2(a)$) characterizes the set of feasible mechanisms. Within this set, the decentralized outcome is the one satisfying the additional condition $E_2(b)$ on buyers’ optimal price-posting. Hence, it follows immediately that a social planner who is not subject to condition $E_2(b)$ can always do weakly better than the decentralized equilibrium.

As long as the monotonicity condition is satisfied (Assumption 1), the decentralized outcome
has to be the least-cost fully separating equilibrium. However, a fully separating equilibrium, which implies distortions on trading rates, depends only on the support of the distribution and is clearly not necessarily efficient (both in terms of total welfare or Pareto efficiency). To see this, consider an asset distribution that has a very thin left tail but a high expected mean. In this case, a high-type seller can be better off in a pooling market, where the price is valued at the average asset quality, than in a separating equilibrium, in which he suffers a high delay cost due to the thin left tail. Clearly, whenever a high-type seller is better off in a pooling equilibrium, so is a low-type seller, since the latter simply receives a subsidy from a high-type seller. Hence, a fully separating equilibrium can be Pareto-dominated by a partial pooling equilibrium under certain distributions.  

Do disclosure of Sellers’ Distress Position improve efficiency? Recall that the monotonicity condition is automatically holds when sellers’ trading motives are observed. Hence, in an environment in which pooling is more desirable, traders can be better off when the private value of holding an asset is unobservable. This can be seen from the fire-sale example. All sellers with a low holding cost are better off when the holding cost is unobserved, regardless of their asset quality. This is because a seller with relatively low-quality assets effectively receives a subsidy from sellers with better assets in the pooling market. On the other hand, a seller with relatively high-quality assets \((s - c_L > x^*)\) is better off since the underlying range effectively decreases and therefore implies a smaller distortion. For a similar reason, a seller with a relatively low-quality asset but a high holding cost \((c_H)\) is also better off. Hence, the only type that might suffer when the holding cost is unobserved is the type with a good asset but a high holding cost.

By the same logic discussed earlier, whether this high-type seller is better off or not depends on which distortion is more costly: a price discount or a distortion in the trading rate. If the distribution is such that pooling is more desirable, the high-type seller is then better off in the pooling equilibrium. If that is the case, then, counterintuitively, all types of sellers achieve higher equilibrium utilities when the holding cost is unobserved.

6 Conclusion

This paper develops an information-based theory of illiquidity. It analyzes how two-dimensional asymmetric information of sellers leads endogenously to two different illiquidity patterns: (i) limited market participation and a dry-up in trading volumes, and (ii) a high trading volume at a depressed price (i.e., fire-sales). It establishes that fire sales can occur only when sellers’ distress positions (private value of the asset) are unobserved by the market and such motives dominates the common value component.

44This point then explains why the competitive search equilibrium is Pareto-inefficient for some parameter values in Guerrieri et al. (2010).
A Appendix

A.1 Model with Dynamic Evolution of Distribution

This section considers an environment where both sellers and buyers leave the market once the trade takes place, instead of being replaced by an identical agent. At time 0, there is a unit measure one of sellers. Let $G^0(s, c) : S \times C \to [0, 1]$ denote the initial joint cumulative distribution of asset quality and holding cost. There is a large measure of buyers, who can choose to enter the market by paying a flow cost $k > 0$. Hence, at each point of time, the measure of buyers entering the market is determined by the free-entry condition.

One convenient feature of this setup is that one can construct stationary equilibria, in which the set of offered prices (denoted by $P^*$), the trading rate in each market $\theta(p)$, and the composition of sellers in each market are time-invariant. The existence of stationary equilibria $\{P^*, \theta(p), \mu(\cdot|p)\}$ can be seen directly from the payoff function. Heuristically, stationary sellers' trading strategies and the random matching in each market (i.e., all sellers in the same market trade with the same rate) imply that the composition of sellers within each market is stationary. As a result, the buyers' expected matching value in each market is also time-invariant. Given that buyers can enter and exit instantaneously to generate the correct ratio $\theta(p)$ in each market at each point in time (the free-entry condition), a stationary equilibrium can therefore be sustained.

As a result, the solution $\{P^*, \theta(p), \mu(\cdot|p)\}$ derived in the main model can be directly applied here. The only difference is that the aggregate distribution does evolve over time and leads to non-stationary dynamics of aggregate prices and volumes. Specifically, given any initial distribution of asset qualities and distress positions, the trading volume $\vartheta_t$ and the weighted trading price $\varphi_t$ at each point in time $t$ are then given, respectively, by:

$$\vartheta_t \equiv \int m(\hat{\theta}(s, c))dg^t(s, c)dsdc,$$

$$\varphi_t \equiv \int \hat{p}(s, c) \cdot m(\hat{\theta}(s, c))g^t(s, c)dsdc \cdot \frac{dg^t(s, c)}{\vartheta_t}.$$

Given that $g^0(s, c) = \frac{\partial^2 G^0(s, c)}{\partial s \partial c}$ and the law of motion $dg^t(s, c) = -m(\hat{\theta}(s, c))g^t(s, c)$, we thus have $g^t(s, c) = g^0(s, c)e^{-m(\hat{\theta}(s, c))t}$.

The time series movement of the price and volume is straightforward in this environment: the volume decreases over time and converges to zero asymptotically, while the weighted price increases over time, given that most assets being sold earlier have a lower price.
A.2 Condition for Monotonicity

To guarantee that the function \( h(x) = E[s|x] \) is increasing in \( x \), the following proposition provides a sufficient condition on the joint distribution of \( s \) and \( c \). I now restrict the case to when the variable \( c \) also has a continuous distribution. Let \( g(s, c) : [s_L, s_H] \times [c_L, c_H] \rightarrow \mathbb{R} \) denote the joint density of the random variable \((s, c)\), and assume that \( g \) is strictly positive and twice continuously differentiable.

**Lemma 5** If the joint distribution \( g \) satisfies the following condition:

\[
g_s \cdot g_c + (g_c)^2 - (g_{sc} + g_{cc}) \cdot g \geq 0,
\]

where \( g_v \) denotes the partial derivative with respect to the variable \( v \), then \( h(x) \) is increasing in \( x \). If \( s \) and \( c \) are independent and the marginal distribution for \( c \) is log-concave, then \( h(x) \) is increasing in \( x \).

**Proof.** Set \( z = s \) and \( x = s - c \). By the Jacobian of the transformation, the joint density of \((z, x)\) yields:

\[
\frac{\partial}{\partial z \partial x} \ln g(z, x) = \frac{1}{g^2} \cdot \{g_c(g_s + g_c) - g \cdot (g_{sc} + g_{cc})\}.
\]

It is well known that \( E[s|x] \) is increasing in \( x \) if \( \frac{\partial}{\partial z \partial x} \ln g(z, x) \geq 0 \), that is, if the variables \( s \) and \( x \) are affiliated (see Milgrom and Weber (1982)). When \((s, c)\) are independent, \( \frac{\partial}{\partial z \partial x} \ln g(z, x) \propto [g'(c)]^2 - g''(c)g(c) \geq 0 \iff \ln g(c) \) is concave. ■

The condition on the distribution borrows from the standard affiliation condition discussed in Milgrom and Weber (1982).\(^{45}\) Note that \( E[s|x] \) can be rewritten as \( E[s|x] = x + E[c|x] \). This shows that whether the monotonicity condition holds depends on the properties of \( E[c|x] \).

A.3 Equilibria with Ironing

Given a non-monotonic function \( h(x) \), one can pool certain types of sellers and reconstruct a new function \( \tilde{H}(x) \) so that the buyers’ valuation function \( \tilde{H}(x) \) is continuous and weakly increasing, as shown in Figure 4.

In the first region, since buyers’ value is strictly increasing, the buyer-seller ratio function can be obtained as before. That is, \( \theta^*(x) \) solves (5) with the initial condition \( \theta^{FB}(x_L) \). At the

\(^{45}\)The commonly used distributions with log-concave density functions are, for example: uniform, normal, and exponential. On the other hand, some distributions (such as the beta and gamma) have log-concave density functions only if their parameters fall into certain ranges. (For more detailed discussions, see Bergstrom and Bagnoli (2005)).
pooling interval, given $\theta^*(x_1)$ obtained from the first region, set $\theta^*(x) = \theta^*(x_a)$ for $\forall x \in [x_a, x_b]$. This means that allocations are the same among this set of sellers, and buyers pay for the expected asset quality in this pool. Hence, by construction the free entry condition is satisfied automatically in this pooling region.

In the last region, one can let $\theta^*(x)$ solve (5) with the initial condition $\theta^*(x_b) = \theta^*(x_a)$, since $\bar{H}(x)$ is again strictly increasing (i.e., A1 holds). Hence, if $\theta^*(x_b) \leq \theta^{FR}(x_b)$, (5) ensures that $\frac{d\theta^*}{dx} < 0$ and thus all previous results remain intact. Figure 7 illustrate this construction for the simple example with two types of holding cost.

More generally, if the underlying function $h(x)$ has multiple non-monotonic regions, there are different ways to iron the underlying functions, and there may be multiple ironing intervals. The logic is nevertheless the same. Let $x_i^a$ and $x_i^b$ denote the starting point and the ending point for the $i$th ironing interval. The ironed valuation $\bar{H}(x)$ can be used to construct an semi-pooling equilibrium as long as the following condition holds:

**Lemma 6** The ironing equilibrium exists if (1) one can construct a function $\bar{H}(x)$ that is continuous and weakly increasing with ironing intervals $[x_i^a, x_i^b]$ and (2) $\theta^*(x_i^b) \leq \theta^{FR}(x_i^b)$ for all $i$. In this case, $\theta^*(x)$ solves (5) for non-ironing intervals with initial condition $\theta^*(x_i^a)$ and $\theta^*(x) = \theta^*(x_a)$ for all ironing intervals $\forall x \in [x_i^a, x_i^b]$.

**A.4 Omitted Proofs**

**A.4.1 Proof of Lemma 1:**

**Proof.** Consider a direct mechanism $\{p^*(x), \theta^*(x)\}$. Define $V^*(x) = \max_{\tilde{x} \in X \cup \emptyset} V(p^*(\tilde{x}), \theta^*(\tilde{x}), x)$, where $V(p, \theta, x) = \frac{x + p^*(x) - m(\theta^*(x))}{r + m(\theta^*(x))}$. Observe that the partial derivative of $V$ with respect to $x$
exists and has a finite upper bound: \( V_{\epsilon}(p, \theta, x) = \frac{1}{r+m(\theta)} \leq \frac{1}{r} \). Following the mechanism literature (see Milgrom and Segal (2002)), the agent’s utility must satisfy the envelope theorem (the integral condition) for any incentive-compatible mechanism:

\[
V^*(x) = V^*(x_L) + \int_{x_L}^{x} V_x(p^\alpha(\bar{x})), \theta^\alpha(\bar{x}), \bar{x}) d\bar{x} = V^*(x_L) + \int_{x_L}^{x} \frac{1}{r + m(\theta)} d\bar{x}
\]

This means that (IC) is the necessary condition for any IC contract. Furthermore, the function \( V(p, \theta, x) \) can then be rewritten as \( v(t, x) = xt + \Gamma(t) \), where \( t = \frac{1}{r+m(\theta)} \) and \( \Gamma(t) = \frac{p(m^{-1}(\frac{1}{t-r}))}{(1/t)} \). Then, one can then easily see that \( v(t, x) \) satisfies the strict single crossing difference property (SSCD). For any \( t' > t \) and \( x' > x \):

\[
v(t', x') - v(t', x) + v(t, x) - v(t, x') > 0.
\]

According to the monotonic selection theorem (See Theorem 4.1 in Milgrom (2004)), \( v(t, x) \) satisfies the SSCD condition if and only if every optimal selection \( t^*(x) \in \arg \max_x v(t, x) \) is nondecreasing in \( x \). Hence, \( t(x) = \frac{1}{r+m(\theta(x))} \) has to be solved subject to the nondecreasing constraint. That is, the trading rate function \( \theta^\alpha(\cdot) \) has to be nonincreasing. Furthermore, from Theorem 4.2 (Milgrom (2004)), given that \( v(t, x) \) is continuously differentiable and satisfies SSCD, nondecreasing \( \theta^\alpha(x) \) and the envelope formulas (IC) are sufficient conditions for the achievable outcome.

It is useful to characterize the set of allocations\( \{p^\alpha(x), \theta^\alpha(x)\} \) which satisfy the free entry condition. To facilitate the rest of the proofs, I now introduce the following notation. Since the price must satisfy the free-entry condition for each market, given the buyer’s valuation in a given market (denoted by \( \tilde{h} \)), sellers’ utilities can be effectively rewritten as a function of buyer-seller ratio and the buyer’s valuation \( \tilde{h} \). The expression is given by

\[
\tilde{V}(x, \theta, \tilde{h}) \equiv \frac{x + m(\theta)(\frac{\tilde{h}}{r} - \frac{k\theta}{m(\theta)})}{r + m(\theta)}.
\]

That is, \( \tilde{V}(x, \theta, \tilde{h}) \) denotes the utility of a type \( x \) seller when he enters the market with buyerseller ratio \( \theta \), given the expected value to buyers \( \tilde{h} \). Given a concave matching function, one can show that, for any value \( \tilde{h} \), \( \tilde{V}(x, \theta, \tilde{h}) \) is a concave function in \( \theta \). A seller’s utility is therefore maximized when the buyer-seller ratio is chosen optimally. Let \( \theta^{FB}(x, \tilde{h}) \equiv \arg \max_\theta \tilde{V}(x, \theta, \tilde{h}) \) denote the optimal level of buyer-seller ratio for any given \( \tilde{h} \).

Since \( \tilde{V}(x, \theta, \tilde{h}) \) is strictly increasing in \( \theta \), \( \forall \theta \in [0, \theta^{FB}(x, \tilde{h})] \), and reaches its maximum at \( \theta^{FB}(x, \tilde{h}) \), let \( \theta^D(V; x, \tilde{h}) \) be the buyer-seller ratio below \( \theta^{FB}(x, \tilde{h}) \) such that the type-x seller enjoys the utility level \( V \), given buyers’ value \( \tilde{h} \). That is, \( \tilde{V}(x, \theta^D(V; x, \tilde{h}), \tilde{h}) = V \) and \( \theta^D(V; x, \tilde{h}) < \theta^{FB}(x, \tilde{h}) \). Similarly, since \( \tilde{V}(x, \theta, \tilde{h}) \) is strictly decreasing for \( \forall \theta > \theta^{FB}(x, \tilde{h}) \), define \( \theta^U(V; x, \tilde{h}) \) to be the buyer-seller ratio above \( \theta^{FB}(x, \tilde{h}) \) such that \( \tilde{V}(x, \theta^U(V; x, \tilde{h}), \tilde{h}) = V \).

33
Since any buyer-seller ratio that is different from \( \tilde{\theta}^{FB}(x, \tilde{h}) \) implies distortion, \( \theta^D(V; x, \tilde{h}) \) and \( \theta^U(V; x, \tilde{h}) \) represent, respectively, the downward and upward distorted buyer-seller ratio that give a lower seller’s utility, \( V < \max_{\theta} \tilde{V}(x, \theta, \tilde{h}) \).

To check the buyers’ optimal price-posting condition \( E2(b) \), I first establish the following Lemma:

**Lemma 7** Consider a market \((p, \theta)\) such that

\[
V(p, x, \theta) < \max_{\theta} \tilde{V}(x, \theta, \tilde{h}) \quad \text{and} \quad p = \frac{\tilde{h}}{r} - \frac{k\theta}{m(\theta)}.
\]

Then, there exists a price \( p' \) and buyer-seller ratio \( \theta' \in (\theta^D(V; x, \tilde{h}), \theta^U(V; x, \tilde{h})) \) that gives buyers a strictly positive profit and leaves sellers indifferent. That is,

\[
V(p', \theta', x) = V(p, \theta, x) \quad \text{and} \quad p' < \frac{\tilde{h}}{r} - \frac{k\theta'}{m(\theta')}
\]

**Proof.** The fact that \( V = V(p, x, \theta) < V^{FB}(x, \tilde{h}) \equiv \max_{\theta} \tilde{V}(x, \theta, \tilde{h}) \) implies that the equilibrium buyer-seller ratio is not chosen optimally and is given by either \( \theta^D(V; x, \tilde{h}) \) or \( \theta^U(V; x, \tilde{h}) \), as shown in Figure 8. Since \((p, \theta)\) is not located at a tangency point between buyers’ iso-profit curve (which is represented by the black line: \( p(\theta) = \frac{\tilde{h}}{r} - \frac{k\theta}{m(\theta)} \)) and sellers’ indifference curve (which is represented by the blue line: \( V(p, \theta, x) = V \)), all combinations of \((p', \theta')\) in the shaded area will make both buyers and sellers better off. Pick any \( \theta' \in (\theta^D(V; x, \tilde{h}), \theta^U(V; x, \tilde{h})) \) and let \( p' \) solve \( V(p', \theta', x) = V \), this pair \((p', \theta')\) will then be on the sellers’ indifference curve while it gives buyers a strictly positive profit.\(^{46}\)

\[\blacksquare\]

### A.4.2 Proof of Lemma 2

Notice that \( p^o(\cdot) \) is nondecreasing for all feasible mechanisms, given the monotonicity condition \((M)\) and \( \theta^o(x) > 0 \ \forall x \). For any \( p' \) outside of the range of \( p^o \) and \( p' > V^*(x) \) (which is the relevant case),\(^{47}\) the buyer-seller ratio \( \tilde{\theta}(p', x) \) solves \( \zeta(p', \tilde{\theta}, x) \equiv V(p', \tilde{\theta}, x) - V^*(x) = 0 \),

\[
\frac{d\tilde{\theta}(p', x)}{dx} = -\frac{d\zeta}{dx} \propto \frac{1}{r + m(\theta^o(x))} - \frac{1}{r + m(\tilde{\theta}(p', x))} = \begin{cases} < 0 \text{ if } p' > p^o(x), (\therefore \tilde{\theta}(p', x) < \theta^o(x)) \\ > 0 \text{ if } p' < p^o(x), (\therefore \tilde{\theta}(p', x) > \theta^o(x)) \end{cases}.
\]

\(^{46}\)One can also use this lemma to understand why the equilibrium must satisfy the tangency condition with complete information. In that case, \( x \) is observable and \( \tilde{h} = h(x) \).

\(^{47}\)In the case where \( p' < V^*(x) \), the function \( \theta(p', x) = \infty \) because \( V(p', \theta, x) \geq V^*(x) \) has no solution. In words, if the deviating price is lower than a type’s equilibrium utility, this type will not come to this market.
Recall that, when posting a new price $p'$, a buyer should expect the lowest buyer-seller ratio, $\theta(p') = \inf_x \{ \tilde{\theta}(p', x) \}$, and the type who will come to this market is given by $T(p') = \arg \inf \{ \tilde{\theta}(p', x) \}$. Since $\tilde{\theta}(p', x)$ is strictly decreasing in $x$ for $p' > p^\alpha(x)$ and strictly increasing in $x$ for $p' < p^\alpha(x)$, this shows that $T(p')$ is unique. The above result then implies that $T(p') = x^+ = \arg \inf \{ x \in x | p' < p^\alpha(x) \} = x_L$. Similarly, if $p' > p^\alpha_H$, $x^- = \arg \inf \{ x \in x | p' > p^\alpha(x) \} = x_H$. Given that $p^\alpha(\cdot)$ is nondecreasing, for any $p'$ outside of the range of $p^\alpha(\cdot)$ and $p' \in [p^\alpha_L, p^\alpha_H]$, $T(p') = x^+ = x^-$.  

A.4.3 Proof of Lemma 3: No Pooling

**Proof.** Proof by contradiction. Take a mechanism $\{p^\alpha(\cdot), \theta^\alpha(\cdot)\}$ which satisfies Lemma 1 and free entry condition, while $p(x) = \tilde{p}$ for $x \in [x_1, x_2] \subset X$. That is, there exists a subset of sellers $x \in [x_1, x_2]$ are in the same market $(\tilde{p}, \tilde{\theta})$. Buyers’ expected value in this market is then given by $H(\tilde{p}) \equiv E[h(x)|p(x) = \tilde{p}]$. Given that $h(x)$ is strictly increasing (Assumption 1), a pooling equilibrium implies that $\frac{H(\tilde{p})}{\tilde{\theta}} < \frac{\tilde{p}_2}{\tilde{p}_1}$. Given that type $x_2$ seller must be locally indifferent, and that $\theta^D(V^*(x_2); x_2, \tilde{h})$ is continuous in $\tilde{h}$, it follows that $\theta(\cdot)$ must jump downward and $p(\cdot)$ must jump upward at $x_2$. Given that there is a jump in the price, a buyer can deviate by posting a

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48The only exception is when some types of sellers are out of the market. In this case, there then exists a marginal type $x^*$ such that $\theta(x^*) = 0$ for $\forall x > x^*$. For any $x > x^*$ and $p' > \frac{\tilde{p}_2}{\tilde{p}_1}$, type-$x$ will come to the market even when $\theta(p', x) \rightarrow 0$. Hence, $T(p')$ is then a set of these types of sellers. Nevertheless, such an exception is not relevant for the equilibrium result since a buyer will deviate even when he expects the worst type within this set, as will become clear later.
new price \( p' = \hat{p} + \varepsilon \). By Lemma 2, he will attract the type-\( x_2 \). 

\[
\frac{H(\hat{p})}{r} - \frac{k\hat{\theta}}{m(\hat{\theta})} - \hat{p} = 0 \implies \frac{x_2}{r} - \frac{k\hat{\theta}}{m(\hat{\theta})} - \hat{p} > 0.
\]

Define \( \tilde{\theta}(p, \nabla, x) \) as the buyer-seller ratio which ensures that the seller of type \( x \) stays on his indifference curve, i.e., \( V(x, p, \tilde{\theta}(p, \nabla, x)) = \nabla \). Since \( \tilde{\theta}(p, \nabla, x) \) is continuous, there exists \( p' \in B_{\varepsilon}(\hat{p}) \) such that

\[
\frac{x_2}{r} - \frac{k\bar{\theta}(p', V^*(x_2), x_2)}{m(\bar{\theta}(p', V^*(x_2), x_2))} - p' > 0.
\]

In words, by posting the price \( p' \), buyers will attract \( x_2 \) and expect the buyer-seller ratio \( \tilde{\theta}(p', V^*(x_2), x_2) \). The above inequality shows that such a deviation is profitable.\(^{49}\)

A.4.4 Proof of Lemma 4: No distortion at the Bottom

**Proof.** I show that there is no distortion at the bottom by contraction. Suppose that there is distortion on the lowest type in a full separating equilibrium, where \( \theta^*(x_L) < \theta^{FB}(x_L) \), \( p^*(x_L) = \frac{h(x_L)}{r} - \frac{k\theta^*(x_L)}{m(\theta^*(x_L))} > p^{FB}(x_L) \), and \( V^*(x_L) < V^{FB}(x_L) \). Consider a deviation \( p' = p(x_L) - \varepsilon \), according to Lemma 2, \( T(p') = x_L \). That is, a buyer can open a new market with a lower price and expect the lowest type to come. Hence, by Lemma 7, such deviation is profitable.

A.4.5 Proof of Proposition 1

**Proof.** Given \( \{p^*(x), \theta^*(x)\} \), one can see that the IR constraint holds for all sellers, since the trading price is higher than the outside option: \( p^*(x) > p^{FB}(x) > \frac{\varepsilon}{r} \). Furthermore, there is no profitable deviation for buyers to open new markets, since any new market \( p' \) and its corresponding \( \theta(p') \), given by (1), lead to a further distortion, implying a negative payoff for buyers. Formally, denote \( (p_L, p_H) \) as the lower bound and the upper bound, respectively, of the image of the function \( p^*(x) \). From Lemma 2, if buyers post a price \( p' > p^H \), only the highest type will come to this market, and if they post a price \( p' < p^L \), only the lowest type will come to the market. In both cases, as shown in Lemma 7, the corresponding \( (p', \theta(p')) \) implies a negative payoff for buyers.

Note that one can easily see why a fully separating equilibrium cannot involve a jump in price. The buyer-seller ratio must be downward distorted and it is therefore given by \( \theta^*(x) = \frac{h^D(V^*(x); x, h(x))}{r} \). Since \( V^*(x) \) must be Lipschitz continuous in \( x \), it follows immediately that \( \theta^*(x) \) is continuous in \( x \), and therefore so is the price function \( p^*(x) = \frac{h(x)}{r} - \frac{k\theta^*(x)}{m(\theta^*(x))} \).

\(^{49}\)This result also holds even when there are no other markets open for all \( x > x_2 \). That is, all higher-type sellers stay in autarky. In that case, although \( T(p') \) is no longer uniquely defined, the possible types who will come to this market are all weakly better than \( x_2 \). Hence, deviation \( p' \) is also profitable.

36
the schedule \( \{ p^*(x), \theta^*(x) \} \) is the unique mechanism that satisfies both \((E1)\) and \((E2)\). In other words, the least-cost separating equilibrium is the unique equilibrium. ■

A.4.6 Proof of Corollary 1

**Proof.** From the FOC of the first-best solution, one can solve \( \frac{d\theta_{FB}(x)}{dx} = \frac{(h'(x)-1)(r'-1)}{km''(r+m(\theta))} \equiv f_{FB}(\theta, x) \). Observe from the differential equation that \( \frac{d\theta^*(x)}{dx} \equiv f(\theta, x) \to -\infty \) at \( f(\theta_{FB}(x), x) \) given \( h_x > 0 \). Hence, we know that \( \theta^*(x) \leq \theta_{FB}(x) \) for some \( x_1 > x_L \). Suppose now that \( \theta^*(x) > \theta_{FB}(x) \) for some \( x \). This implies that these two functions must cross at some point \( (\tilde{x}, \theta_{FB}(\tilde{x})) \) and that the slope of the crossing point must satisfy the following inequality: \( f_{FB}(\theta_{FB}(\tilde{x}), \tilde{x}) < f(\theta_{FB}(\tilde{x}), \tilde{x}) = -\infty \), which is a contradiction. ■

A.4.7 Proof of Proposition 2

**Proof.** I now show that any corresponding buyers function \( \hat{H}(\cdot) \) under any achievable mechanism cannot have an interior strict local maximum. Suppose not and let \( \hat{x} \) denote an interior strict local maximum of \( \hat{H}(\cdot) \). By definition, there exists type \( x < \hat{x} \) such that \( \hat{H}(\hat{x}) > H(x) \). Hence, the trading rate for \( \hat{x} \) must be downward-distorted in order to prevent the types below \( \hat{x} \) to mimic: \( \theta(\hat{x}) < \theta_{FB}(\hat{x}) \).

Given that the function \( \hat{\theta}_{FB}(x, \hat{h}) \) is continuous in \( x, \exists x_1 > \hat{x} \) such that \( \theta(\hat{x}) < \theta_{FB}(x_1, h(\hat{x})) \) and \( x_1 \in B_{\varepsilon}(\hat{x}) \). Below, I show that it is profitable for such a seller to mimic \( \hat{x} \). The utility of seller-\( x_1 \) entering the market yields

\[
\hat{V}(x_1, \theta(\hat{x}), \hat{H}(\hat{x})) = \frac{x_1 + m(\theta(\hat{x}))(\hat{H}(\hat{x}) + \frac{k\theta(\hat{x})}{m(\theta(\hat{x}))})}{r + m(\theta(\hat{x}))} \geq \frac{x_1 + m(\theta(x_1))(\hat{H}(x_1) + \frac{k\theta(x_1)}{m(\theta(x_1))})}{r + m(\theta(x_1))} = V^*(x_1)
\]

The first inequality is given by (1) the monotonicity conditions \( \theta(x_1) \leq \theta(\hat{x}) \) and (2) \( \hat{V}(x_1, \theta, h) \) is decreasing in \( \theta \) when \( \theta < \theta_{FB}(x_1, h(\hat{x})) \). The second inequality is given by \( \hat{H}(\hat{x}) > \hat{H}(x_1) \). This thus shows that the IC condition is violated. Note that this proof relies on the fact that \( \hat{x} \) is a strict local maximum. Otherwise, if \( \hat{H}(\hat{x} - \varepsilon) \neq \hat{H}(\hat{x}) \), then \( \theta(\hat{x}) \) doesn’t need to be downward-distorted. ■

A.4.8 Proof of Proposition 3

**Proof.** One can easily see that the construction satisfies Lemma 1, which guarantees condition \((E1)\). The trading rate in the pooling market is upward-distorted (i.e., \( \theta_q(x^*) > \theta_{FB}(x^*) \)) so
that the monotonicity condition (M) is satisfied. Since the marginal type is indifferent between the pooling market and his first-best market, all seller types below the marginal type then strictly prefer the pooling market. Given that buyers’ valuation is strictly increasing after the marginal type \((h'(x) > 0 \text{ for } \forall x > x^*)\), a separating market for each \(x > x^*\) can then be solved as before, which guarantees that both \((E1)\) and the free-entry condition \(E2(a)\) are satisfied for each \(x > x^*\). Furthermore, the free-entry condition \(E2(a)\) holds for the pooling market by construction. Hence, the construction satisfies both \(E1\) and the free-entry condition \(E2(a)\).

What is left to show is that \(E2(b)\) is satisfied. In particular, notice that there is an upward jump in the equilibrium price at \(x^*\) from \(p_q(x^*)\) to \(p^{FB}(x^*)\). According to Lemma 2, a buyer will only attract the marginal type \(x^*\) if he posts a price \(p' \in (p^{FB}(x^*), p_q(x^*))\). Since the marginal type has already obtained his first-best utility, any price \(p' > p^{FB}(x^*)\) and its corresponding \(\theta(p')\) necessarily generates distortion, and a buyer thus suffers from this additional distortion. This can be seen clearly from Figure 5. Furthermore, a buyer will not benefit from lowering the price \((p' = p_q - \varepsilon)\) to attract \(x_L\), since he cannot do better given condition \((H2)\): \(V^*(p_q, \theta_q, x_L) \geq V^{FB}(x_L)\). Evidently, for the same reason as in the baseline model, any price \(p' = p(x_H) + \varepsilon\) is not profitable since it attracts \(x_H\) while resulting in more distortion. As a result, the scheme in Proposition 3 also satisfies \(E2(b)\). Therefore, it can be decentralized as a competitive equilibrium outcome.

### A.4.9 Existence for fire-sale equilibria

In the simple example, \((s, c)\) are independently distributed and \(c \in \{c_L, c_H\}\). Let \(\theta^{FB}(c)\) denote the optimal buyer-seller ratio when asset quality is observed. Note that, as shown in Equation (2), the optimal ratio is only a function of the gain from trade \(c\), but independent of \(s\).

**Claim 1** There exists \(\hat{\lambda} > 0\), distribution \(\hat{G}_s(s)\), and \(\Delta = c_H - c_L > 0\), such that a fire sale equilibrium \(x^*\) exists and

\[ \theta_q(x^*) > \theta^{FB}(c_H). \]

Furthermore, this result holds for any \(\lambda > \hat{\lambda}\) and for any distribution \(G_s(s)\) that first-order stochastically dominates \(\hat{G}_s(s)\).

**Proof.** To facilitate the analysis, define

\[
\phi(s; \lambda, G_s) \equiv \frac{\lambda}{\lambda + (1 - \lambda)G_s(s)}E_G(\tilde{s}) + (1 - \frac{\lambda}{\lambda + (1 - \lambda)G_s(s)})E[\tilde{s}|\tilde{s} \leq s],
\]

\[\text{In the pooling market, the lower types get subsidies from the higher types, and therefore can achieve higher utility than first-best.}\]
where \(E_G(\tilde{s}) \equiv \int_{s \in S} \tilde{s} dG_s(\tilde{s})\) denotes the expected value of \(s\). Observe that \(\phi(s; \lambda, G_s)\) decreases in \(s\) and increases with \(\lambda\). Moreover, one can see that, if \(G_s\) first-order stochastically dominates \(G_s\), then \(\phi(s; \lambda, G_s') > \phi(s; \lambda, G_s)\).

In the construction of a fire sale equilibrium, the expected quality in the pooling market is then given by \(q(x^*) = \phi(x^* + c_L; \lambda, G_s)\). Let \(\tilde{s}(\lambda, G_s)\) denote the unique solution such that \(\phi(\tilde{s}; \lambda, G_s) = \tilde{s}\) and thus \(\tilde{x} = \tilde{s} - c_L\). Given any \(c_L\), let \(\tilde{s}(c_L, \tilde{s})\) solves the following condition:

\[
Q(\tilde{s}) \equiv \frac{\tilde{s} - c_L + m(\theta^{FB}(c_H)) \tilde{s} - k\theta^{FB}(c_H)}{r + m(\theta^{FB}(c_H))} - \frac{\tilde{s} - c_L + m(\theta^{FB}(c_L)) \tilde{s} - k\theta^{FB}(c_L)}{r + m(\theta^{FB}(c_L))} = 0.
\]

Observe that \(\tilde{s}(c_H, \tilde{s})\) must be smaller than \(\tilde{s}\), given that \(\theta^{FB}(c_H) > \theta^{FB}(c_L)\). Moreover, (1) \(\tilde{s}\) decreases with \(c_H\) since \(\theta^{FB}(c_H)\) increases with \(c_H\); and (2) \(\tilde{s}\) increases with \(\tilde{s}\). Hence, by choosing a higher enough \(\tilde{s}\) and low enough \(c_L\) conditional on \(\tilde{x} = \tilde{s} - c_L > s_H - c_H = \underline{x}\), we can find a value of \(\tilde{s}(c_H, \tilde{s})\) such that \(\tilde{s}(c_H, \tilde{s}) - c_L > s_H - c_H\).

I now show that whenever this condition is satisfied, any \(x^* \in (s_H - c_H, \tilde{s}(c_H, \tilde{s}) - c_L]\) is a fire sale equilibrium and, moreover, \(\theta_q(x^*) > \theta^{FB}(c_H)\) in the pooling market.

First of all, since \(x^* \leq \tilde{s}(c_H, \tilde{s}) - c_L < \tilde{s} - c_L\), the expected quality in the pooling market is given by \(q(x^*) = \phi(x^* + c_L; \lambda, G_s) > x^* + c_L\). That is, condition (H1) holds. Second, for any \(x^* \in (s_H - c_H, \tilde{s}(c_H, \tilde{s}) - c_L]\),

\[
\tilde{V}(x^*, \theta^{FB}(c_H), q(x^*)) - V^{FB}(s; c_L) = \frac{x^* + m(\theta^{FB}(c_H)) \frac{q(x^*)}{r} - k\theta^{FB}(c_H)}{r + m(\theta^{FB}(c_H))} - \frac{x^* + m(\theta^{FB}(c_L)) \frac{\tau + c_L}{r} - k\theta^{FB}(c_L)}{r + m(\theta^{FB}(c_L))} > Q(x^* + c_L) > 0.
\]

The first inequality follows from the fact \(q(x^*) > \tilde{s}\), and the second one follows is driven by \(Q'(s) < 0\). Recall that \(\tilde{V}(x^*, \theta^{FB}(c_H), q(x^*))\) decreases in \(\theta\). Hence, the buyer-seller ratio that makes the marginal type indifferent must be larger than \(\theta^{FB}(c_H)\). This thus establishes that \(\theta_q(x^*) > \theta^{FB}(c_H)\).

Lastly, denote the asset quality of the marginal seller as \(s^* = x^* + c_L \geq s_L\). Below shows that condition condition (H2) also holds:

\[
V(p_q, \theta_q, s_L - c_H) - V^{FB}(s_L, c_H) \geq V(p_q, \theta_q, s^* - c_H) - V^{FB}(s^*, c_H) \geq V(p_q, \theta_q, s^* - c_L) - V^{FB}(s^*, c_L) = 0.
\]

To see this, define \(D(s, c) = V(p_q, \theta_q, s - c) - V^{FB}(s, c)\). The first inequality follows from \(\theta_q(x^*) > \theta^{FB}(c_H)\) and thus \(\frac{\partial D(s, c)}{\partial c} = \frac{1}{r + m(\theta_q)} - \frac{1 + m(\theta^{FB}(c_H))}{r + m(\theta^{FB}(c_H))} \frac{\tau}{\tau + c_L} < 0\). The second inequality is given by \(\frac{\partial D(s, c)}{\partial c} = -\frac{1}{r + m(\theta_q)} + \frac{1}{r + m(\theta^{FB}(c))} > 0\). And the last equality is given by the fact that the marginal seller must be indifferent.
The above thus establishes that \( x^* \) is a fire-sale equilibrium. Let \( \lambda \) and \( \hat{G}_s(s) \) denote its underlying distribution that gives \( \bar{s} \). Fixing \( x^* \), any \( \lambda > \bar{\lambda} \) and any distribution \( G_s(s) \) that first-order stochastically dominates \( \hat{G}_s(s) \) leads to a higher \( q(x^*) \), which increases buyer-seller ratio in the pooling market. This is because \( \theta_q(x^*) = \theta_U(V^{FB}(x^*); x^*, q(x^*)) \) and \( \theta_U(V; x, \tilde{h}) \) increases with \( \tilde{h} \). Hence, under \( \lambda \) and \( G_s(s) \), \( x^* \) satisfies all conditions for a fire sale equilibrium and \( \theta_q(x^*) > \theta^{FB}(c_H) \).

\[\text{A.4.10 Proof of Proposition 4}\]

**Proof.** From Equation (6), \( \theta_q(x^*) \) is the upward distorted buyer-seller ratio when buyers’ valuation is \( q(x^*) \). That is, \( \theta_q(x^*) = \theta_U(V^{FB}(x^*); x^*, q(x^*)) \). Recall that \( \theta_U(V; x, \tilde{h}) \) increases with \( \tilde{h} \) and \( q(x^*) \) also increases with \( \lambda \), \( \theta_q(x^*) \) must increase with \( \lambda \) and, consequently, \( p_q(x^*) \) must decrease with \( \lambda \). As a result, a higher \( \lambda \) suggests more sellers enter the pooling price traded at a higher rate with a lower price, this thus leads to a higher aggregate volume and a lower aggregate price.

Without adverse selection (either \( \sigma = 0 \), or \( s \) is perfectly observable), all assets are then traded at the buyer-seller ratio \( \theta^{FB}(c) \), which only depends on the gains from trade (i.e., the distress position). The difference in the aggregate volume compared to the benchmark is given by:

\[
\vartheta - \vartheta^0 = \lambda \int \left\{ m(\theta_q(x^*)) - m(\theta^{FB}(c_H)) \right\} dG_s(s)
\]

\[
+ (1 - \lambda) \left\{ \int_{s \leq x^* + c_L} m(\theta_q(x^*))dG_s(s) + \int_{s > x^* + c_L} m(\theta^*(s - c_L))dG_s(s) \right\} - \int m(\theta^{FB}(c_L))dG_s(s) \right\}
\]

Hence, when upward distorted buyer-seller ratio is large enough such that \( \theta_q(x^*) > \theta^{FB}(c_H) \), as established in Claim 1, a large enough \( \lambda \) leads to \( \vartheta > \vartheta^0 \). On the other hand, under 1-D asymmetric information model, since \( \theta^*(s; c) < \theta^{FB}(c) \) for \( (s, c) \), it must be the case that \( \vartheta < \vartheta^0 \).

\[\text{A.4.11 Proof of Proposition 5}\]

**Proof.** For all distressed sellers, \( \tilde{p}(s, c_H) = p_q(x^*) \). Sellers with relatively high quality assets \( s > x^* + c_L \), go to the pooling market when if having high holding cost \( \tilde{p}(s, c_H) = p_q(x^*) \) and enter a separate market with low holding cost, where price is given by \( \tilde{p}(s, c_L) = \frac{s}{r} - \frac{k\theta(s, c_L)}{m(\theta(s, c_L))} \). Since \( \tilde{p}(s, c_L) \) is independent of \( \lambda \) but \( p_q(x^*) \) decreases with \( \lambda \), this thus explains the distress discount increases with \( \lambda \). On the other hand, under 1-D asymmetric information model, they always enter a separating market and thus \( \tilde{p}(s, c_L) \) and \( \tilde{p}(s, c_H) \) are independent of the aggregate distribution.
Under perfect information, the price discount simply reflects the difference in the holding cost, which is given by

$$\Delta \hat{p}_{FB}(s) = \frac{k\theta_{FB}(c_H)}{m(\theta_{FB}(c_H))} - \frac{k\theta_{FB}(c_L)}{m(\theta_{FB}(c_L))}$$

The difference of the distress discount in a fire sale equilibrium vs. the perfect information case yields:

$$\Delta \hat{p}(s) - \Delta \hat{p}_{FB}(s) = \left(\frac{s}{r} - \frac{E[s|x < x^*]}{r}\right) + \left(\frac{k\theta_q(x^*)}{m(\theta_q(x^*))} - \frac{k\theta_{FB}(c_H)}{m(\theta_{FB}(c_H))}\right) + \left(\frac{k\theta_{FB}(c_L)}{m(\theta_{FB}(c_L))} - \frac{k\theta(s, c_L)}{m(\theta(s, c_L))}\right).$$

The first term is pooling discount, which is positive for sellers with relatively high quality assets. The second term is positive when buyer-seller ratio is upward distorted so that $\theta_q(x^*) > \theta_{FB}(c_H)$. Recall that $\theta_{FB}(s; c)$ is only a function of $c$, so this term is independent of $s$. The last term is always positive due to downward distorted buyer-seller ratio in the separating market, and is increasing in $s$. Hence, there exist $\bar{s}$ such that $\Delta \hat{p}(s) - \Delta \hat{p}_{FB}(s) > 0$ for $s > \bar{s}$. ■

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