Relationships and Growth: On the Dynamic Interplay between Relational Contracts and Competitive Markets in Economic Development

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First version received 25 November 2011; Final version accepted 4 July 2015

Abstract

This paper presents a dynamic general equilibrium model to investigate how different contracting modes based on formal and relational enforcements emerge endogenously and are linked dynamically with the process of economic development. Formal contracts are enforced by third-party institutions (courts), whereas relational contracts are self-enforcing agreements without third-party involvement. The novel feature of our model is that it demonstrates the co-evolution of these different enforcement modes and market equilibrium conditions, all of which are jointly determined. We then characterise the equilibrium paths of such dynamic processes and show the time structure of relational contracting in the process of development. In particular, we show that relational contracting fosters the emergence of a market-based economy in the growth phase of development; however, its role declines as the economy enters a mature phase.

Keywords: dynamic general equilibrium, economic development, arm’s length contract, relational contract

JEL Classification Numbers: D86, E10, O11

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*I am very grateful to the editor, Philipp Kircher, and three anonymous referees of the journal for their useful comments and suggestions, which significantly improved my paper. Of course, all remaining errors are my own.

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1 Introduction

Informal contract arrangements, which we call relational contracting in this paper, are common during the developing stages of economies. These arrangements are not based on formally written contracts but rather on long-term relationships, implicit agreements, and reputation mechanisms via personal ties and connections, as typically observed in tribal and ancient societies as well as in emerging and transition economies.\footnote{See Greif (2006) and Milgrom, North and Weingast (1990) for how the merchant trade system functioned as a reputation device in medieval times. For the roles of relational contracting in emerging and transition economies, see McMillan and Woodruff (1999a, 1999b), who investigate trade credits in Vietnam, and Johnson, McMillan and Woodruff (2002), who focus on informal contracting in Russia.} Among other informal contract arrangements, one well-documented example is relationship (insider) lending based on personal relationships between borrowers and lenders, typically banks, which is known to be an important financial arrangement during the developmental stages when market-based economies are immature (see Lamoreaux (1994)).

This paper develops a dynamic general equilibrium model with long-term relationships to understand the interactions of relational contracting with the process of economic development during which the anonymous credit market expands so that falling market interest rates foster capital investment and boost aggregate output. Long-term relationships have mostly been analysed in the partial equilibrium framework in the literature of repeated games (see, for example, Mailath and Samuelson (2006)). However, despite much historical evidence showing the important roles of long-term relationships in the process of economic development (which will be discussed in more detail below), few theoretical studies have considered the macroeconomic implications of long-term relationships as addressed in this paper.

Lamoreaux (1994) found evidence that during the early 19th century New England banks lent a large portion of their funds to their boards of directors and those who had close personal ties with these banks, contributing to New England’s economic growth in that period.\footnote{Lamoreaux (1994) also reports that such insider lending was a phenomenon observed not only in New England but also in other U.S. states during the early 19th century before the expansion of the U.S. markets.} A related historical fact is that major German banks such as Commerzbank, Dresdner and Deutsche grew rapidly in the 19th century by developing long-term relationships with industry enterprises, offering them low interest rates and being represented as directors on the boards of these firms. Such close and lasting relationships between large banks and firms contributed to the rapid expansion of the German economy between the late 19th century and the First World War (see Allen (2001)).\footnote{Maurer and Haber (2007) provide related empirical evidence about Mexican banks during similar periods.}

These facts point towards a positive view that relational contracting is not a substitute but a complement to the market economy wherein the former fosters the latter, raising the aggregate outputs of economies during the stages in which market-based
transactions do not function well.4

Another view on relational contracting holds that its roles decrease as the economy grows and reaches more developed stages. For example, the New England banks that lent to closely related persons (for example, directors of these banks) in the early 19th century eventually began to lend to ‘outside’ borrowers, whom they did not personally know well, as the economy shifted from capital-poor to capital-rich stages, thus expanding the anonymous credit market in the late 19th century (Lamoreaux (1994)). A related argument is that relationship-based financial systems were dominant in Asian countries such as Korea and Japan after the Second World War, but these countries have been recently changing to market-based financial systems (see Rajan and Zingales (2000)). Demirgüç-Kunt and Levine (2004) provide evidence that the ratio of bank finance to equity finance is negatively associated with per capita GDP levels across countries, suggesting that bank finance, which is often characterised as long-term lending relationships between particular banks and firms, becomes less important and is replaced by market-based finance in developed countries.

Our model is able to account for the above dynamic patterns concerning the rise and fall of relational contracting over the course of economic development. A novel feature of our model is how it embeds relational contracts supported by long-term relationships into a dynamic general equilibrium model in a tractable manner.

In our model economy, producers who finance their capital investments have incentive to default after they borrow funds from lenders. We consider two means that prevent producers from defaulting. First, a formal or arm’s length contract uses public and costly enforcement in the competitive credit market. When lenders lend their funds to the credit market, they must incur some enforcement costs to write formally enforceable contracts that prevent borrowers from defaulting. Second, a relational contract uses implicit and informal contracting arrangements to protect lenders from default. The producers and lenders form long-term relationships over successive generations of their dynasties and engage in relational contracts for financing capital investment without using formally written contracts. The relationship pairs of producers and lenders can avoid the strategic default problem via self-enforcing agreements that ensure that the producers voluntarily honour the agreed upon repayments. A relational contract becomes self-enforcing if the one-shot gain from deviating from the agreement is not larger than the loss caused by the future punishment. The new insight of our paper is that we investigate how such self-enforcing conditions are endogenously related to the development process in a dynamic general equilibrium framework.

We then demonstrate that equilibrium paths involve dynamic transformations from a relationship-based system relying on relational contracts to a market-based system relying on arm’s length contracts. In any equilibrium path, a unique switching period exists before which the self-enforcing constraint becomes less stringent so that producers who engage in relational contracts invest and produce more than producers who

4For more on the positive roles of informal institutions, see the papers contained in Aoki and Hayami (2000).
engage in arm’s length contracts, but after which it becomes more stringent so that the latter invests and produces more than the former. Thus, in the early stages of economic development characterised by high market interest rates, relational contracting contributes more to the rise of aggregate outputs than arm’s length contracts, and contributes less than the latter during the mature stages of development characterised by low interest rates. Consequently, the role of relational contracts diminishes as the economy enters the mature phase of development.

In addition, we demonstrate that relational contracting helps to promote the output expansion of arm’s length contracts. In fact, we find that an economy that hosts a larger number of relationships engaging in relational contracts can foster a greater expansion of market-based transactions, increasing the output of arm’s length producers and promoting a larger GDP in the long run, than an economy that involves fewer such relationships. Thus relational contracting plays a positive role in supporting the rise of market-based economies, leading to long-run development.

The key element behind these results is the dynamic general equilibrium interaction between the self-enforcing condition of relational contracting and determination of market interest rates. When interest rates decline more rapidly over time, producers in relationships will face lower costs to raise capital; they expect to gain higher future profits, which makes it less profitable to deviate from honouring the current period’s agreement. The self-enforcing condition then becomes less severe, resulting in these producers making greater capital investments during the growth phase in which interest rates are high but decline rapidly. However, as the economy enters the mature phase in which market interest rates are low enough and do not decrease much further, the self-enforcing constraint becomes more stringent so that producers in relationships eventually switch to investing less in capital than arm’s length producers. Thus relational contracting plays a smaller role as the economy develops. However, since relational contracting can contribute to raising the aggregate output of an economy as a whole without incurring enforcement costs, more resources become available to finance capital investment by arm’s length producers. Thus, the overall impact of relational contracting on long-run development becomes positive and promotes a larger GDP in the long run than the benchmark case wherein all financing is market-based.

This dynamic switching pattern from a relationship-based system relying on relational contracts to a market-based system relying on arm’s length contracts is linked to the argument by Polanyi (1947) that Western societies experienced a ‘great transformation’ from non-market systems to market-based systems when their economies expanded rapidly in the 19th century. In addition, our result also confirms the historical fact that relationship-based systems decline as the economy develops (see Lamoreaux (1994), Rajan and Zingales (2000) and Demirgüç-Kunt and Levine (2004) as already mentioned). Moreover, our result that relational contracting promotes the output expansion of producers who trade in the anonymous credit market and fosters long-term development is consistent with the aforementioned historical evidence that relational contracting supports the rise of a market-based economy (Aoki and Hayami (2000) and Lamoreaux (1994)).
Related literature. Three strands of the literature are related to our paper. First, many papers address relational contracting (see Malcomson (2012) for a recent survey); however, most of these focus on the partial equilibrium framework and few studies examine its macroeconomic implications via dynamic general equilibrium models. Francois and Roberts (2003) examine how long-term employment relationships affect long-run economic growth in an R&D-based endogenous growth model. However, their analysis is restricted to the steady state so that transitional dynamics, which is our main interest, is not considered. Second, some papers have compared enforcement of informal contracting, such as reputation, with formal and legal enforcement (see Kranton (1996), Dhilon and Rigolini (2006), and Francois (2011)). Related to this, some papers have attempted to study the interactions of several markets that affect the incentive constraints for borrowing (see Jeske (2006) for a general model of a related issue, and Bose (1998), Floro and Ray (1997), and Hoff and Stiglitz (1998) for the interactions between formal and informal (rural) financial sectors). Third, there is the literature on models of competitive economies with endogenous debt constraints that prevent borrowers from defaulting (see, for example, Jeske (2006), Kehoe and Levine (1993), and Hellwig and Lorenzoni (2009) for international borrowing and lending with default risk). Our paper shares some features with these papers in that financial arrangements regarding the default problem are endogenously investigated in general equilibrium models. However, our paper departs from the existing literature as we investigate the time structure of relational contracting, such as in what phases of development it becomes less stringent. We also investigate dynamic issues regarding whether or not relational contracting promotes the rise of market-based transactions in the endogenous course of development. Specifically, we address the hitherto neglected issues of how and when relationship-based economic systems change to market-based systems during the endogenous process of economic development.

The remaining sections are organised as follows. Section 2 establishes the basic model and Section 3 investigates the optimal relational contracts by considering interest rates to be exogenously given. Section 4 characterises the set of equilibrium paths and demonstrates that relational contracting contributes more to economic development in the early stages; however, its role becomes more limited as the economy enters mature stages of development. Section 5 discusses an extension of the model that allows endogenous formation of relationships. Section 6 presents our conclusions. All proofs are relegated to the Appendix (some lengthy proofs and extensions are given in

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5See also Baker, Gibbons and Murphy (2002), Itoh and Morita (2015), Levin (2003), MacLeod and Malcomson (1998), and Ramey and Watson (2003) for related models of relational contracting.

6Many studies in this literature consider punishments to deter countries' default by assuming that those who default are excluded from future trades in international markets forever. The basic model in our paper has a similar feature, but we extend the model to allow individuals who default to re-start their relationships in future periods (see Section 5).

7Fafchamps (2003) addresses the dynamic issue of how markets spontaneously emerge.
the Online Appendix).

2 Model

2.1 Economic Environment and Preference

We consider an overlapping generations (OLG) economy with discrete time \( t = 0, 1, 2, \ldots \).

The economy comprises a single good, taken as numéraire with its price normalised to 1, which is used both for consumption and investment. Every period, a new continuum of individuals is born; each individual lives through two periods: young and old. In each generation, one young individual is born from each old individual, and we use \( i \) to denote both the individual \( i \) and the dynasty to which individual \( i \) belongs. The newly-born individuals consist of one unit mass of young lenders and one unit mass of young producers (who become borrowers). For simplification and as is often assumed in OLG models, we assume that each individual (lender or producer) does not consume when young but consumes his or her entire income only when old. In what follows we use the masculine pronoun for producers and feminine pronoun for lenders.

Each individual (lender or producer) has an altruistic preference of the consumption levels of his or her descendants in his or her dynasty.\(^8\) We introduce altruistic preference to ensure that each individual has a reputation concern. If an individual reneges on an implicitly agreed upon contract, future generations of his or her dynasty may be punished by not having the best trading opportunities and hence may consume less in future periods, which is a utility loss for the deviating individual in the current period. Without an altruistic concern regarding consumption by future generations, no individuals will honour implicit promises as will be apparent in the following analysis.

More specifically, consider an individual in a dynasty who was born in period \( t = 1 \) and whose consumption when old (in period \( t = 0 \)) is denoted by \( C_0 \). Then, we assume that his or her utility \( u_{t-1} \) depends not only on his or her own consumption level \( C_t \) when old in period \( t \), but also on the consumption levels of future generations of the same dynasty, \( C_s \) for \( s \geq t + 1 \), as follows:

\[
u_{t-1} = C_t + \sum_{s=t+1}^{\infty} \delta^{s-t} C_s, \tag{1}\]

where \( \delta \in (0, 1) \) represents the parameter value regarding the degree to which each individual is altruistic toward the consumption of future generations in the same dynasty.

\(^8\)One might wonder why we use a two-period-lived OLG model framework with altruistic preference instead of using an infinitely-lived agent model. The main reason for this is to simplify each individual’s saving decision, which makes the credit market equilibrium condition easier to handle. In an infinitely-lived agent model, we need to keep track of the Euler equation of saving decisions as an additional dynamic equation that would make the dynamic analysis of self-enforcing agreement more complicated. Our OLG model of a two-period-lived agent with altruistic preference avoids such complications while allowing us to incorporate self-enforcing agreements into a dynamic general equilibrium in a simpler way.
We call $\delta$ the discount factor when no confusion arises, because it plays a similar role to the discount factor in the literature on repeated games.

In the basic setting, we assume that no bequest transfers are made across generations in each dynasty. Thus, each individual consumes all of his/her old income for himself/herself such that $C_t$ is equal to the entire income level an individual earns when old. Because the utility function $u_{t-1}$ is linear with respect to the consumption of future generations, this will actually be the case when the discount factor is not so large because then leaving a bequest to child yields a smaller future utility than the current loss of consumption. We discuss an extension of the basic model to allow bequests in the Online Appendix.

2.2 Production

Each old producer in dynasty $i$ can produce the output $Y_{i,t}$ in period $t$,

$$Y_{i,t} = k_{i,t}^\alpha,$$

by investing in capital $k_{i,t} \geq 0$ when young in period $t-1$ (one period prior to production), where $\alpha \in (0, 1)$. We denote by $Y_t \equiv \int_0^t Y_{i,t} di$ the aggregate output of the economy produced by old producers in period $t$. We assume that capital fully depreciates after use within one period as commonly assumed in OLG models. Each young producer is born without any endowment so that he needs to finance his capital investment $k_{i,t}$.

Each young lender is endowed with an amount $w_t$ of the good when she is born in period $t$. We then suppose that the lender’s endowment $w_t$ depends positively on the economy’s aggregate output $Y_t$ produced by old producers in period $t$. For example, each young lender is endowed with a skill to produce $w_t$ by herself in home production and such skills depend on aggregate knowledge of the economy that might be accumulated through learning-by-doing over the course of production (Arrow (1962)). Specifically, the young lender’s endowment $w_t$ is proportional to the social knowledge embodied in the economy’s aggregate output $Y_t$ as follows:

$$w_t = Y_t.$$

Thus, as the economy develops a larger aggregate output $Y_t$, more knowledge is accumulated by learning-by-doing and young lenders’ endowments correspondingly increase.

The above specification of the lender’s endowment is tractable enough to link the young lender’s income $w_t$ to the aggregate output $Y_t$. Although this seems to lack a microeconomic foundation, we can provide a more sophisticated model to endogenise the lender’s endowments by introducing the labour market.\footnote{Suppose that each young lender is endowed with one unit of labour that is inelastically supplied to the competitive labour market, and she earns the market wage. The final good of the economy is produced by many intermediate goods and labour. Each intermediate good is produced by a monopolistic producer. We can endogenise the young lender’s income as the equilibrium market wage, which is proportional to the aggregate outputs of the final good. See the Online Appendix for more details.}
### 2.3 Strategic Default and Contracting Modes

Each young producer wants to finance his capital investment $k_{i,t}$ from young lenders, but there is the strategic default problem that any producer can run away and fail to make repayments after he borrows from lenders. Anticipating such a strategic default, young lenders thus never lend to young producers.

We consider two alternative means of alleviating this problem. One is the arm’s length contract, which is enforced publicly between producers (borrowers) and lenders in the competitive credit market but is costly. We call the producers and lenders who engage in arm’s length contracts A-producers and A-lenders, respectively, in what follows. The other is the relational contract, which is an implicit informal agreement between producers (borrowers) and lenders who trade via long term relationships. We call the producers and lenders who engage in relational contracts R-producers and R-lenders, respectively, in what follows.

We now explain these contracting modes in more detail.

**Arm’s Length Contract.** In the economy, the competitive credit market exists where borrowing and lending occur between A-producers and A-lenders at a given interest rate $r_t$ in the spot fashion. The credit market works as a standard competitive market in the sense that A-producers and A-lenders take the market interest rate $r_t \geq 0$ as given in any period $t$. However, each young A-lender must spend $1 - \lambda$ units of the good ($0 < \lambda < 1$) to lend one unit to an A-producer and prevent him from committing strategic default. Here, $1 - \lambda$ is called the enforcement cost per unit lending in the credit market, which includes the costs of making the information disclosure credible, collecting hard evidence, hiring professionals such as lawyers and accountants who help write formal contracts, and using outside institutions such as courts. Then, when a young A-lender lends her endowment $w_t$ to the credit market in period $t$, she needs to spend $(1 - \lambda)w_t$ for contract enforcement and will thus earn the interest income $r_t w_t$ in the next period $t + 1$ by lending the remaining amount $\lambda w_t$.

We let $x_t \geq 0$ denote the capital investment of a young A-producer (that is, $k_{i,t} = x_t$ for A-producer $i$). An old A-producer earns the profit $x_t^\alpha - r_t x_t$ in period $t$, which is equal to his consumption, where recall that we normalise the price of the good to 1 and that each old producer spends all this profit for his own consumption without leaving a bequest. We denote by $\pi_t \equiv \max_{x_t \geq 0}\{x_t^\alpha - r_t x_t\}$ the maximum profit attained by an old A-producer in period $t$ where the capital investment $x_t$ satisfies the following optimality condition:

$$\alpha x_t^{\alpha - 1} = r_t. \quad (4)$$

Thus the maximum profit of an A-producer in period $t$ is given by

$$\pi_t \equiv (1 - \alpha)x_t^\alpha, \quad (5)$$

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10 We can also allow A-producers to incur costs that prevent themselves from committing strategic default. For example, each A-producer makes some investment in enforcement technology so that he commits himself to not default. Because such a possibility cannot alter the main results substantially, we will not pursue this in what follows.
where \( x_t = x(r_t) = (r_t/\alpha)^{1/(\alpha-1)} \). In what follows and where no confusion arises, we will use the shorthand notation \( x_t \equiv x(r_t) \) to denote the optimal investment of each A-producer that achieves the above maximum profit \( \pi_t \).

**Relational Contract.** A relational contract is an informal bilateral agreement between an R-producer and an R-lender that governs the financial arrangements between them. More specifically, a relational contract means that R-producers in a dynasty finance their capital investments directly from R-lenders in a dynasty without explicit contractual agreements, in contrast to arm’s length contracts.

In the basic model, we make the simplifying assumption that \( N \) relationship pairs are formed between \( N \) old R-producers and \( N \) old R-lenders in the initial period \( t = 0 \), which are carried over to successive generations of their dynasties. However, no opportunities arise for new relationships to form in any subsequent period \( t \geq 1 \). Thus the number of relationship pairs who engage in relational contracts is fixed at \( N \) in any period. This assumption can be interpreted as our model economy containing two sectors, namely, the formal sector in which lending and borrowing are conducted in the competitive market, and the informal (or rural) sector in which only particular members can form relationships via personal connections and can trade without access to the outside credit market (see Hoff and Stiglitz (1998) and Bose (1998) for related models). We will maintain this assumption to eschew introducing complications regarding endogenous relationship formation into the basic model. Section 5 extends the model to endogenise the number of agents who engage in relational contracts over time.

We also suppose that any R-producer and any R-lender can always exercise the option to quit their relationship (called the *quitting option*) in the beginning of any period. In that case they engage in arm’s length contracts in the anonymous credit market. Their descendants will then also have no choice but to engage in arm’s length contracts in future periods because no opportunities exist to start new relationships in the future. Only when neither exercises the quitting option do relationship pairs implement an agreed upon relational contract.

In what follows, we let \( z_{t} \geq 0 \) denote the capital investment of a young R-producer (\( k_{i,t} = z_{t} \) for R-producer \( i \)). Each young R-producer born in period \( t - 1 \) can directly finance his capital investment \( z_{t} \) from his partner, that is, a young R-lender, in exchange for making repayment \( R_{t} \) to the latter when old in period \( t \). Because the repayment \( R_{t} \) is not secured owing to a possible strategic default by the R-producer, such an agreement, \( \{ z_{t}, R_{t} \} \), must be implicit and self-enforceable. We assume here that R-producers and R-lenders in any relationship pair can observe the history of what capital investments and repayments their ascendants have made in all past periods.

### 2.4 Initial condition \( (t = 0) \)

In the initial period \( (t = 0) \), old generations of producers and lenders exist with one unit mass each. The initial number of R-producers (R-lenders, respectively) in the economy, denoted by \( N \), is given exogenously. Thus the remaining \( 1 - N \) agents are
old A-producers (old A-lenders, respectively) in period 0. Each initial old R-producer (old A-producer, respectively) owns an initial capital stock, $z_0$ ($x_0$, respectively), which is assumed to be given historically. Because the old producers in the initial period do not need to raise funds for capital investment, they can produce $z_0$ and $x_0$ with no production costs. Thus the initial young lenders can each earn the endowment $w_0 = Y_0 = Nz_0 + (1 - N)x_0$. Because only the initial income $w_0$, rather than $z_0$ and $x_0$, matters for determining the equilibrium path, in what follows we normalise that $z_0 = x_0$ for notational simplicity.

3 Relational Contracts: Self-Enforcing Conditions

This section derives the set of constraints that determine the optimal relational contract, by assuming that the market interest rate $r_t$ is fixed exogenously. In this sense, this section presents a partial equilibrium analysis because we do not consider equilibrium interactions between the interest rates and the optimal relational contract. After deriving the optimal relational contract given $r_t$, we embed it into the general equilibrium framework in the next section to endogenise the market interest rate $r_t$.

We formally define a relational contract as a sequence of capital investments and repayments $\{z_t, R_t\}_{t=0}^{\infty}$ agreed by a relationship pair of an R-producer and an R-lender in the initial generation. The descendants in their dynasties implicitly follow these agreements in subsequent periods. A relational contract indicates that a young R-producer born in period $t$ borrows $z_t$ from the corresponding young R-lender born in period $t$ for investing in capital $z_t$, and the former then makes repayment $R_t$ to the latter when old in period $t$.

Three constraints need to be satisfied for a relational contract agreement $\{z_t, R_t\}_{t=0}^{\infty}$ to be self-enforceable: the incentive compatibility (IC) condition, the individual rationality condition for R-producers (IRP) and the individual rationality condition for R-lenders (IRL), which we explain in more detail below.

Incentive Compatibility (IC). For the moment, we suppose that each young R-lender born in period $t - 1$ has sufficient endowment $w_{t-1}$ to cover the capital investment demand $z_t$ of the R-producer, that is, $w_{t-1} \geq z_t$. This will be shown to be true in any equilibrium (see Lemma 2 below).

A young R-producer born in period $t - 1$ obtains the following value or utility $V_t$ by honouring the agreed upon repayment $R_t$ to the R-lender when old in period $t$ and by expecting that successive R-producers and R-lenders in future generations of their dynasties will also honour the relational contract agreements $\{z_s, R_s\}_{s \geq t+1}$:

$$V_t = z_t^\alpha - R_t + \delta V_{t+1}. \quad (6)$$

Here, an R-producer earns profit $z_t^\alpha - R_t$ and consumes this profit when old, while the next generation obtains the future value $V_{t+1}$. A current R-producer evaluates the value of his child $V_{t+1}$ by the discount factor $\delta \in (0, 1)$. 
When a young R-producer quits the relationship, he then borrows from the anonymous credit market via an arm’s length contract and his descendants must also do so in future periods because of the assumption that no opportunities exist to start new relationships in future periods. Thus they must act as A-producers forever. Because any A-producer can earn the profit \( \pi_t \) (see (5)) in period \( t \) via an arm’s length contract, the R-producer who quits a relationship obtains the following utility or value:

\[
\Pi_t = \pi_t + \delta \Pi_{t+1},
\]

which becomes the reservation value for each R-producer.

Because any old R-producer can renege on the repayment \( R_t \) and exercise the quitting option, he does not renege on the repayment \( R_t \) specified in the relational contract in period \( t \) only if the IC constraint is satisfied as follows:

\[
V_t = z_t^\alpha - R_t + \delta V_{t+1} \geq z_t^\alpha + \delta \Pi_{t+1}.
\]

(\text{IC}_t)

Here the right hand side denotes the payoff to the old R-producer who reneges on the repayment \( R_t \) and exercises the quitting option, which increases his profit to \( z_t^\alpha \) and results in the dissolution of the relationship so that all his descendants will engage in arm’s length contracts in all future periods, yielding the value \( \Pi_{t+1} \). The above constraint can be modified to

\[
\delta\{V_{t+1} - \Pi_{t+1}\} \geq R_t.
\]

(\text{IC}_t)

This has the reasonable interpretation that each old R-producer honours the repayment \( R_t \) only when the cost of doing so is less than the future losses from dissolving the relationship, as captured by the left hand side \( \delta\{V_{t+1} - \Pi_{t+1}\} \).

**Individual Rationality (IR).** Each young R-lender born in period \( t - 1 \) is endowed with \( w_{t-1} \), from which she lends \( z_t \) directly to the R-producer trading with her and lends the remaining amount \( w_{t-1} - z_t \geq 0 \) to the anonymous credit market (recall that we are assuming \( w_{t-1} \geq z_t \)). Because one unit of lending in the anonymous credit market requires //\( 1 - \lambda \)\ // units of enforcement costs, the R-lender obtains the interest income \( r_t \lambda (w_{t-1} - z_t) \) from lending \( w_{t-1} - z_t \) to the anonymous credit market.

By continuing the relationship rather than exercising the quitting option in period \( t - 1 \), each young R-lender born in period \( t - 1 \) obtains the following value:

\[
W_t = R_t + \lambda r_t (w_{t-1} - z_t) + \delta W_{t+1}.
\]

(8)

Here, the R-lender obtains the repayment \( R_t \) from the R-producer trading with her in addition to the interest income \( \lambda r_t (w_{t-1} - z_t) \) from lending \( w_{t-1} - z_t \geq 0 \) to the anonymous credit market. Her descendant will obtain the future value \( W_{t+1} \). On the other hand, if she quits and lends all her endowment \( w_{t-1} \) to the anonymous credit market at the market interest rate \( r_t \), she obtains the following value:

\[
U_t = \lambda r_t w_{t-1} + \delta U_{t+1},
\]

(9)
because lending \( w_{t-1} \) via an arm’s length contract requires enforcement costs of \((1 - \lambda)w_{t-1}\). Thus \( U_t \) is the reservation value for each young R-lender born in period \( t - 1 \).

Each young R-lender decides not to exercise the quitting option only if

\[
W_t \geq U_t, \tag{IRL_t}
\]

which can be rearranged as

\[
R_t - \lambda r_t z_t + \delta(W_{t+1} - U_{t+1}) \geq 0. \tag{IRL_t}
\]

We call this the individual rationality constraint for R-lenders.\(^{11}\)

In addition, each young R-producer prefers the relationship to continue over exercising the quitting option only if

\[
V_t \geq \Pi_t, \tag{IRP_t}
\]

which we call the individual rationality constraint for R-producers.

We say that a relational contract \( \{z_t, R_t\}_{t=0}^{\infty} \) is self-enforceable for a relationship pair if it satisfies IC\(_t\), IRL\(_{t-1}\) and IRP\(_{t-1}\) for all \( t \geq 1 \), so that all the generations in their dynasties optimally follow these agreements given the reservation values \( \{U_t, \Pi_t\}_{t=1}^{\infty} \) and interest rates \( \{r_t\}_{t=1}^{\infty} \). Conversely, if a self-enforceable relational contract \( \{z_t, R_t\}_{t=0}^{\infty} \) exists, then a relationship pair can actually implement such an agreement by using a trigger strategy specifying that the R-producer and the R-lender born in period \( t - 1 \) implement the agreed upon relational contract \( \{z_t, R_t\} \) as long as all the ascendants in their dynasties followed the agreements in all past periods, while otherwise they exercise the quitting option simultaneously.\(^{12}\) As is well known in the literature on repeated games, the game has multiple subgame-perfect equilibria. To avoid the multiplicity of equilibria, we focus on the optimal relational contract where each relationship pair in the initial generation chooses their relational contract to maximise the value for the initial R-producer \( V_0 \) subject to IC\(_1\), IRP\(_{t-1}\) and IRL\(_{t-1}\) for all \( t \geq 1 \).\(^{13}\)

Formally, the optimal relational contract \( \{z_t, R_t\}_{t=0}^{\infty} \) solves the following problem:

\(^{11}\)As long as IRL\(_t\) is satisfied, any old R-lender also has no incentive to renege on a repayment \( R_t \). This follows from the fact that, because \( U_t \) is not smaller than \( \lambda r_t (w_{t-1} - z_t) + \delta U_{t+1} \), then IRL\(_t\) implies that \( R_t + \lambda r_t (w_{t-1} - z_t) + \delta W_{t+1} \geq \lambda r_t (w_{t-1} - z_t) + \delta U_{t+1} \) which yields \( R_t + \delta W_{t+1} \geq \delta U_{t+1} \). This means that every old R-lender wants to make the repayment \( R_t \) and continue the relationship for the next generation rather than quitting the relationship, provided she has already lent \( z_t \) to the R-producer from her income \( w_{t-1} \) when young.

\(^{12}\)By our definitions of IC\(_t\), IRL\(_{t-1}\) and IRP\(_{t-1}\), there are no profitable deviations from this strategy. Also, it always becomes a continuation equilibrium for both parties of each relationship to exercise the quitting option simultaneously. Thus the strategy specified above becomes a subgame-perfect equilibrium.

\(^{13}\)See Acemoglu et al. (2008) for a related model. In addition, we need to impose the condition \( p_t z_t \geq R_t \geq -\lambda r_t (w_{t-1} - z_t) \) for all \( t \geq 1 \) and \( \tau_0 \geq R_0 \geq 0 \). This ensures that the consumptions of each old R-producer and old R-lender are non-negative in each period, and is called the condition of non-negative consumption (NNC\(_t\)). However, we can solve the optimal relational contracts without NNC\(_t\), and can then check that the optimal relational contract actually satisfies NNC\(_t\). See the proof of Lemma 1 below.
Problem (RC):

$$\max_{\{z_t, R_t\}_{t=0}^\infty} \quad V_0 = \sum_{t=0}^\infty \delta^t(z_t^\alpha - R_t)$$

subject to IC, IRL and IRP for all $t \geq 1$, given the reservation values $\{\Pi_t, U_t\}_{t=1}^\infty$ and interest rates $\{r_t\}_{t=1}^\infty$.

Then we can obtain the following result.

**Lemma 1.** The optimal relational contract $\{z_t, R_t\}_{t=0}^\infty$ has the following features:

(i) IRL is always binding, so that $R_t = \lambda r_t z_t$ holds in any period $t \geq 1$.

(ii) Let denote $\hat{\lambda} = \lambda^{1/(\alpha-1)}$ and let $x_t = x(r_t)$ satisfy (4). Then an R-producer’s capital investment $z_t$ in period $t \geq 1$ is determined as follows:

i) $z_t$ maximises $z_t\alpha - \lambda r_t z_t$, that is, $z_t = \hat{\lambda} x_t$ if IC is not binding.

ii) $z_t < \hat{\lambda} x_t$ holds only if IC is binding.

(iii) IC can be written as

$$\delta^t\{z_{t+1}^\alpha - \lambda r_{t+1} z_{t+1} + \delta(V_{t+2} - \Pi_{t+2}) - \pi_{t+1}\} = \lambda r_t z_t. \quad (IC_t)$$

Lemma 1 (i) follows from the fact that the R-producer has full bargaining power in making a relational contract and hence the R-lender’s payoff is always reduced to her reservation value $U_t$, which in turn implies that the repayment $R_t$ is set to the opportunity cost of lending $z_t$ to the R-producer, $\lambda r_t z_t$, in each period $t$. Here, each R-lender could earn the interest income $\lambda r_t$ if she lent one unit to the anonymous credit market instead of lending it to the R-producer. Thus $\lambda r_t$ can be interpreted as the opportunity cost of lending one unit via a relational contract. Lemma 1 (ii) then follows directly because the optimal capital investment $z_t$ must maximise the R-producer’s profit $z_t\alpha - R_t = z_t\alpha - \lambda r_t z_t$ when it is not constrained by IC. When IC becomes binding, the capital investment choice $z_t < \hat{\lambda} x_t$ is distorted downward as compared with the case where IC is slack (thus $z_t = \hat{\lambda} x_t$). Lemma 1 (iii) is obtained from Lemma 1 (i) and IC.

Before investigating the full equilibrium of the model economy, it will be helpful to see from Lemma 1 how the interest rate $r_t$ affects the optimal relational contract. To this end, consider the case that IC is binding in every period, which will actually be shown to be true in any equilibrium (see Lemma 3 below). Then, by using IC$_{t+1}$ with equality, we can rewrite IC$_t$ as $S_t = \delta\{z_{t+1}^\alpha - \pi_{t+1}\} = \lambda r_t z_t$, or equivalently $S_t/r_t x_t = \lambda (z_t/x_t)$, where $S_t$ denotes the discounted value of the R-producer’s future surplus evaluated in period $t$ (note that $S_t = \delta\{V_{t+1} - \Pi_{t+1}\} = \delta\{z_{t+1}^\alpha - \pi_{t+1}\}$ due to binding IC$_{t+1}$). By using (4) and (5), this IC$_t$ condition can be arranged as $\lambda(z_t/x_t) = \lambda^{1/(\alpha-1)} (z_t/x_t)$.
\((1-\alpha)/\alpha)S_t/\pi_t\), which means that the relative capital \(z_t/x_t\) of the R-producer in period \(t\) is determined by the \textit{relative continuation value} of the R-producer’s discounted surplus \(S_t\) in terms of an A-producer’s profit \(\pi_t\) in period \(t\). Here \(S_t/\pi_t\) can be expanded as follows:

\[
S_t/\pi_t = \delta \left( \gamma_{t+1}^\alpha - \lambda r_{t+1} z_{t+1} - \pi_{t+1}^\alpha + \lambda r_{t+1} z_{t+1} \right)/\pi_t
\]

where \(\gamma_{t+1}^\alpha - \lambda r_{t+1} z_{t+1} - \pi_{t+1}^\alpha\) denotes the R-producer’s net surplus in period \(t+1\), which is shown to be non-negative in equilibrium. Then \(S_t/\pi_t \geq \delta(S_{t+1}/\pi_{t+1})(\pi_{t+1}/\pi_t)\), where the right-hand side is larger than \(S_{t+1}/\pi_{t+1}\) if \(\pi_{t+1}/\pi_t > \delta\), in which case \(S_t/\pi_t > S_{t+1}/\pi_{t+1}\) implying that the relative continuation value of the R-producer decreases over time when the A-producer’s profit grows faster.

Suppose now that interest rates \(r_t = \alpha x_t^{\alpha-1}\) fall rapidly over time, that is, the A-producer’s profit \(\pi_t\) grows according to \(\pi_{t+1}/\pi_t = (x_{t+1}/x_t)^\alpha > 1/\delta\). Then, as we have seen above, R-producers expect a larger relative continuation value \(S_t/\pi_t\) in the current period \(t\) than in the next period \(t+1\), \(S_{t+1}/\pi_{t+1}\), which implies that \(z_t/x_t > z_{t+1}/x_{t+1}\). In words, as interest rates fall sharply over time, R-producers gain a higher growth in the future surplus relative to A-producers in the current period \(t\) than in the next period \(t+1\), which makes IC \(_t\) less stringent than IC \(_{t+1}\). This results in larger capital investments of R-producers relative to A-producers in early periods than in later periods in the course of economic development when interest rates are declining. We confirm this intuition more rigorously in the next section by endogenising the market interest rate \(r_t\).

Note that the optimal relational contract depends on the value for A-producers \(\Pi_t\), which is the reservation value for R-producers, through the IC \(_t\) constraint. Thus, if the interest rate \(r_t\) is fixed exogenously, an A-producer’s profit \(\pi_t\) becomes constant over time and hence \(\Pi_t\) is also constant. In such cases, the optimal contract becomes stationary so that there are no non-stationary equilibrium dynamics in which relative magnitudes of arm’s length and relational contracts change over time. Thus an endogenous change in the interest rate \(r_t\) plays a crucial role in obtaining the dynamic interplay between relational contracts and the competitive credit market during economic development.

### 4 Equilibrium Dynamics

In this section we embed the optimal relational contract derived in the previous section into the general equilibrium framework in which we endogenously determine the market interest rate \(r_t\).

#### 4.1 Credit market equilibrium

At the end of period \(t-1\), there are \(N\) young R-producers who each invest \(z_t\) in capital in period \(t-1\) and produce \(z_t^\alpha\) when old in period \(t\). There are also \(1-N\) young
A-producers who each invest $x_t$ in capital in period $t-1$ and produce $x_t^\alpha$ when old in period $t$. Moreover, each of the $1-N$ young A-lenders saves her entire income $w_{t-1}$ in the anonymous credit market. Because A-lenders incur enforcement costs of $1 - \lambda$ per unit of lending, their actual credit supply to the anonymous credit market is $\lambda w_{t-1}(1-N)$ after subtracting the enforcement costs $(1 - \lambda)w_{t-1}$ from the savings $w_{t-1}$ of each A-lender. There are also $N$ young R-lenders each of whom finances the capital investment $z_t$ of their trading partners (R-producers). Each young R-lender then lends the remaining amount of her income $w_{t-1} - z_t \geq 0$ to the anonymous credit market (recall that we are assuming that $w_{t-1} \geq z_t$). Thus, the total credit supply in the credit market in period $t-1$ is $\lambda(1-N)w_{t-1} + \lambda N(w_{t-1} - z_t)$. On the other hand, in period $t-1$ there are $1-N$ young A-producers who each need to finance a capital investment of $x_t$. This becomes the credit demand in the credit market in period $t-1$.

Thus the credit supply $\lambda(1-N)w_{t-1} + \lambda N(w_{t-1} - z_t)$ and the credit demand $(1-N)x_t$ are equal in the anonymous credit market when

$$\lambda w_{t-1} = \lambda N z_t + (1 - N)x_t, \quad \text{(CME}_{t-1}\text{)}$$

which we call the credit market equilibrium (CME$\text{}_{t-1}$) in period $t-1$.

### 4.2 Fall and Rise of Relational Contracts

We now provide a formal definition of an equilibrium path in this model economy.

**Definition.** A sequence $\{x_t, z_t, r_t\}_{t=1}^{\infty}$ is said to be an equilibrium path in the economy if the following conditions are satisfied:

(i) Each young A-producer born in period $t-1$ chooses his capital investment $x_t$ that satisfies optimality condition (4).

(ii) Each relationship pair in the initial generation chooses the optimal relational contract $\{z_t, R_t\}_{t=1}^{\infty}$ so as to solve Problem (RC).

(iii) The values $V_t$ and $W_t$ for R-producers and R-lenders are determined by (6) and (8) while the values $\Pi_t$ and $U_t$ for A-producers and A-lenders are determined by (7) and (9).

(iv) The credit market equilibrium (CME$\text{}_{t-1}$) is satisfied:

$$\lambda w_{t-1} = \lambda N z_t + (1 - N)x_t.$$

Here, the initial endowment $w_0$ is given exogenously.

As already explained, conditions (i) and (ii) state the optimal capital choices of A-producers and R-producers respectively. Condition (iv) requires that the credit market clears in each period. The economy’s initial condition is given by the young lenders’ initial endowment $w_0 = Nz_0^\alpha + (1 - N)x_0^\alpha$. 

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We define economic development as the process during which the anonymous credit market expands so that market interest rates \( r_t \) fall over time. As the economy develops, A-producers can more easily access the anonymous credit market, thus incurring lower costs for financing their capital investments and resulting in the rise of a market-based economy. Our central question then becomes how relational contracting affects and is affected by this process of development. In what follows, we use the term growth phase to refer to the development phase in which market interest rates are relatively high but decline rapidly over time. On the other hand, we use the term mature phase, to refer to the development phase in which market interest rates are steadily low and do not decrease much further.

We begin with preliminary results that will be useful for characterising equilibrium paths.

First, we have so far assumed that each R-producer does not invest more in capital than the funds available to the R-lender trading with him, that is, \( w_{t-1} \geq z_t \). We now show that this is actually the case in any equilibrium. If this is not the case in some period \( t \), we have \( z_t > w_{t-1} \) and hence each R-producer needs to finance the remaining amount \( z_t - w_{t-1} \) from the credit market after borrowing \( w_{t-1} \) directly from the R-lender trading with him. The R-producer’s profit in period \( t \) then changes to \( z_t r_t \left( z_t - w_{t-1} \right) \), which is maximised at \( z_t = x_t \equiv x(r_t) \). Thus each R-producer must choose \( z_t = x_t \) in the optimal relational contract because, if \( z_t > x_t \), decreasing \( z_t \) slightly can increase his profit without violating the IC constraint. However, then because \( w_{t-1} < z_t \leq x_t \), the total credit supply \( (1 - N)\lambda w_{t-1} \) becomes smaller than the total credit demand of \( N(z_t - w_{t-1}) + (1 - N)x_t \), which is the sum of the credit demands \( z_t - w_{t-1} > 0 \) for each R-producer and \( x_t \) for each A-producer. The equilibrium interest rate \( r_t \) must then go up so that the credit market can be cleared. This reduces the capital investment \( z_t \) by an R-producer until the condition \( w_{t-1} \geq z_t \) is restored in the credit market equilibrium. We thus obtain the following lemma.

Lemma 2. In any equilibrium path, \( z_t \leq w_{t-1} \) must hold in any period \( t \).

Second, we can show that IC\(_t\) is always binding in any equilibrium path. This follows from the result that \( (1/\lambda)x_t \geq z_t \) holds in any period \( t \). To demonstrate this, suppose contrary to this claim that \( z_t > (1/\lambda)x_t \) holds in some period \( t \). Then, we can see from CME\(_t\) that \( w_{t-1} < z_t \), that is, each R-producer must invest more in capital than the funds available to the R-lender trading with him. This, however, never happens by Lemma 2. Thus \( z_t = (1/\lambda)x_t \) must hold. This in turn shows that \( z_t \leq (1/\lambda)x_t < \lambda x_t \) where \( \lambda \equiv \lambda^{1/(\alpha-1)} \), implying that each R-producer chooses a lower capital \( z_t \) than the optimal choice without IC\(_t\) (that is, \( z_t < \lambda x_t \)). Lemma 1 then implies that IC\(_t\) must be binding in any equilibrium.

Lemma 3. In any equilibrium, \( z_t \leq (1/\lambda)x_t \) holds in any period \( t \). Therefore, IC\(_t\) becomes binding in any period \( t \) in any equilibrium path.
Lemmas 1 and 3 then imply that \( \text{IC}_t \) can be rewritten as
\[
\delta \{z_{t+1}^\alpha - \pi_{t+1}^\alpha\} = \lambda r_t z_t,
\]
because \( \text{IC}_{t+1} \) is binding and \( R_t = \lambda r_t z_t \) for all \( t \geq 1 \). Any equilibrium path in the economy can therefore be described as a sequence \( \{z_t, x_t\}_{t=1}^\infty \) that satisfies the binding \( \text{IC}_t \) and \( \text{CME}_t \):
\[
\delta \{z_{t+1}^\alpha - \pi_{t+1}^\alpha\} = \lambda r_t z_t, \quad t = 1, 2, ..., \quad (\text{IC}_t)
\]
\[
\lambda w_t = \lambda N z_{t+1} + (1 - N)x_{t+1}, \quad t = 0, 1, ..., \quad (\text{CME}_t)
\]
where \( r_t = \alpha x_t^{\alpha-1} \) (see (4)) and \( w_t = Y_t \), that is,
\[
w_t = N z_t^\alpha + (1 - N)x_t^\alpha.
\]

We define a new variable
\[
y_t \equiv z_t/x_t
\]
for an \( R \)-producer’s relative capital, which measures the ratio between the capital investments of an \( R \)-producer and of an \( A \)-producer. In what follows, we use \( y_t \) as a state variable rather than using an \( R \)-producer’s capital \( z_t \) directly, because we can then simplify the dynamical system of equilibrium paths.

By substituting \( r_t = \alpha x_t^{\alpha-1} \) into \( \text{IC}_t \), and using \( \pi_{t+1} = (1 - \alpha)x_{t+1}^\alpha \) (see (5)) and the \( R \)-producer’s relative capital \( y_t \equiv z_t/x_t \) defined above, we can rewrite the above \( \text{IC}_t \) as follows:
\[
\left(\frac{x_{t+1}}{x_t}\right)^\alpha \delta \{y_{t+1}^\alpha - (1 - \alpha)\} = \lambda \alpha y_t, \quad t = 1, 2, ..., \quad (10)
\]

Note that \( \pi_{t+1}/\pi_t = (x_{t+1}/x_t)^\alpha \). Thus, a higher growth in \( A \)-producer’s capital \( x_t \) (hence, his profit \( \pi_t \)) raises the \( R \)-producer’s relative capital \( y_t \) in the current period \( t \) as we have seen in Section 3. Also, \( \text{CME}_t \) can be written as
\[
\lambda(N y_t^\alpha + (1 - N))x_t^\alpha = x_{t+1}(N \lambda y_{t+1}^\alpha + (1 - N)). \quad (11)
\]

Thus, an equilibrium path in the economy is a sequence \( \{y_t, x_t\}_{t=1}^\infty \) that satisfies (10) and (11), where \( x_1 \) and \( y_1 \) satisfy the credit market equilibrium (CME0) in the initial period \( t = 0 \) as follows:
\[
\lambda w_0 = x_1(N \lambda y_1 + (1 - N)). \quad (12)
\]

Here, the capital in period 1, \( (x_1, y_1) \), is indeterminate because any combination of \( x_1 \) and \( y_1 \) satisfying (12) can be consistent with the equilibrium conditions (10) and (11). However, we will show that the equilibrium path is unique once we fix \( (x_1, y_1) \) satisfying (12) (see the proof of Proposition 1 below) and that any equilibrium path converges to a unique steady-state in the long run regardless of its starting value \( (x_1, y_1) \) (see Proposition 3 below). Thus, indeterminacy of the first-period capital \( (x_1, y_1) \) does not matter for the long-run features of equilibrium paths.

**Benchmark.** Before characterising equilibrium paths, we first consider the benchmark case where no relationship pairs are active in production in the economy. Specifically, suppose that \( R \)-producers cannot invest at all, that is, \( z_t = 0 \) for all \( t \) for some exogenous
reason, and that all production in the economy is made only by A-producers. Thus all financing and production is market-based, that is, based solely on arm’s length contracts in the anonymous credit market. In this case, we set \( y_t = y_{t+1} = 0 \) in (11) to obtain

\[ x_t^0 = x_{t+1}, \quad t = 0, 1, 2, \ldots, \]

which has the unique steady state \( \bar{x} \equiv \lambda^{1/(1-\alpha)} \). Because we are interested in the process of economic development during which \( x_t \) increases and hence interest rates \( r_t = \alpha x_t^{\alpha-1} \) decrease over time, we suppose that the economy starts with low initial capital \( x_0 \) as assumed in the following:

**Assumption 1.** \( x_0 < \bar{x} \).

Under Assumption 1, the equilibrium capital \( x_t \) in the benchmark case increases over time until it converges to the steady state \( \bar{x} \).

We now return to the original setting where R-producers are active (\( y_t > 0 \)) and characterise the equilibrium paths described by (10), (11), and (12). To this end, we define the following value of the discount factor \( \tilde{\delta} \), which plays a critical role in what follows:

\[ \tilde{\delta} = \frac{N + \lambda^\alpha (1 - N)}{\lambda (1 - \alpha)^{1/\alpha} N + (1 - N)} \left[ \frac{\alpha}{1 - \lambda^\alpha (1 - \alpha)} \right]. \quad (13) \]

We then make the following assumption:

**Assumption 2.** \( \delta > \tilde{\delta} \).

Assumption 2 states that the discount factor \( \delta \) is not so small that R-producers are greatly concerned with the consumption of their descendants.\(^{14}\) This ensures a large gain from honouring relational contracting (thus the left-hand side of IC\(_t\) increases), allowing R-producers to invest more capital than A-producers. When the discount factor is small, in contrast with Assumption 2, IC\(_t\) may be so strict that R-producers cannot invest more than A-producers.

Under Assumptions 1 and 2, we can obtain the following result.

**Proposition 1.** Suppose that Assumptions 1 and 2 are satisfied. Then any equilibrium path has a unique period \( T \geq 1 \), which we call the switching period, such that:

(i) \( y_t > 1 \) for all \( t \leq T - 1 \) and \( y_t < 1 \) for all \( t \geq T \).

(ii) if \( y_1 > 1 \) so that \( T \geq 2 \), then an R-producer’s relative capital \( y_t \) decreases over time until the switching period \( T \), that is, \( y_1 > y_2 > \cdots > y_{T-1} > 1 > y_T \).

\(^{14}\)Because \( \delta < 1 \), Assumption 2 is meaningful only when \( \tilde{\delta} < 1 \). We can see that \( \delta \) is less than 1 when the enforcement cost \( 1 - \lambda \) is not so small (and thus \( \lambda \) is small) and \( \alpha \) is small. Alternatively, if \( N \) is small, then \( \delta \) is close to \( \alpha \lambda^\alpha / (1 - \lambda^\alpha (1 - \alpha)) \), which is less than 1.

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Regarding the effects of relational contracting on an A-producer’s capital investment (and output), we can obtain the following result.

**Proposition 2.** Suppose that Assumptions 1 and 2 hold, and let \( T \) be a switching period. Then we have the following:

(i) An A-producer’s capital \( x_t \) increases (that is, \( x_{t+1} > x_t \)) and the interest rate \( r_t \) falls (that is, \( r_t > r_{t+1} \)) over time for all \( t \leq T \) and for some \( T \leq T^* \).

(ii) An A-producer’s capital \( x_t \) eventually becomes larger than the steady state capital \( \pi \) in the benchmark case without relational contracting (that is, \( x_t > \pi \) for all \( t \geq T^* \) and for some period \( T^* \)).

Propositions 1 and 2 are the main results of this paper, and they support the aforementioned views regarding the roles of relational contracting in the process of economic development.

First, each R-producer invests more in capital and produces more outputs than each A-producer in the early stages before the switching period \( T \) (\( y_t > 1 \) for any \( t \geq T \)). Also, each R-producer’s relative capital \( y_t \) becomes larger in the early periods with high interest rates (when each A-producer’s capital is low) than in the later periods with low interest rates (when each A-producer’s capital is high); that is, \( y_t > y_{t+1} \). In the growth phase before the switching period \( T \), each A-producer’s capital \( x_t \) grows owing to falling interest rates and it eventually exceeds the steady state capital \( \pi \) in the benchmark case where all financing is market-based. Second, each R-producer switches to invest less capital than each A-producer after the economy passes the switching period \( T \) and enters the mature phase during which interest rates are steadily low and do not decrease much.

The intuition behind these results is explained as follows. Suppose that R-producers expect that interest rates will fall rapidly over time, that is, the growth rate of an A-producer’s profit (capital) \( \frac{x_{t+1}}{x_t} = (x_{t+1}/x_t)^\alpha \) is high. Then, as we have seen in Section 3, R-producers obtain a larger continuation value of the future surplus relative to A-producers evaluated in the current period \( t \) than in the next period \( t + 1 \) (that is, \( S_t/\pi_t > S_{t+1}/\pi_{t+1} \)). Thus, R-producers invest more capital relative to A-producers in the earlier period than they do in the later period, \( y_t > y_{t+1} \). In particular, when the growth rate \( x_{t+1}/x_t \) is expected to be high enough in the current period \( t \), \( y_t > 1 \) holds.

To see that the above expectation about the high growth rate \( x_{t+1}/x_t \) is self-fulfilled and that this can be consistent with the supposed path with \( y_t > 1 \) and \( y_t > y_{t+1} \), we examine how an R-producer’s relative capital affects the growth rate of an A-producer’s capital by rearranging the credit market equilibrium (CME) as

\[
\lambda x_t^{\alpha-1}(Ny_t^\alpha + (1 - N)) = (x_{t+1}/x_t)(\lambda Ny_{t+1}^\alpha + (1 - N)). \tag{CME''}
\]
This condition says that the growth rate of an A-producer’s capital $x_{t+1}/x_t$ increases when (i) the marginal return of the A-producer’s capital $\alpha x_t^{\alpha-1}$ is larger, (ii) the contribution of each R-producer’s capital to the aggregate output $y_t^\alpha$ is larger, and (iii) the demand for the R-producer’s capital for the next period $\lambda y_{t+1}$ is lower. First, a high output from R-producers, $y_t^\alpha > 1$, in current period $t$ raises the total credit supply (the left-hand side of CME$_t$ above), financing more capital investments by A-producers while the R-producer’s demand $\lambda y_{t+1}$ for capital in the next period $t + 1$ crowds out capital investments by A-producers. However, because R-producers do not incur enforcement costs $1 - \lambda$ and demand only $\lambda y_{t+1} < 1$ for the next period, the former credit expansion effect dominates the latter crowding out effect, that is, $y_t^\alpha > 1 \geq \lambda y_{t+1}$, contributing to increasing the growth rate. Thus, CME$_t$ implies that $\lambda x_t^{\alpha-1} < (x_{t+1}/x_t)$, which shows that the growth rate $x_{t+1}/x_t$ becomes high when the current capital $x_t$ is sufficiently low owing to the diminishing marginal returns of capital. Therefore, $y_t > 1$ and $y_t > y_{t+1}$ actually constitute an equilibrium path in the growth phase in which capital $x_t$ is low and the interest rate is high before the switching period $T$.

In this way, relational contracting promotes the capital investments of A-producers in the growth phase when each A-producer’s capital is low. Both $y_t^\alpha > 1 \geq \lambda y_{t+1}$ and CME$_t$ together imply that $\lambda x_t^{\alpha-1} < x_{t+1}$ holds for any $t < T$, showing that each A-producer’s capital $x_t$ grows to exceed the benchmark capital $\overline{x}$. However, as the A-producer’s capital $x_t$ increases, its growth rate tends to slow due to diminishing marginal returns of capital, so that interest rates do not decrease much more over time corresponding to the mature phase. In this phase, IC$_t$ becomes more stringent so that R-producers choose lower levels of capital investment. In this manner, each R-producer eventually switches to invest less capital than each A-producer.

These features of equilibrium dynamics can be better understood by using the phase diagram in Figure 1.\textsuperscript{15} Figure 1 depicts the two loci $\Delta y_{t+1} = y_{t+1} - y_t = 0$ and $\Delta x_{t+1} = x_{t+1} - x_t = 0$, which leave the R-producer’s relative capital $y_t$ and A-producer’s capital $x_t$ unchanged over time in (10) and (11), respectively.\textsuperscript{16} The intersection, denoted by $(\tilde{x}, \tilde{y})$ and indicated by the point $E$ in Figure 1, between these two loci determines a unique steady state of the economy. More formally, the steady state capitals $\tilde{x}$ and $\tilde{y}$ are derived by setting $x_{t+1} = x_t = \tilde{x}$ and $y_{t+1} = y_t = \tilde{y}$ in (10) and (11) as follows:

$$\delta \{\tilde{y}^\alpha - (1 - \alpha)\} = \lambda \alpha \tilde{y}, \quad (14)$$

$$\lambda (\Delta \tilde{y}^\alpha + (1 - N))\tilde{x}^\alpha = (N \lambda \tilde{y} + (1 - N))\tilde{x}. \quad (15)$$

\textsuperscript{15}We use Figure 1 to present only a rough intuition behind Propositions 1 and 2. Figure 1 is not a precise description of our dynamical system (10) and (11), because it is in discrete time and so $x_t$ and $y_t$ do not change continuously.

\textsuperscript{16}By setting $y_{t+1} = y_t$ in (10) and (11) and substituting $x_{t+1}$ from (11) into (10), we derive the relationship between $y_t$ and $x_t$ that yields the locus $\Delta y_{t+1} = 0$. Also, by setting $x_{t+1} = x_t$ in (10) and (11) and substituting $y_{t+1}$ from (10) into (11), we derive the relationship between $x_t$ and $y_t$ that yields the locus $\Delta x_{t+1} = 0$. 

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The arrows in Figure 1 denote the directions in which \( x_t \) and \( y_t \) change over time. We can verify that both \( y_t \) and \( x_t \) move inward so that they tend toward the steady state. Also, the first period values of capital \( y_1 \) and \( x_1 \) must satisfy the credit market equilibrium (12) in the initial period \( \lambda w_0 = x_1(N\lambda y_1 + (1 - N)) \) (such a point \( (x_1, y_1) \) is indicated as \( I \) in Figure 1). Then an equilibrium path \( \{x_t, y_t\}_{t=1}^{\infty} \) is given by a trajectory, shown as the connected arrows in Figure 1 starting from an initial point \( I \). We can also see that the steady state capital \( \bar{x} \) in the benchmark without relational contracting (\( y_t = 0 \) for all \( t \)) is on the locus \( \Delta x_{t+1} = 0 \) for some \( y_t = \hat{y} \in (0, y) \) where \( \hat{y} \equiv (1 - \alpha)^{1/\alpha} \).\(^{17}\)

Consider an equilibrium path starting with each A-producer having low capital \( x_1 \) (and thus a high interest rate \( r_1 = \alpha x_1^{\alpha-1} \)) and each R-producer having high relative capital \( y_1 > 1 \), as depicted in Figure 1. Each R-producer’s relative capital is higher in the earlier periods than in the later periods before the switching period \( T \); in the equilibrium trajectory shown in Figure 1, \( y_t \) decreases and \( x_t \) increases over time. Then \( y_t < 1 \) eventually holds after the switching period \( T \). Following this, the equilibrium path converges to the steady state \((\bar{x}, \hat{y})\) in the long run (see Proposition 3 below for detailed conditions for obtaining the convergence result). In the steady state, each R-producer’s relative capital \( \hat{y} \) contributes \( \hat{y}^\alpha \) to the aggregate output, which outweighs the increase in capital demand \( \lambda \hat{y} \) in future periods as we have discussed. That is, \( \hat{y}^\alpha > \lambda \hat{y} \), which in turn together with (15) implies that \( \lambda \hat{x}^\alpha < \bar{x} \) and so \( \bar{x} > \bar{x} \). In this way, each A-producer’s capital \( \bar{x} \) becomes larger in the long run than the steady state capital \( \bar{x} \) in the benchmark case wherein all financing is market-based. Consequently, relational contracting helps to promote expansion of the A-producers’ outputs in the long run (see Proposition 4 below for more details about this result).

To investigate the uniqueness and stability of the steady state, we next define the following value of the discount factor:

\[
\hat{\delta} \equiv \lambda \left[ \frac{\lambda N + (1 - N)}{\lambda(1 - \alpha)^{1/\alpha} N + (1 - N)} \right]^2.
\]

We then strengthen Assumption 2 as follows.\(^{18}\)

**Assumption 3.** \( \delta > \max\{\hat{\delta}, \hat{\delta}\} \).

We can now show the following:

**Proposition 3.** Suppose that Assumption 3 holds. Then the steady state capital \((\bar{x}, \hat{y})\)

\(^{17}\)\(\hat{y}\) is formally defined as the value \( y \) that satisfies \( \delta\{(y^n/\lambda)^n - (1 - \alpha)\} = \lambda \alpha y \). Then \( y^n = \lambda y_{t+1} \) in (11) implies that \( x_{t+1} = x_t = \bar{x} \) satisfies (11). By substituting \( y_{t+1} = \hat{y}^n/\lambda \) into (10) and setting \( x_{t+1} = x_t = \bar{x} \) in (10), we can see that when \( y_t = \hat{y} \) holds we have \( x_t = \bar{x} \) on the locus \( \Delta x_{t+1} = 0 \).

\(^{18}\)When \( \lambda \) and \( \alpha \) are small, \( \hat{\delta} < 1 \) and \( \hat{\delta} < 1 \) hold. Thus Assumption 3 is more likely to be satisfied for small values of \( \alpha \in (0, 1) \) and small values of \( \lambda > 0 \) as in Assumption 2.
defined in (14) and (15) is unique. Furthermore, any equilibrium path converges to the steady state \((\tilde{x}, \tilde{y})\) for any \((x_1, y_1)\) satisfying the initial condition (12).

Proposition 3 implies that the capital in the first period \((x_1, y_1)\) does not matter for long-run features of the economy, such as steady state capital, steady state GDP and steady state welfare, although it affects the transitional path of the equilibrium dynamics.

Finally, we discuss the long-run effect of relational contracting upon the outputs of A-producers and GDP (total income) of the economy in the steady state. The economy’s GDP or equivalently its total income in the steady state is defined as the aggregate output from all producers:

\[
\tilde{Y} = \tilde{x}^\alpha [N\tilde{y}^\alpha + (1 - N)].
\]

We have the following result:

**Proposition 4.** Suppose that Assumption 3 is satisfied. Then increasing the number of relationship pairs \(N\) raises each A-producer’s capital \(\tilde{x}\) and the economy’s GDP (total income) \(\tilde{Y}\) in the steady state.

Proposition 4 shows that an economy that relies more on relationship lending (larger \(N\)) can attain a larger GDP in the long run than an economy that relies less on relationship lending (smaller \(N\)) as long as \(N\) is sufficiently small to satisfy Assumption 3.\(^{19}\) This result reflects the fact that relational contracting contributes to raising the aggregate output more than to the increase in capital demand as we have already mentioned. This contributes to increasing the long-run GDP of the economy.

The above results (Proposition 1-4) are supported by several pieces of historical evidence regarding the interaction between relational contracting and economic development.

First, Propositions 2 and 4 confirm the positive view of relational contracting wherein non-market-based systems (relationship-based systems) relying on relational contracting are not an impediment to the emergence of a market-based economy but rather promote the evolution of such a system. Lamoreaux (1994) reports evidence that relationship-based lending became effective in financing and promoting economic growth in New England in the early 19th century in a situation where financial market immaturity rendered market-based financing. Maurer and Haber (2007) also support the positive view of relationship lending in that, between 1888 - 1913, Mexican bankers largely engaged in relationship lending as the optimal response to high enforcement costs, but they did not loot their own banks at the expense of outside shareholders.\(^{20}\)

\(^{19}\)For Assumption 3 to be satisfied, \(N\) must not be too large. When \(N\) is close to 1, Assumption 3 is violated. Thus the comparative statics result of Proposition 4 is conducted for the range of small values of \(N\) for which Assumption 3 holds.

\(^{20}\)See the papers collected in Aoki and Hayami (2001) for evidence that personal relationships perform a variety of valuable functions in developing economies.
Second, Proposition 1 also implies the other side of relational contracting, in which relationship-based systems eventually decline as the economy becomes richer through the process of development. Specifically, in the mature phase after the switching period $T$, each R-producer who engages in relationship lending eventually invests less than each A-producer who engages in market lending. Demirgüç-Kunt and Levine (2004) report a related fact that bank finance, sometimes characterised as long-term relationships between particular banks and firms, becomes less popular than market-based finance, such as equity, in more-developed countries (see Rajan and Zingales (2000) for a related argument).

4.3 Effects of enforcement costs $1 - \lambda$

The level of enforcement costs $\lambda$ reflects how society establishes well-functioning institutions such as courts and a transparent accounting system that support the enforcement of formally written contracts. We address the issue of how equilibrium paths respond when enforcement costs decrease because of the establishment of more effective institutions for enforcing arm’s length contracts at a lower cost.

For this purpose, we perform a comparative statics exercise on the effects of the enforcement costs $1 - \lambda$ on the steady state capital $(\tilde{x}, \tilde{y})$. Suppose that the enforcement cost $1 - \lambda$ decreases (thus $\lambda$ increases). Then we can see from (14) and (15) that the following statement holds.

**Corollary.** Suppose that Assumption 3 holds. Suppose also that the enforcement costs $1 - \lambda$ decrease. Then (i) the steady state R-producer’s capital $\tilde{y}$ increases, and (ii) the steady state A-producer’s capital $\tilde{x}$ increases when $\lambda$ is sufficiently small that $\lambda \leq \alpha(1 - N)/(1 - \alpha N)$ is satisfied.

When the enforcement costs $1 - \lambda$ decrease, the opportunity cost $\lambda r_t$ required by R-producers to raise one unit of capital increases. Because the IC$_t$ constraint becomes binding in the steady state (see (14)), the current gain from relational contracting (the left-hand side in (14)) must also increase to offset the increase in the opportunity cost. Thus each R-producer’s relative capital $\tilde{y}$ must increase. On the other hand, when both $\lambda$ and $\tilde{y}$ increase, the young lender’s income $w = \tilde{Y} = \tilde{x}^\alpha (N \tilde{y}^\alpha + (1 - N))$ and total savings (and hence the credit supply) $\lambda \tilde{Y}$ increase, reducing interest rates. This allows A-producers to finance more capital investment, which has the positive effect of increasing the A-producer’s capital $\tilde{x}$. However, the increase in the R-producer’s capital investment $z = \tilde{y}\tilde{x}$ also reduces the credit supply $\lambda(w - z)$ from each young R-lender to the anonymous credit market. This decrease in the credit supply raises interest rates, which reduces the A-producer’s capital $\tilde{x}$. Because these two effects are in opposition, their combined effect is ambiguous. However, when $\lambda$ is small, the decrease in the credit supply $\lambda(w - z)$ from each R-lender becomes small enough that the former effect dominates the latter, resulting in an increase in the A-producer’s capital $\tilde{x}$.

Thus, lower enforcement costs $1 - \lambda$ may raise both A-producers’ and R-producers’
capitals in the steady state. One implication of this comparative statics exercise is that, as society establishes more effective formal institutions such as courts and a transparent accounting system that reduce the enforcement costs \(1 - \lambda\), both relational and arm’s length contracts may function to complement each other to increase capital investments (see Aoki and Hayami (2000) for related evidence on the complementary role of informal institutions in the emergence of market-based economies).

5 Endogenous Relationship Formation

In this section, we endogenise the evolution process of arm’s length and relational contracts in terms of both intensive and extensive margins: capital investment by R-producers changes relative to that of A-producers in the intensive margin, and the number of relationship pairs changes in the extensive margin.

To this end, suppose that a matching market exists where young producers and young lenders born in each period search for each other and the matches between them are determined randomly according to the matching function described below. Each young producer who wants to enter the matching market incurs the disutility of searching, called the search cost and denoted by \(d > 0\). On the other hand, we assume that young lenders incur no search costs in the matching market. When a young producer is successfully matched with a young lender in the matching market, he and his descendants start new relationships with the lenders. We call these producers R-producers and we call the corresponding lenders R-lenders as in the previous section. Any young producer or young lender who is unmatched in the matching market must engage in arm’s length contracts in the anonymous credit market. We refer to these producers (lenders, respectively) as A-producers (A-lenders, respectively) as before.

We use \(N_t\) to denote the number of relationship pairs (equivalently old R-producers and old R-lenders) who engage in relational contracts for production in period \(t\). We assume that the production technology owned by the producers in each dynasty becomes obsolete in each period with probability \(1 - \rho\), \(\rho \in (0, 1)\), in which case the future generations of that dynasty cannot produce any outputs. Thus each young producer inherits production technology from his parent with probability \(\rho \in (0, 1)\) in each period. We also assume that \(1 - \rho\) new producer dynasties are born in each period. Thus producer dynasties whose technologies became obsolete are replaced by

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\(\text{Footnotes:}\)

21 We can interpret this search cost as the opportunity cost for each young producer to learn his production technology before production. When each young producer exerts a limited effort to learn the production technology only, he will incur a low disutility normalised to zero. However, when he spends a limited effort for both learning the production technology and searching for a young lender in the market at the same time, he incurs a larger disutility denoted by \(d > 0\).

22 This may be because young lenders do not need to learn the production technology and hence can concentrate on searching for young producers. See footnote 21 above. When we introduce the search costs of young lenders, the model becomes more complicated. We do not pursue this extension further in this paper.

23 The assumption \(\rho \in (0, 1)\) ensures that there will always be A-producers; that is, we can guarantee \(N_t < 1\) in every period.
newly created dynasties in each period. The total population of producers is therefore constant over time. Therefore, \( \rho N_t \) of the young R-producers inherit the relationships from their parents (old R-producers) while \( 1 - \rho N_t \) of the young producers have no matching partners in period \( t \).

The matching process within period \( t \) proceeds as follows:

- We denote by \( M_t \) the number of young producers who decide to enter the matching market in period \( t \), where \( 0 \leq M_t \leq 1 - \rho N_t \) (thus \( 1 - M_t - \rho N_t \) remaining producers rely on the anonymous credit market.) On the other hand, all \( 1 - \rho N_t \) young lenders go to the matching market seeking new matches because they incur no search costs and are hence weakly better off searching for R-producers rather than immediately engaging in arm’s length contracts in the anonymous credit market.\(^{24}\)

- In the matching market in period \( t \), each young producer is matched with a young lender with probability \( \eta_t \in [0, 1] \) while each young lender is matched with a young producer with probability \( \theta_t \in [0, 1] \).

- An R-producer and an R-lender who form a new relationship or who inherit it from the previous generation engage in the relational contract \( \{z_t, R_t\}_{t=m+1}^{\infty} \), where \( z_t \) and \( R_t \) denote the capital investment and the repayment in period \( \tau \), respectively. They continue the relationship in the next period with probability \( \eta_t \). On the other hand, those who fail to be matched engage in arm’s length contracts through the anonymous credit market as A-producers and A-lenders.

We assume here that the matching probability \( \eta_t \) is a function of the market tightness \( (1 - \rho N_t)/M_t \) defined as the ratio between the numbers of young lenders and young producers who enter the matching market, that is, \( \eta_t = \eta((1 - \rho N_t)/M_t) \). As in the standard literature of search models, we assume that \( \eta \) is increasing with \( \eta(0) = 0 \) and \( \eta(\infty) \equiv \bar{\eta} < 1 \). Analogously, we assume that \( \theta_t \) is a function of the market tightness, defined by \( \theta_t = \theta((1 - \rho N_t)/M_t) \), which is decreasing with \( \theta(0) \equiv \bar{\theta} < 1 \) and \( \theta(\infty) = 0 \), and \( \eta_t M_t = \theta_t(1 - \rho N_t) \) holds.\(^{25}\)

The dynamic equation governing the number of relationship pairs \( N_t \) is then given by

\[
N_{t+1} = \eta_t M_t + \rho N_t,
\]

because \( M_t \) young producers decide to enter the matching market and each of them finds a new young lender with probability \( \eta_t \), while \( \rho N_t \) relationships are inherited from the previous generation. By the definition of \( \eta_t \), we obtain \( \eta_t = \eta((1 - \rho N_t)/M_t) \), which implies that \( (1 - \rho N_t)/M_t = \eta^{-1}(\eta_t) \) where \( \eta^{-1} \) denotes the inverse function of \( \eta \). We substitute this into \( \theta_t = \theta((1 - \rho N_t)/M_t) \) to obtain \( \theta_t = \psi(\eta_t) \equiv \theta(\eta^{-1}(\eta_t)) \), which is

\(^{24}\)This follows from the individual rationality constraint for R-lenders. See the Online Appendix.

\(^{25}\)For example, these properties are satisfied by a standard CES matching function such as \( m(M_t, 1 - \rho N_t) = [(\pi M_t)^{-\xi} + (\theta(1 - \rho N_t))^{-\xi}]^{-1/\xi} \) for \( \xi > 0 \), \( \eta \in (0, 1) \) and \( \bar{\theta} \in (0, 1) \). Here we have \( \eta(n_i) \equiv m(1, n_i) \) and \( \theta(n_i) \equiv m(1/n_i, 1) \) for \( n_i \equiv (1 - \rho N_i)/M_t \).
decreasing in $\eta$ with $\psi(0) < 1$ and $\psi(\infty) = 0$. Then, because $\eta M_t = \theta_t(1 - \rho N_t)$, we can rewrite (16) as

$$N_{t+1} = \psi(\eta)(1 - \rho N_t) + \rho N_t. \tag{17}$$

The credit market equilibrium condition is modified slightly from the basic setting in (11) to the following:

$$\lambda x_t^0 [N_t y_t^0 + (1 - N_t)] = x_{t+1} [\lambda N_{t+1} y_{t+1} + (1 - N_{t+1})], \tag{18}$$

where we replaced $N$ by $N_t$ and $N_{t+1}$. Recall here that $N_t$ denotes the number of R-producers who invest when young in period $t$ and produce outputs when old in period $t$.

As in the basic model, we use $\Pi_{t+1}$ to denote the value, called the arm’s length contract value, expected by a young producer who goes to the anonymous credit market in period $t$ to borrow via an arm’s length contract. On the other hand, when a young producer decides to search for a young lender in the matching market in period $t$ and to incur the search cost $d$, he expects to obtain the search value $J_{t+1}$.

The arm’s length contract value $\Pi_{t+1}$ is defined as

$$\Pi_{t+1} = \pi_{t+1} + \rho \delta \max\{\Pi_{t+2}, J_{t+2}\}, \tag{19}$$

while the search value for entering the matching market is given by

$$J_{t+1} = -d + \eta_t V_{t+1} + (1 - \eta_t)\{\pi_{t+1} + \rho \delta \max\{\Pi_{t+2}, J_{t+2}\}\}. \tag{20}$$

With matching probability $\eta_t$, each young producer can meet a young lender in the matching market in period $t$, in which case he can engage in a relational contract with her and expects to obtain the match value $V_{t+1}$ from production when old in period $t + 1:

$$V_{t+1} = z_{t+1}^0 - R_{t+1} + \rho \delta V_{t+2}.$$  

With probability $1 - \eta_t$ he fails to meet a young lender in the matching market, in which case he earns the arm’s length contract profit $\pi_{t+1}$ when old in period $t + 1$ and obtains the largest value between the search value $J_{t+2}$ and the arm’s length contract value $\Pi_{t+2}$ in period $t + 2$.

Each old R-producer then has no incentive to renege on the agreed upon repayment $R_t$ in period $t$ when the IC constraint $z_t^0 - R_t + \rho \delta V_{t+1} \geq z_t^0 + \rho \delta \max\{J_{t+1}, \Pi_{t+1}\}$ is satisfied, that is,

$$\rho \delta \{V_{t+1} - \max\{J_{t+1}, \Pi_{t+1}\}\} \geq R_t. \tag{M-IC_t}$$

We can then extend the results from the basic setting to show that M-IC$_t$ becomes binding in every period $t$, and that $R_t = \lambda r_t z_t$ holds in every period $t$ owing to the binding individual rationality constraint on R-lenders.

We make the assumption that the search cost $d$ is not so large but also not so small (Assumption B1 in the Online Appendix). When $d$ is in such intermediate range, some but not all young producers find it optimal to enter the matching market and search.
for young lenders with whom to match. In other words, a young producer born in period $t - 1$ is indifferent to the choice between entering the matching market and the anonymous credit market, because these choices give him the same value $J_t = \Pi_t$.

By using this indifference condition $J_t = \Pi_t$ together with $R_t = \lambda r_t z_t$ and $r_t = \alpha x_t^{\alpha-1}$, we can rewrite M-IC$_t$, which is binding, as

$$\rho \delta \left( \frac{x_{t+1}}{x_t} \right)^\alpha \{ y_{t+1}^\alpha - (1 - \alpha) \} = \lambda \alpha y_t,$$

which is essentially same as IC$_t$ in the basic setting. Thus we can apply the same reasoning as in the basic model to show that our main results, Propositions 1 and 2, can be extended to allow endogenous relationship formation:

(i) There exists a switching period $T$ such that each R-producer invests less in capital than each A-producer from $T$ onward, that is, $1 > y_t$ for all $t \geq T$. Also, $y_t$ decreases until the economy passes the switching period ($t < T$).

(ii) There exists some cutoff value for $N_t$, denoted by $\bar{N}$, such that the number of relationship pairs $N_t$ is eventually below $\bar{N}$. Also, when the economy starts with a large initial value $N_0$, $N_t$ declines over time until the economy passes the switching period ($t < T$).

Although the formal argument for this result is relegated to the Online Appendix, we give the intuition behind this extension below.

First, as we can see from the M-IC$_t$ above, in the growth phase when interest rates are high but decline sharply over time so that the growth rate of an A-producer’s capital $x_{t+1}/x_t$ increases, M-IC$_t$ becomes less stringent so that an R-producer’s relative capital $y_t$ increases. However, as the growth rate $x_{t+1}/x_t$ decreases in the mature phase due to diminishing marginal returns of capital, each R-producer invests less than each A-producer does. This captures the switching between the two contracting modes in the intensive margin as shown in Proposition 1.

Second, the number of relationship pairs $N_t$ that engage in relational contracts changes together with the R-producer’s relative capital $y_t$ over time. In particular, $N_t$ eventually becomes smaller than some cutoff, denoted by $\bar{N}$ (see Figure 2). Thus relational contracting declines over time not only in the intensive margin but also in the extensive margin as the economy develops. The reason for this result is that, as the economy develops and enters the mature phase in which the interest rate $r_t$ does not decrease much, the M-IC$_t$ constraint becomes stricter so that relational contracting becomes more constrained and hence the R-producer’s match value $V_{t+1}$ decreases relative to the arm’s length contract value $\Pi_{t+1}$. Fewer young producers then enter the matching market, so that the number of new matches formed in the matching market $\eta_t M_t = \psi(\eta_t)(1 - \rho N_t)$ decreases in this mature phase. In Figure 2, we depict the case that, as an R-producer’s relative capital $y_t$ declines over time, the fraction of matches $\psi(\eta_t)$ decreases over time, resulting in a smaller number of relationships. Thus the
number of relationships $N_t$ must eventually fall below some cutoff $\overline{N}$ after the economy passes the switching period.

The above results show that relational contracting contributes more to the aggregate outputs of the economy in both the intensive and extensive margins in the early periods than in the later periods of economic development.

6 Conclusion

This paper has investigated a dynamic general equilibrium model that involves dynamic changes in contract enforcement modes over time from relational contracts to arm’s length contracts. We have shown that relational contracting plays an important role in sustaining production during the early stages of economic growth in which arm’s length contracting does not function well to support capital investment because market interest rates are high. In subsequent periods, however, producers find it profitable to use arm’s length contracts because the economy is well-enough developed that interest rates fall. Thus, as the economy enters its mature stages, relationship-based systems decline and may be partially replaced by market-based systems. In this paper, we have focused on relational contracting between borrowers and lenders, as this approach is valuable for relating our theoretical results to the historical evidence on relationship-based financing. This is one of the modelling choices that capture relational contracting in dynamic general equilibrium frameworks. It is important to investigate how the economic development process is dynamically linked with long-term relationships in different contexts such as firm–worker relationships, inter-firm relationships, and government–public relationships.

7 Appendix: Proofs for Lemmas and Propositions

Proof of Lemma 1.

Consider Problem (RC) which defines the optimal relational contract chosen by each R-producer in the initial generation $\{R_t, z_t\}_{t=0}^\infty$, given the reservation values $\{\Pi_t, U_t\}_{t=1}^\infty$ and the interest rates $\{r_t\}_{t=1}^\infty$.

For the moment, we ignore the non-negativity constraints of consumption. Then, $R_t$ should be set in order to make R-lenders indifferent for accepting the relational contract and rejecting it in each period $t$: $W_t = U_t$ where $W_t = R_t + \lambda r_t (w_{t-1} - z_t) + \delta W_{t+1}$ and $U_t = \lambda r_t w_{t-1} + \delta U_{t+1}$. Thus, IRL$_t$ must be binding in any period, that is, $R_t = \lambda r_t z_t$ holds in any period $t$. The consumption of R-lender in period $t$ is then given by $R_t + \lambda r_t (w_{t-1} - z_t) = \lambda r_t w_{t-1} > 0$ which becomes actually positive. Also IC$_t$ is written by $\delta (V_{t+1} - \Pi_{t+1}) \geq R_t = \lambda r_t z_t$. Thus the optimal relational contract involves the capital investment $z_t$ which maximizes the value of R-producer in each period $t$, $V_t = z_t^\Pi - R_t + \delta V_{t+1} = z_t^\Pi - \lambda r_t z_t + \delta V_{t+1}$ subject to the above IC$_t$ constraint. Thus $z_t$ should be set as the one maximizing $z_t^\Pi - \lambda r_t z_t$ when IC$_t$ is slack whereas $z_t$ should satisfy $\delta (V_{t+1} - \Pi_{t+1}) = \lambda r_t z_t$ when IC$_t$ is binding respectively. This proves Lemma 1.
Proof of Lemma 2.  
Suppose contrary to the claim that \( z_t > w_{t-1} \) holds in some period \( t \) in some equilibrium path. Thus, each R-producer born in period \( t-1 \) obtains a profit \( z_t^0 - R_t - r_t(z_t - w_{t-1}) \) when old in period \( t \) because he borrows \( w_{t-1} \) from the R-lender trading with him and the remaining amount \( z_t - w_{t-1} > 0 \) from the anonymous credit market. The R-lender born in period \( t-1 \) earns \( R_t \) when old in period \( t \) because she lends all the income \( w_{t-1} \) to the R-producer trading with her. Thus the individual rationality of the R-lender is modified to
\[
W_t = R_t + \delta W_{t+1} \geq U_t = \lambda r_t w_{t-1} + \delta U_{t+1}
\]
in the supposed equilibrium. This yields \( R_t = \lambda r_t w_{t-1} \) because the individual rationality constraint of R-lenders becomes binding in the equilibrium (which can be shown by a similar argument to Lemma 1 (i)). Then the incentive compatibility constraint in period \( t \) is modified to
\[
\delta(V_{t+1} - \Pi_{t+1}) \geq \lambda r_t w_{t-1}.
\]
This is not affected by \( z_t \) at all. Thus, the optimal choice of \( z_t \) must maximize the profit of the R-producer in period \( t \), \( z_t^0 - r_t z_t + (1 - \lambda)r_t w_{t-1} \), which implies that \( z_t = x_t \equiv x(r_t) \).

However, then CME\(_{t-1}\) is modified as follows: there are \( (1-N) \) young A-lenders who lend \( w_{t-1} \) each while there are \( N \) R-producers who need to finance \( z_t - w_{t-1} \) each and \( 1-N \) A-producers who need to finance \( x_t \) each. Thus the credit market clears in period \( t-1 \) when
\[
(1-N)\lambda w_{t-1} = N(z_t - w_{t-1}) + (1-N)x_t.
\]
Since \( \lambda < 1 \), the above condition implies that \( w_{t-1} \geq N z_t + (1-N)x_t \). Then, since \( z_t = x_t \), this shows that \( w_{t-1} \geq x_t = z_t \) which is however a contradiction. Q.E.D.

Proof of Lemma 3.  
By Lemma 2, we know that \( w_{t-1} \geq z_t \) in any equilibrium. Therefore, CME\(_{t-1}\) implies that \( \lambda(1-N)w_{t-1} + \lambda N(w_{t-1} - z_t) = (1-N)x_t \) and hence \( \lambda w_{t-1} \leq x_t \). Since \( z_t \leq w_{t-1} \) by Lemma 2, we then obtain \( z_t \leq w_{t-1} \leq (1/\lambda)x_t \). Lemma 1 shows that IC\(_t\) must be binding when \( z_t < \lambda x_t \) where \( \lambda \equiv \lambda^{1/(\alpha-1)} \). Since \( \lambda > (1/\lambda) \), we then obtain \( z_t \leq (1/\lambda)x_t \) which shows that IC\(_t\) must be binding. Q.E.D.

Proof of Proposition 1.  
Suppose that Assumption 2 holds. From (11) we obtain
\[
x_{t+1}^{\alpha} \lambda(Ny_t^0 + (1-N)) \lambda N y_{t+1} + (1-N) = \left( \frac{x_{t+1}}{x_t} \right)^\alpha
\]
which we substitute into (10) to derive
\[
x_{t+1}^{\alpha-1} \lambda(Ny_t^0 + (1-N)) \lambda N y_{t+1} + (1-N) \delta\{y_{t+1}^0 - (1-\alpha)\} = \lambda \alpha y_t, \quad t = 1, 2, ...
\]
When $y_t < 1$, we define $T = 1$. Now let $y_t \geq 1$. Then we show that there exists some $T \geq 2$ such that $y_t > y_{t+1}$ for all $t < T$ and $y_{T-1} \geq 1 > y_T$.

First, note that $y_t^\alpha - (1 - \alpha) \geq \lambda \alpha y_t$ must be satisfied for all $t \geq 1$: If this is not the case in some period $t$, $z_t^\alpha - \lambda r_t z_t < \pi_t = (1 - \alpha)x_t^\alpha$. Then, the relationship pair of some dynasty can modify the relational contract as follows: trade via arm’s length contract in the anonymous credit market in period $t$ without dissolving the relationship but return back to the original relational contract from period $t + 1$ onward. This deviation contract yields the value $V_t' = \pi_t + \delta V_{t+1}$ to the R-producer in $t$-th generation of the deviating dynasty which is larger than the equilibrium value $V_t = z_t^\alpha - \lambda r_t z_t + \delta V_{t+1}$, a contradiction.\textsuperscript{26}

Then, since $1/\lambda \geq y_t$ for all $t \geq 1$ (Lemma 3) and $y_t^\alpha \geq (1 - \alpha)$ for all $t \geq 1$ by the above result, (11) implies that

$$\lambda(N \lambda^{-\alpha} + (1 - N))x_t^\alpha \geq x_{t+1}^{\alpha}(\lambda(1 - \alpha)^{1/\alpha} N + (1 - N)) \text{ for all } t \geq 0.$$  

Also, Assumption 1 implies that $x_0 < \overline{x} \equiv \lambda^{1/(1-\alpha)} < \overline{X} \equiv \Gamma^{1/(\alpha-1)}$ holds where

$$\Gamma \equiv \frac{\lambda(1 - \alpha)^{1/\alpha} N + (1 - N)}{\lambda(\lambda^{\alpha} N + (1 - N))} < 1/\lambda.$$  

Then we can verify that $x_{t+1} \leq \overline{X}$ for all $t \geq 0$\textsuperscript{27} Thus, since $x_{t+1}^\alpha \leq \overline{X}^\alpha$, (A1) implies that

$$\Gamma \delta \{y_{t+1}^\alpha - (1 - \alpha)\} \leq \frac{\lambda \alpha y_t}{\lambda N y_t + (1 - N)}.$$  

(A2)

Since $\lambda y_{t+1} \leq 1$ due to Lemma 3, we can show that for all $y_t \geq 1$, (A2) implies that

$$\Gamma \delta \{y_{t+1}^\alpha - (1 - \alpha)\} \leq \alpha y_t, \quad t = 1, 2, ...$$  

(A3)

Since the function $f(y) \equiv \alpha y/(y^\alpha - (1 - \alpha))$ is increasing in $y \in [1, 1/\lambda]$, we can verify that $\Gamma \delta > f(y)$ holds for all $y \in [1, 1/\lambda]$ if $\Gamma \delta > f(1/\lambda)$ which is satisfied under Assumption 2. Thus the left hand side of (A3) is larger than its right hand side at $y_{t+1} = y_t \in [1, 1/\lambda]$. Thus, when $y_t \in [1, 1/\lambda]$, we must have $y_{t+1} < y_t$: if not (thus $y_{t+1} \geq y_t \geq 1$), $\Gamma \delta \{y_{t+1}^\alpha - (1 - \alpha)\} \leq \alpha y_t \leq \alpha y_{t+1}$ which however contradicts to the above result that $\Gamma \delta > f(y)$ for all $y \in [1, 1/\lambda]$. Therefore, any equilibrium path $\{y_t\}_{t=1}^\infty$ is decreasing whenever $y_t \geq 1$. Then, if $y_t \geq 1$ holds for all $t \geq 1$, $y_t$ must be eventually lower than some $y < 1$ satisfying (A3) as equality: $\delta \Gamma \{y^\alpha - (1 - \alpha)\} = \alpha y$. This is a contradiction. Thus, there must exist some period $T$ such that $y_{T-1} \geq 1 > y_T$, provided $y_1 \geq 1$.

\textsuperscript{26}IC$_{t-1}$ is not violated by such deviation as well: $\delta(V_t' - \Pi_t) > \delta(V_t - \Pi_t) \geq R_{t-1}$. IRL$_t$ is also not affected because the value of any R-lender is reduced to her reservation value $\Pi_t$.

\textsuperscript{27}Since $(1/\Gamma)x_t^\alpha \geq x_{t+1}$ for all $t = 0, 1, 2, ...$ and $\overline{X} \geq \overline{x} > x_0$, we have $\overline{X} = (1/\Gamma)\overline{X}^\alpha > (1/\Gamma)x_0 > x_1$, and hence $\overline{X} > x_1$. Repeating this, $\overline{X} > (1/\Gamma)x_1^\alpha \geq x_2$ which implies $\overline{X} > x_2$, and so on. When $z_0 \neq x_0$, we can slightly modify Assumption 1 as follows: $\lambda w_0/(1 - N) < \overline{x}$ Then CME$_0$ implies that $x_1 < \lambda w_0/(1 - N) < \overline{x}$ and hence $x_t \leq \overline{x} = \overline{X}$ for all $t \geq 1$ again.
We can next show that the switching period $T$ defined above becomes unique once $y_1$ is fixed (hence $x_1$ is fixed by CME0: $\lambda w_0 = x_1(\lambda N y_1 + (1 - N))$). Given $x_t$ and $y_t$, we can derive from (11) that

$$x_{t+1} = \frac{\lambda [N y_t^\alpha + (1 - N)] x_t^\alpha}{\lambda N y_{t+1} + (1 - N)}.$$  

We substitute this into (10) to obtain

$$\delta [x_t^{\alpha - 1} \lambda (N y_t^\alpha + (1 - N))]^{\alpha} \frac{y_{t+1}^\alpha - (1 - \alpha)}{[\lambda N y_{t+1} + (1 - N)]^{\alpha}} = \lambda \alpha y_t.$$

Here the function $f(y_{t+1}) = \frac{y_{t+1}^\alpha - (1 - \alpha)}{[\lambda N y_{t+1} + (1 - N)]^{\alpha}}$ is strictly increasing in $y_{t+1}$. Thus, $y_{t+1}$ satisfying the above equation must be unique for given $x_t$ and $y_t$. Then $x_{t+1}$ is uniquely determined by (11) once $y_{t+1}$ as well as $x_t$ and $y_t$ are uniquely determined. Thus an equilibrium path becomes unique once $(x, y_1)$ is determined. Finally, we show that such unique path must have the property that $y_t < 1$ implies $y_{t+1} < 1$, which proves that the switching period $T$ becomes unique. To see this, consider some period $\tau$ such that $y_{\tau} < 1$ but $y_{\tau+1} \geq 1$. Note that (A1) implies that

$$x_{\tau+1}^{\alpha - 1} \delta (y_{\tau+1}^\alpha - (1 - \alpha)) \leq \frac{\alpha y_{\tau}}{N y_{\tau}^\alpha + (1 - N)}$$

due to $\lambda y_{\tau+1} \leq 1$ (Lemma 3). Then the above inequality together with $x_{\tau+1} \leq \bar{x} \equiv \Gamma^{1/(\alpha - 1)}$ implies that $\Gamma \delta \leq 1$ because $y_{\tau+1} \geq 1$ and $y_{\tau}/(N y_{\tau}^\alpha + (1 - N)) \leq 1$ by $y_{\tau} < 1$. Since $\delta \Gamma > 1$ by Assumption 2, this is a contradiction. Thus $y_{\tau} < 1$ must imply that $y_{\tau+1} < 1$. This shows that the switching period $T$ becomes unique in any equilibrium path.

**Proof of Proposition 2.**

(i): By Proposition 1 we know that $y_t$ decreases over time before the switching period $T$: $y_1 > y_2 > \cdots y_{T-1} > 1 > y_T$. Then in the growth phase in which $y_t \geq 1$, (11) implies that

$$\lambda x_t^\alpha \leq x_{t+1}, \quad \forall \ t \leq T - 1$$

because $\lambda y_{t+1} \leq 1$ in any period $t$ (Lemma 3). Since $x_0 < \bar{x} \equiv \lambda^{1/(1-\alpha)}$ (Assumption 1), we can show from $\lambda x_t^\alpha \leq x_{t+1}$ for $t \leq T - 1$ that $x_t < x_{t+1}$ for all $t \leq T < T$ for some $\hat{T} < T$. Since $r_t = \alpha x_t^{\alpha - 1}$, $r_t$ decreases for all $t \leq \hat{T}$.

(ii) Note that (11) implies that $\lambda x_t^\alpha \leq x_{t+1}$ holds if and only if $y_t^\alpha \geq \lambda y_{t+1}$. Now we show that, when $y_t^\alpha \leq \lambda y_{t+1}$, we have $x_t \geq \bar{x} \equiv \lambda^{1/(1-\alpha)}$. Suppose contrary to this claim that $y_t^\alpha \leq \lambda y_{t+1}$ but $x_{t+1} > \bar{x}$ holds in some period $t$. Then $t \geq \hat{T}$ must hold for a switching period $T$ because otherwise $y_t > 1$ and $y_t > y_{t+1}$ hold for $t < T$ due to Proposition 1 so that $y_t^\alpha \geq \lambda y_t > \lambda y_{t+1}$ because of $y_t \leq \lambda$ (Lemma 3). Thus $t \geq \hat{T}$.
holds so that $y_t < 1$. Second, recall that $y_t^0 - (1 - \alpha) \geq \lambda_0 y_t$ must hold in any period $t \geq 1$. Then, by using $x_{t+1} \leq \bar{x}$ and $y_t^0 - (1 - \alpha) \geq \lambda_0 y_{t+1}$, (A1) shows that

$$\frac{\delta x^{\alpha-1}}{\lambda y_{t+1} + (1 - N)} \leq \frac{y_t^0}{N y_t^0 + (1 - N)} y_t^{1-\alpha}.$$ 

Letting $\xi(s) \equiv s/(sN + (1 - N))$ and noting that $\xi$ is increasing in $s$, we have $\xi(\lambda y_{t+1}) \geq \xi(y_t^0)$ due to $\lambda y_{t+1} \geq y_t^0$. Thus the above inequality implies that $\delta x^{\alpha-1} \xi(\lambda y_{t+1}) \leq \xi(y_t^0) y_t^{1-\alpha} \leq \xi(\lambda y_{t+1}) y_t^{1-\alpha}$ so that $\delta \leq \lambda y_t^{1-\alpha} \leq \lambda$ due to $y_t \leq 1$. However, since Assumption 2 implies that $\delta > \lambda$, this is a contradiction. Thus $y_t^0 \leq \lambda y_{t+1}$ must imply that $x_{t+1} > \bar{x}$.

In summary, for all $t < T$, we have $y_t^0 \geq \lambda y_t > \lambda y_{t+1}$ due to $y_t > 1$ and $y_t > y_{t+1}$. Thus $\lambda x_t^0 < x_{t+1}$ for all $t < T$. For all $t \geq T$, $\lambda x_t^0 < x_{t+1}$ holds if $y_t^0 > \lambda y_{t+1}$ and $x_{t+1} > \bar{x}$ otherwise respectively. In the latter case, we have $\lambda x_t^0 \geq \bar{x}$, which shows the following: if $y_t^0 > \lambda y_{t+1}$, then $\lambda x_t^0 \leq x_{t+2}$ so that $\bar{x} < x_{t+2}$. On the other hand, if $y_t^0 < \lambda y_{t+1}$, then $x_{t+2} > \bar{x}$ again. Repeating this, we must have $x_t > \bar{x}$ for all $\tau \geq t + 1$ once $x_{t+1} > \bar{x}$. In this way, it must be either that $\lambda x_t^0 < x_{t+1}$ holds for all $t$ or that $\lambda y_t^0 \leq \lambda y_{t+1}$ holds in some period $t \geq 1$, which implies that $x_t > \bar{x}$ holds for all $\tau \geq t + 1$. In either case $x_t > \bar{x}$ must be eventually satisfied for all large $t$ in any equilibrium path. Q.E.D.

**Proof of Proposition 3.**

Suppose that Assumption 3 holds. The steady state capital $(\bar{x}, \bar{y})$ satisfies

$$\delta \{\bar{y}^0 - (1 - \alpha)\} = \lambda \alpha \bar{y}$$

(A4)

and

$$\lambda [N \bar{y}^0 + (1 - N)] \bar{x}^\alpha = \bar{x} [\lambda N \bar{y} + (1 - N)]$$

(A5)

Under Assumption 3 ($\delta > \hat{\delta}$), we have $\delta > \lambda^0 \alpha/(1 - \lambda^0 (1 - \alpha))$. Then we can readily show that there exists a unique $\bar{y} \in (\bar{y}, 1/\lambda)$ such that (A4) holds. Then, by (A5) $\bar{x}$ is also unique.

The proof about the convergence of equilibrium paths to the steady state $(\bar{x}, \bar{y})$ is lengthy and hence relegated to the Online Appendix (Appendix A). Q.E.D.

**Proof of Proposition 4.**

In the steady state $(\bar{x}, \bar{y})$, (A5) implies that

$$\lambda \bar{x}^\alpha = \left[ \frac{\lambda N \bar{y} + (1 - N)}{N \bar{y}^0 + (1 - N)} \right] \bar{x}.$$ 

(A6)

Here, $\lambda N \bar{y} + (1 - N) < N \bar{y}^0 + (1 - N)$ because of $\bar{y} < \hat{\lambda} \equiv \lambda^{1/(\alpha-1)}$. Thus $\bar{x} > \bar{x}$. Furthermore, we can show that $\lambda N \bar{y} + (1 - N)$ is decreasing in $N$. Thus $\bar{x}$ increases

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Since $\lambda \Gamma < 1$ and $\delta \Gamma > f(1) = 1$, we have $\delta > \lambda \delta \Gamma > \lambda$ so that $\delta > \lambda$.  

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with $N$. By using (A6) for $\tilde{x}$ and substituting it into $\tilde{Y}$, we can re-write the steady state GDP level $\tilde{Y}$ as

$$\tilde{Y} = \lambda^{\alpha/(1-\alpha)}[N\tilde{y}^\alpha + (1 - N)]^{1/(1-\alpha)}[\lambda N\tilde{y} + (1 - N)]^{-\alpha/(1-\alpha)}$$

Then we can show that $d\tilde{Y}/dN > 0$ if $(1 - \alpha)(\tilde{y}^\alpha - 1)(\lambda\tilde{y} - 1)N > -[\tilde{y}^\alpha - \lambda\alpha\tilde{y} - (1 - \alpha)]$. Assumption 3 ensures that $\delta > \lambda$, which in turn implies that $\tilde{y} < 1$. Combining this with $\lambda\tilde{y} < 1$ and $\tilde{y}^\alpha - (1 - \alpha) \geq \lambda\alpha\tilde{y}$ (due to IC in the steady state), the above inequality $(1 - \alpha)(\tilde{y}^\alpha - 1)(\lambda\tilde{y} - 1)N > -[\tilde{y}^\alpha - \lambda\alpha\tilde{y} - (1 - \alpha)]$ holds so that $d\tilde{Y}/dN > 0$. Q.E.D.

**Proof of Corollary.**

We can readily see from (A4) that, since $\tilde{y} < 1$ and $\delta > \lambda$ implied by Assumption 3,

$$\frac{\partial \tilde{y}}{\partial \lambda} = \frac{\alpha \tilde{y}}{\delta \alpha \tilde{y}^{\alpha-1} - \lambda \alpha} > 0.$$ 

We define

$$\omega(y) \equiv N\tilde{y}^\alpha + (1 - N)\lambda \tilde{y} + (1 - N)$$

where we verify that $\omega'$ has the same sign as $\tilde{\omega}(\hat{y}) \equiv \lambda N y^\alpha (\alpha - 1) + (1 - N)(\alpha y^{\alpha-1} - \lambda)$ and that $\tilde{\omega}' < 0$. Also, $\tilde{\omega}(0) = +\infty$ and $\tilde{\omega}(1) \geq 0$ if and only if $y \leq \alpha(1 - N)/(1 - \alpha N)$ which we assume. Thus $\omega' > 0$ for all $y \in [0, 1]$ under our assumption. Also, from (A5) we have $\tilde{x} = [\lambda \omega(\tilde{y})]^{1/(1-\alpha)}$. Then we can compute that

$$\frac{d\tilde{x}}{d\lambda} = \frac{\partial \tilde{x}}{\partial \lambda} + \frac{\partial \tilde{x}}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial \lambda} > 0$$

because $\partial \tilde{x}/\partial \lambda > 0$, $\partial \tilde{x}/\partial \tilde{y} > 0$ (due to $\lambda \leq \alpha(1 - N)/(1 - \alpha N)$) and $\partial \tilde{y}/\partial \lambda > 0$. Q.E.D.

**References**


