Name Your Own Price at Priceline.com: 
Strategic Bidding and Lockout Periods

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Abstract

A buyer suggests prices to \( N \) sellers in a time period and buys from the seller who accepts the bid first. The number of bidding rounds is determined by how frequently the buyer can make an offer. We show that with no limit on the frequency and without discounting, the price path is either kept flat initially with large jumps at the end or increasing steadily over time. Which class of path occurs in equilibrium depends on the buyer’s trade-off between committing to a price ceiling versus finely screening the sellers’ costs. With discounting, limiting the number of rounds mitigates the delay caused by the reluctance to raise bids in the first class of equilibrium, and therefore can benefit the buyer. This result suggests why, in reality, bargaining parties often take measures to make their offers rigid and consequently force themselves to make fewer offers.

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1 Introduction

Priceline.com, known for its Name Your Own Price (NYOP) system, is a website devoted to helping travelers to obtain discount rates for travel-related items such as airline tickets and hotel stays. The NYOP mechanism works as follows. First, a customer enters a bid that specifies the general characteristics of the desired item (travel dates, location, hotel rating, etc.) and the price that he is willing to pay. Next, Priceline.com either communicates the customer’s bid to participating sellers or accesses their private database to determine whether Priceline.com can satisfy the customer’s specified terms and the bid price. If a seller accepts the bid, the offer cannot be cancelled. If no seller accepts the bid, the customer can rebid either by changing the desired specifications or by waiting for a minimum period of time, the lockout period, before submitting a new, higher price offer.\(^1\)

To represent the Priceline auction, we use a dynamic model in which a single buyer suggests prices to \(N\) potential sellers for a finite number of rounds. The number of rounds \(T\) determines the length of the lockout period. By letting \(T\) go to infinity, we can consider the case of no lockout period. For simplicity, we assume that the buyer’s valuation is known. The sellers’ costs are privately known and independently drawn from a common distribution. We characterize the equilibrium bidding path and the timing of the transaction. We also show that the lockout period may often benefit the buyer, because it allows the buyer to commit to fewer rounds of bidding.

It is important to note that, although we build on the NYOP mechanism as a motivating example, the implications obtained here are not necessarily limited to this trading platform. The current analysis can be applied to a broad range of bargaining settings in which an uninformed buyer needs to screen out potential trading partners competing for providing a good/service: for instance, when a company chooses an outsourcing partner among several external providers, when a government agency procures goods and services among a pool of potential sellers, or when an employer makes offers to job candidates. In many of those situations, there is a deadline: after all, few negotiations, if any, can continue forever. Moreover, in virtually any bargaining process, it is time-consuming to make and/or respond to an offer because it takes time to fully understand possible consequences of the offer or because it

\(^1\)To make a bid at Priceline, a customer must enter his credit card number, billing address, phone number, and email address. This requirement helps Priceline to identify each customer and makes it difficult for a customer to create fake identities.
takes time to reach a consensus within the party (when the party making offers is an organization like a firm). At first glance, more chances to make offers should help the uninformed party to better screen the informed parties, and hence the uninformed party should try hard to alleviate any frictions which might cause delays in making offers. In reality, however, this does not seem to be the case, as the bargaining parties often take measures to make their offers more rigid: the parties hire intermediaries and let them bargain, often without conferring full authority; negotiations often take place during periodic meetings while communication outside the meetings is prohibited (or prohibitively costly). This rigid nature of bargaining is analogous to the lockout period in Priceline’s case. Our analysis suggests that the mechanisms that reduce the chances of making offers may actually enhance the welfare of the uninformed party, as it allows the party to commit to fewer rounds of offer revisions.\textsuperscript{3}

By reversing the roles of buyer and seller in the model, we can also consider the case where a seller with an indivisible unit makes price offers to several interested buyers, so the seller is confronted with a dilemma similar to the one in a durable goods monopoly. However, there are two differences between our setting and a durable goods monopoly. First, there is a deadline in our environment, so that the seller can wait until the end to set a monopoly price and is thus endowed with some commitment power. Second, the seller cannot produce additional units, so the scarcity of the good causes competition among buyers. Due to these two differences, our model has very different equilibrium paths than in models of the Coase conjecture and yields a positive profit for the seller.\textsuperscript{4} Applications include an airline or a cruise line selling a seat to travelers, or a landlord renting his apartment to tenants, etc. In those applications, there exists a deadline after which the object becomes valueless. It used to be the case that the seller could only advertise the price of the object in newspapers or on flyers; nowadays, the seller can post the price on his own website or other bulletin websites such as Craigslist. Hence, the

\textsuperscript{2}For instance, a car dealership hires salesmen without delegating much authority to slow down the bargaining process with customers. Similarly, a business firm hires a professional debt collector to collect its debts. There is now a large literature on delegated bargaining as a commitment device; see Schelling (1960) for more examples of delegated bargaining.

\textsuperscript{3}This type of reasoning can also be applied to other situations where the lockout period is exogenously imposed. For instance, an unsuccessful takeover bidder is prohibited from launching a new bid for a period of twelve months in countries such as the United Kingdom, Austria, Sweden, and China. While it seems to be a restriction on the bidder, it could actually benefit him as a commitment device.

\textsuperscript{4}The Coase conjecture asserts that with no constraint on the rate of sales, the monopolist’s price will fall immediately to the marginal cost in an infinite-period setting.
price can be updated much more frequently than before. Our result provides an indication of how the price paths can be expected to change, e.g., why last-minute deals have become more prevalent with the advent of the Internet.

Our paper has three main findings. First, we start the paper by showing that without a lockout period and with no discounting, the equilibria can be classified into two classes:

- the fully screening equilibrium in which the buyer keeps raising the bid until a seller accepts or until the price reaches the maximum cost level;
- the price ceiling equilibrium in which the buyer keeps the bids close to the price accepted by the minimum-cost seller until the very end and then reaches a price ceiling, which is lower than the maximum cost level, by jumps.

In the fully screening equilibrium, as the number of rounds $T$ goes to infinity, sellers’ types are almost fully separated through a sequence of gradually increasing bids by the buyer. That is, as $T$ increases, sellers’ types are separated more and more finely, and will be fully separated in the limit. The equilibrium is therefore efficient in the sense that the object is allocated to the seller with the highest valuation for the object. By contrast, in the price ceiling equilibrium, the bidding path is convexly increasing with most of the trades (if any) realized at the end. Sellers with costs bounded away from the minimum cost level are roughly partitioned into a finite number of groups depending on the interval in which a seller’s cost falls, even in the limit where $T$ goes to infinity. This equilibrium, which is inefficient, will be our main focus.

Which class of path arises in equilibrium is determined by the buyer’s trade-off between finely screening the sellers’ costs and successfully committing to a price ceiling. Ideally, to maximize his payoff, the buyer would like to commit to a strategy in which he gradually raises the bid to price-discriminate among the sellers and stops at the optimal reserve price, much like in a Dutch auction with a reserve price, but in reverse. The implementation of this optimal mechanism, however, requires full commitment power on the buyer’s part. In the absence of commitment power, if the buyer gradually raises the bids and screens out the sellers finely, he inevitably obtains information about the distribution of the sellers’ costs along the way, thereby making him unable to stop at the optimal reserve price. The only way to avoid this and successfully commit to the reserve price is by keeping the bids low until the
very end and making serious bids only in the last few rounds: that way, the buyer does not obtain much useful information to react to in early rounds. This strategy obviously comes with a cost, as it virtually wastes early bidding opportunities to screen out the sellers. The price ceiling equilibrium emerges when the benefit of committing to a reserve price outweighs the cost of wasting bidding opportunities.

The net benefit of committing to a reserve price largely depends on two factors, the buyer’s valuation for the object and the number of sellers. The price ceiling equilibrium emerges either when the buyer’s valuation is sufficiently low or when the number of sellers is sufficiently small. When the buyer’s valuation is low, he cares less about not obtaining the object, which essentially lowers the cost of setting a reserve price.\textsuperscript{5} When the number of sellers is small, competition among the sellers becomes less intense; in that case, a reserve price is valuable since it forces sellers to accept below what otherwise would have been accepted from the competition.

After characterizing the equilibrium path, we next show that without a lockout period, the expected payoff of a buyer is weakly higher than that in a first-price reverse auction (where sellers submit their bids to a buyer) without a reserve price, but lower than that in a first-price reverse auction with the optimal reserve price. The first part of the result (that the expected payoff is higher than a first-price auction without a reserve price) comes from the fact that even though the buyer cannot commit to any bidding path, the presence of a deadline endows him with some commitment power, which enables him to achieve a payoff (weakly) higher than in a simple first-price auction.\textsuperscript{6} Moreover, the expected payoff is strictly higher than that in a reverse auction without a reserve price when the price ceiling equilibrium occurs.

Finally, we show that the lockout period affects the process of information revelation by reducing the number of bidding rounds. The lockout period restriction makes the buyer bid more aggressively early on, because the buyer does not need to be as concerned about the detrimental effects of learning the sellers’ information when he only has a few bidding opportunities left. This can be especially valuable if the buyer discounts the future enough that he wants to learn early about bookings. Thus, the lockout period can be advantageous to the buyer because it permits the buyer to commit to fewer rounds of bidding. The result that the lockout period can

\textsuperscript{5}Note that the probability of not obtaining the object increases as the reserve price decreases.

\textsuperscript{6}The second part (that the expected payoff is lower than a first-price auction with the optimal reserve price) should be obvious from the discussion made thus far, as it simply indicates that the buyer cannot implement the optimal mechanism.
be valuable is in line with McAdams and Schwarz’s (2007) view that an intermediary can create value by offering a credible commitment device.

The inability to make a commitment has been studied in the literature on contract theory and mechanism design. In contract theory, the possibility of renegotiation is shown to slow down the speed of information revelation, because information revealed through contract execution leads to recontracting opportunities that are detrimental to the principal from the \textit{ex ante} point of view (see, e.g., Freixas, Guesnerie, and Tirole (1985), Laffont and Tirole (1987, 1988), Hart and Tirole (1988), and Dewatripont (1989)). This intention to suppress information revelation is also observed in the price ceiling equilibrium characterized in our model. In mechanism design, McAfee and Vincent (1997) and Skreta (2006, 2011) examine the situation where an uninformed seller can commit to a mechanism for the current round, but cannot make a commitment for future rounds if the object is not sold. McAdams and Schwarz (2007) further consider a seller who cannot commit to a mechanism even for the current round; the seller in their model conducts a first-price auction, but after observing the buyers’ bids, the seller cannot commit not to asking for more rounds of bids. They show that when the cost of delay is substantial but not very large, the lack of commitment results in multiple rounds of bidding and causes significant loss to the seller due to delay.\footnote{When the cost of delay is very large, the seller can credibly commit to a first-price auction; and when the cost of delay is very small, the outcome approximates that of an English auction.} With the roles of buyer and seller reversed in our model, our paper is another exploration of the environment in which the seller cannot commit to the current mechanism. The seller in our setting conducts a Dutch auction in a given time period; it takes the seller a short amount of time to revise his asking price, so that the seller has a finite, but large, number of chances to revise the price. While in a Dutch auction the optimal price path simply consists of prices steadily increasing to the optimal reserve price level, this paper shows that without a commitment to the optimal path, the path realized will be very different from the optimal path, and as in McAdams and Schwarz (2007) the seller suffers from the inability to commit.

In addition to the literature on contract theory and mechanism design with limited commitment, our analysis is related to two other strands of the literature, namely, bargaining and the Coase conjecture. With regard to bargaining, the convexly increasing path characterized in the price ceiling equilibrium is related to the deadline effect observed in many bargaining processes and also in experiments (see,
e.g., Roth, Murnighan, and Schoumaker (1988)). The experimental evidence shows that a high percentage of agreements are reached just before the deadline, and the frequency of disagreements is non-negligible. Among the explanations provided in the literature, the environments and rationales proposed by Hart (1989) and Spier (1992) on strikes and pretrial negotiations are the closest to ours. Both consider bilateral sequential bargaining models with one-sided incomplete information and a deadline. Hart (1989) assumes the existence of a crunch, after which a deadline (when the firm becomes valueless) arrives with positive probability. The stochastic arrival of a deadline is equivalent to having a higher discount factor after the crunch, and so the deadline effect disappears when the time interval goes to zero. By contrast, both Spier’s model and ours assume a fixed deadline, which is essential in giving rise to the deadline effect even when the time interval approaches zero.\footnote{See Fuchs and Skrzypacz (2011) for a discussion on the essential role of a stochastic deadline in this type of setup.}

With regard to the Coase conjecture, the environment studied here is similar to a durable goods monopoly, but with the roles of buyer and seller reversed. To avoid confusion, let us call the uninformed side, which is also the side determining the price, “the principal” and the other side “the agents”. Much of the durable goods theory explores the conditions under which a monopolist makes a positive profit. Kahn (1986) and McAfee and Wiseman (2008) show that with a capacity cost (i.e., an increased cost of increased production speed), the principal has the ability to restrict future sales and thus makes a positive profit. Nevertheless, the static monopoly profit cannot be reached because the principal cannot help but eventually trade with every type of agent. In our setting, the unit demand of the principal is like a capacity constraint, which endows the principal with commitment power and induces competition among agents. Moreover, the presence of a deadline further helps the principal to commit to stopping the trade even though his demand is not fulfilled.\footnote{Stokey’s (1981) discrete-time model also considers the case with a deadline and shows that the Coase conjecture still holds when the length of the period shrinks. Her conclusion is different from ours because in our model, (i) the concern over losing a chance to trade serves as a screening device, and (ii) an agent derives the same utility regardless of when the good is received, whereas in Stokey’s model, (i) an agent discounts the future so that time behaves as a screening device, and (ii) an agent derives less utility if the trade occurs close to the deadline, and this reduces the inclination to trade at the end.} The two features in our setting, competition among agents due to the principal’s capacity constraint and the presence of a deadline, put the principal in a favorable position and allow for a profit higher than the static monopoly profit.
Lastly, our paper is closely related to Horner and Samuelson (2011). While our papers were developed independently, both identify and characterize the two classes of equilibrium paths. Assuming that the agents’ types are uniformly distributed, Horner and Samuelson prove the uniqueness of the equilibrium and show that the price ceiling equilibrium arises when the number of agents is large enough. Our paper proves that the equilibrium paths must be one of the two classes for a large set of distributions including the uniform distribution, and show that the price ceiling equilibrium arises when the principal’s (buyer’s) valuation is sufficiently low. We also study discounting and discuss the inefficiency caused by late transactions, which is an important feature of the class of the price ceiling equilibrium.

The remainder of this paper is organized as follows. Section 2 describes the model and presents an example that motivates our research. Section 3 derives the equilibrium path. Section 4 characterizes the equilibrium bidding behavior. Section 5 incorporates waiting cost into the model to characterize the optimal lockout period for the buyer. Section 6 discusses extensions and concludes.

2 The Model and an Example

There are \( N \geq 2 \) sellers and one buyer in the market. The buyer demands one unit of the good provided by one of the sellers. The buyer’s reservation value for the good is \( v \), which is common knowledge. Seller \( i \) privately knows his cost \( \theta^i \) of providing the good. Each \( \theta^i \) is independently and identically distributed on the interval \( [c, \bar{c}] \), where \( c \geq 0 \) and \( \bar{c} \leq v \), according to a distribution function \( F \). \( F \) is smooth, i.e., of class \( C^\infty \), and has a density \( f \) with full support. We further assume that \( x + \frac{F(x)}{f(x)} \) strictly increases in \( x \). The buyer’s payoff is \( v - b \), where \( b \) is his payment to the seller, if he gets the object, and zero otherwise. All the players are risk neutral.

There are \( T \) rounds. In each round the buyer can make exactly one offer to all sellers. In round \( t \), the buyer announces a price, the bid, and each seller decides whether to accept it or not. As soon as a seller accepts a bid, the game ends. If \( n \) sellers accept the bid in a round, each of them gets to provide the good with probability \( \frac{1}{n} \). If no seller accepts and \( t < T \), the process proceeds to the next round, with the buyer submitting a new bid. If \( t = T \), the market closes.
2.1 Equilibrium concept

The equilibrium concept used in this paper is the Perfect Bayesian Equilibrium (PBE). Only symmetric pure strategy equilibria are considered. Let $p_t$ be the bid that the buyer offers the sellers in round $t$, and denote by $h_t = (p_1, p_2, \cdots, p_t)$ the history of the bids submitted by the buyer in the first $t$ rounds.

The buyer’s strategy is a set of functions $\{b_t(h_{t-1})\}_{t=1}^{T}$, where $b_t(h_{t-1})$ is the bid that the buyer plans to submit in round $t$ given the price history $h_{t-1}$ and the fact that no seller accepts in the first $t-1$ rounds. One can show that in equilibrium, if a seller with cost $x$ accepts in round $t$, a seller with cost $x'<x$ also accepts in round $t$ (see Appendix B for detail). Therefore, a seller’s strategy can be summarized by a set of cutoff values of cost, $\{x_t(h_t)\}_{t=1}^{T}$, where $x_t(h_t)$ is such that in round $t$, given $h_t$, a seller accepts the buyer’s offer if and only if his cost is less than $x_t(h_t)$. Conditional on the fact that no seller accepts in the first $t-1$ rounds, let $y_t(h_{t-1})$ be the greatest lower bound of a seller’s cost believed by the other players given history $h_{t-1}$. Denote by $u_t^0(b, x \mid h_{t-1}, y_t(h_{t-1}))$ the buyer’s expected utility, and $u_t^i(b, x^{-i}, x^i \mid h_t, \theta^i, y_t(h_{t-1}))$ seller $i$’s expected utility, where $x^{-i}$ is the other sellers’ strategy, $x^i$ is seller $i$’s strategy, and $\theta^i$ is the realization of seller $i$’s cost.\footnote{10$x^{-i}$ is a tuple consisting of the other sellers’ strategies. In a symmetric equilibrium, $x^{-i}$ can be represented by a single function.}

Definition 1 A symmetric equilibrium is a $(b, y, x)$ that satisfies

(a) $y_{t+1}(h_t) = \max \{x_t(h_t), x_{t-1}(h_{t-1}), \cdots, x_1(h_1)\}, \forall t, h_t,$ and

(b) $u_t^0(b, x \mid h_{t-1}, y_t(h_{t-1})) \geq u_t^0(b', x \mid h_{t-1}, y_t(h_{t-1}))$ and

$u_t^i(b, x, x \mid h_t, \theta^i, y_t(h_{t-1})) \geq u_t^i(b, x, x' \mid h_t, \theta^i, y_t(h_{t-1})), \forall b', x', t, h_t, h_{t-1}.$

A seller with cost below $\max \{x_t(h_t), x_{t-1}(h_{t-1}), \cdots, x_1(h_1)\}$ would have accepted a bid occurring in the first $t$ rounds. So if seller $i$ has not accepted any bid in the first $t$ rounds, the other players believe that his cost is above $\max \{x_t(h_t), x_{t-1}(h_{t-1}), \cdots, x_1(h_1)\}$, as characterized in Condition (a). Condition (b) means that players cannot achieve more by deviating from the equilibrium strategy.

2.2 An Example

We use a simple example in which $N = 2$, $v = 1$, and $F$ is a uniform distribution on $[0, 1]$ to illustrate the derivation of the equilibrium path and the main results developed in the following sections. Given this setting, in a first-price reverse auction,
the buyer’s expected payoff is $\frac{1}{3}$. If the buyer is allowed to set a reserve price, on the other hand, he sets the price at $\frac{1}{2}$ and the expected payoff is $\frac{5}{12}$, which is the same as the expected payoff in Myerson’s optimal mechanism.

**T=2:** First we consider the case where there are two rounds, i.e., $T = 2$. We solve the equilibrium path from the second round. Suppose that, in the first round, the buyer offers $b_1$ and no seller accepts. Then, a seller’s cost is believed to be above $x_1(b_1)$, and the updated belief about the distribution of a seller’s cost is uniform on $[x_1(b_1), 1]$. In the second round which is also the last round, a seller accepts as long as the bid $b_2$ is higher than his cost. Thus, $x_2(b_1, b_2) = b_2$. In expecting the sellers’ strategy $x_2(b_1, b_2)$, the buyer bids at $b_2(b_1) = 1 - \frac{1-x_1(b_1)}{\sqrt{3}}$ to maximize his expected payoff.

In the first round, suppose that the buyer has submitted a bid $b_1$. A seller, seller $i$, with cost $x$ must decide whether to accept $b_1$ in this round or wait until the next round. Seller $i$ expects that the other seller, seller $j$, will accept if seller $j$’s cost is below $x_1(b_1)$. If seller $i$ accepts in this round, with probability $x_1(b_1)$, seller $j$ accepts too, and each of them gets to sell with probability $\frac{1}{2}$; and with probability $1 - x_1(b_1)$, seller $j$ does not accept, and seller $i$ gets to sell with probability one. Therefore, seller $i$’s expected payoff is

$$ (b_1 - x) \left[ \frac{1}{2} x_1(b_1) + (1 - x_1(b_1)) \right]. \quad (1) $$

If seller $i$ waits, with probability $1 - x_1(b_1)$, the game moves to the next round. In the second round, the buyer is expected to submit $b_2(b_1)$, and seller $i$ will accept if his cost is below $b_2(b_1)$. If seller $i$ accepts, with probability $\frac{b_2(b_1) - x_1(b_1)}{1 - x_1(b_1)}$, seller $j$ also accepts, and each of them gets to sell with probability $\frac{1}{2}$; and with probability $\frac{1 - b_2(b_1)}{1 - x_1(b_1)}$, seller $j$ does not accept, and seller $i$ gets to sell with probability one. Therefore, seller $i$’s expected payoff is

$$ (1 - x_1(b_1)) (b_2(b_1) - x) \left[ \frac{1}{2} \frac{b_2(b_1) - x_1(b_1)}{1 - x_1(b_1)} + \frac{1 - b_2(b_1)}{1 - x_1(b_1)} \right]. \quad (2) $$

By comparing the two payoffs in (1) and (2), a seller is better off to accept $b_1$ in the first round if and only if his cost is below $x_1(b_1) = 1 - \frac{-3b_1 + \sqrt{9b_1^2 + 12(1-b_1)}}{2}$. In expecting the sellers’ strategy $x_1(b_1)$, the buyer chooses $b_1$ to maximize his total expected revenue in the two rounds. The buyer’s problem in the first round is
Table 1: Equilibrium outcomes for $T = 1, 2, 3, 4, 5$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>Buyer’s Payoff</th>
<th>$E(\tau)$</th>
<th>$x_{T-4}$</th>
<th>$x_{T-3}$</th>
<th>$x_{T-2}$</th>
<th>$x_{T-1}$</th>
<th>$x_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.38490</td>
<td>0</td>
<td>0.4225 (0.4225)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.40024</td>
<td>0.2972</td>
<td>0.1709 (0.4214)</td>
<td>0.5212 (0.5212)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.40111</td>
<td>0.4563</td>
<td>0.0597 (0.4099)</td>
<td>0.2165 (0.4538)</td>
<td>0.5475 (0.5475)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.40115</td>
<td>0.5826</td>
<td>0.0154 (0.4007)</td>
<td>0.0597 (0.4127)</td>
<td>0.2165 (0.4538)</td>
<td>0.5475 (0.5475)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.40115</td>
<td>0.6626</td>
<td>0.0070 (0.3990)</td>
<td>0.0154 (0.4021)</td>
<td>0.0597 (0.4127)</td>
<td>0.2165 (0.4538)</td>
<td>0.5475 (0.5475)</td>
</tr>
</tbody>
</table>

defined as

$$\max_{b_1} \left[ 1 - b_1 \right] \left[ 1 - (1 - x_1(b_1))^2 \right] + \left[ 1 - b_2(b_1) \right] \left[ (1 - x_1(b_1))^2 - (1 - b_2(b_1))^2 \right].$$

So, in equilibrium, $b_1 = 0.4214$, $b_2 = 0.5212$, $x_1 = 0.1709$, $x_2 = 0.5212$, and the buyer’s payoff is 0.40024.

$T=1\sim5$: Table 1 shows the equilibrium paths of the cutoff $x_t$ and the bid $b_t$, and the buyer’s expected payoffs when the number of rounds $T = 1, 2, 3, 4, 5$. Column $E(\tau)$ lists the expected transaction time conditional on the transaction occurring. We assume that the game begins at time 0 and ends at time 1. If the buyer’s bid in the $t$th round is accepted, the transaction occurs at $(t-1)/T$. There are several points worth noting:

1. The cost cutoff in round $T-\tau$, $x_{T-\tau}$, increases in $T$ and converges when $T$ goes to infinity.\(^{11,12}\) The buyer’s payoff increases in $T$, but the increment shrinks as $T$ increases. Therefore, the profit from having one more bidding chance shrinks as $T$ increases.

2. Given $T$, the cutoff path $x_t$ and the bidding path $b_t$ are increasing. With a larger $T$, however, the rate of increase is small in the first few rounds, and

\(^{11}\)The numerical results have four digits of precision, so the numbers in the table might not be sufficiently accurate to show small differences.

\(^{12}\)This is proved in Lemma 1 in Appendix A.
large jumps occur in the last few rounds. This represents one class of the equilibrium path. If we consider another example with $v = 1.5$, then the sequence of $x_t$ when $T = 5$ becomes \( \{0.1318, 0.2823, 0.4600, 0.6678, 0.9125\} \). This represents another class of the equilibrium path along which $x_t$ increases steadily over time (as characterized in Theorem 1).

3. The payoff is lower than the payoff in a reverse auction with the optimal reserve price for all values of $T$; and when $T$ is large enough (in this example, when $T \geq 1$), the payoff is higher than the payoff in a reverse auction with no reserve price (as shown by Theorem 2).

4. In equilibrium, the buyer does not get the object only if both sellers’ costs are above $x_T$. Therefore, the probability of the buyer getting the object increases in $T$, but the increment shrinks as $T$ increases. From the table, we see that when $T$ increases from 3 to 4, and to 5, neither the buyer’s payoff nor the probability that the buyer gets the object increases much. However, the expected transaction time is much later. This fact suggests that if the buyer has waiting cost and prefers earlier transactions, having fewer rounds might be good for him. The analysis in Section 5 confirms the conjecture.

3 Derivation of the Equilibrium Path

In this section, we briefly describe how we obtain the equilibrium path of this dynamic game and show its existence. The complete construction of the equilibrium strategy and belief is shown in the online appendix.

First, let

$$
\bar{F}(x) = 1 - F(x)
$$

be the counter-cumulative distribution function. In the last round $t = T$, a seller accepts the last-round bid $b_T$ as long as his cost is below $b_T$, so the cost cutoff $x_T = b_T$. Knowing this, and given the belief that all the sellers have costs higher than $x_{T-1}$, the buyer chooses $b_T$ to maximize his payoff:

$$
V_T(x_{T-1}) = \max_{b_T \in [x_{T-1}, x_T]} (v - b_T) P(x_{T-1}, x_T) \\
s.t. b_T = x_T.
$$

$$
P(x_{T-1}, x_T) \equiv \bar{F}(x_{T-1})^N - \bar{F}(x_T)^N
$$

is the probability that all the sellers’ costs
are above $x_{T-1}$ and that the demand is fulfilled, given that sellers with costs between $x_{T-1}$ and $x_T$ are willing to provide the good. The maximizers, $b_T(x_{T-1})$ and $\pi_T(x_{T-1})$, are the buyer’s bid and the sellers’ cost cutoff in the continuation equilibrium given belief $x_{T-1}$:

$$\left(b_T(x_{T-1}), \pi_T(x_{T-1})\right) \in \arg \max_{b_T, \pi_T \in [x_{T-1}, \pi]} (v - b_T)P(x_{T-1}, x_T) \quad \text{(P1)}$$

$$\text{s.t. } b_T = x_T.$$  

Next we proceed backward to round $t = T - 1$. Given belief $x_{t-1}$, suppose that the buyer chooses $b_t \in [x_{t-1}, \pi]$. The cost cutoff $x_t$ in round $t$ is determined by finding the type of seller who is indifferent between accepting in round $t$ and round $t + 1$:

$$x_t \in \chi(b_t, x_{t-1}) \equiv \left\{ x \in [x_{t-1}, \pi] \mid (b_t - x) \frac{G(x_{t-1}, x)}{NF(x_{t-1})^{N-1}} = \frac{C_{t+1}(x)}{NF(x_{t-1})^{N-1}} \right\}, \quad \text{(3)}$$

where $G(x_{t-1}, x) \equiv \sum_{n=0}^{N-1} \hat{F}(x)^{N-1-n} \hat{F}(x_{t-1})^n$ and $C_{t+1}(x) = (\bar{b}_{t+1}(x) - x)G(x, \pi_{t+1}(x))$. Given belief $x_{t-1}$, $\frac{G(x_{t-1}, x)}{NF(x_{t-1})^{N-1}}$ is the conditional probability that a seller with cost $x_t$ gets to sell the good if he accepts in round $t$: $\frac{C_{t+1}(x_t)}{NF(x_{t-1})^{N-1}}$ and $(b_t - x_t) \frac{G(x_{t-1}, x_t)}{NF(x_{t-1})^{N-1}}$ are the expected payoffs of a seller with cost $x_t$ if he accepts in round $t + 1$ and in round $t$, respectively. By expecting the other sellers to apply the cutoff strategy with cutoff $x_t \in \chi(b_t, x_{t-1})$, a seller with cost $x$ gets expected payoffs $(\bar{b}_{t+1}(x_t) - x) \frac{G(x_t, \pi_{t+1}(x_t))}{NF(x_{t-1})^{N-1}}$ and $(b_t - x) \frac{G(x_{t-1}, x)}{NF(x_{t-1})^{N-1}}$ if accepting in round $t + 1$ and in round $t$, respectively. In comparing the two payoffs, a seller is better off by accepting $b_t$ in round $t$ if and only if his cost is below $x_t$.\footnote{For some values of $b_t$, $\chi(b_t, x_{t-1})$ in (3) might be empty. In the online appendix, we construct a continuation equilibrium for the case when $\chi(b_t, x_{t-1})$ is empty and show that choosing $b_t \in \{b \mid \chi(b, x_{t-1}) = \emptyset\}$ is not an optimal strategy for the buyer. So, without loss of generality, we can focus on $b_t \in \{b \mid \chi(b, x_{t-1}) \neq \emptyset\}$ when deriving the equilibrium path.}

By expecting (3), the buyer chooses $\bar{b}_t(x_{t-1})$ to maximize his expected payoff.
V_t(x_{t-1}):

\[
V_t(x_{t-1}) = \max_{b_t, x_t \in [x_{t-1}, \overline{x}]} (v - b_t) P(x_{t-1}, x_t) + V_{t+1}(x_t) \\
\text{s.t. } (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t),
\]

\[
(\overline{b}_t(x_{t-1}), \overline{x}_t(x_{t-1})) \in \operatorname{arg\ max}_{b_t, x_t \in [x_{t-1}, \overline{x}]} (v - b_t) P(x_{t-1}, x_t) + V_{t+1}(x_t) \quad (P2)
\text{s.t. } (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t).
\]

Applying the same procedure backward, we obtain a sequence of \(\overline{b}_t(x_{t-1})\) and \(\overline{x}_t(x_{t-1})\), for \(t = 1, 2, \cdots, T\). However, we need to make sure that solutions to \((P1)\) and \((P2)\) exist. This is proved in the following proposition. Note that there might be multiple solutions to programs \((P1)\) and \((P2)\). In that case, only those that ensure the existence of equilibrium can be candidates for \(\overline{b}_t(x_{t-1})\) and \(\overline{x}_t(x_{t-1})\) (see the proof of Proposition 1 for more details).

**Proposition 1** There exists a set of solutions \(\{\overline{b}_t(x_{t-1}), \overline{x}_t(x_{t-1})\}_t\) that solves programs \((P1)\) and \((P2)\) for all \(t\).

**Sketch of Proof.** The details of the proof are in Appendix A. Here is the outline. First, by Berge’s maximum theorem, \(V_T(x_{T-1})\) is continuous, and the solution set of \(x_T\) for program \((P1)\) is upper hemi-continuous. Therefore, we are able to pick \(\overline{x}_T(x_{T-1})\) from the solution set such that \(C_T(x_{T-1})\) is lower semi-continuous. Next, by substituting the constraint into the objective function in round \(T - 1\) in program \((P2)\), the objective function is graph-continuous as defined in Leininger (1984), and by Leininger’s generalized maximum theorem, \(V_{T-1}\) is upper semi-continuous, and the solution set of \(x_{T-1}\) exists and is upper hemi-continuous. Applying the same procedure backward, we guarantee the existence of a solution to each round-\(t\) program.

The following remark summarizes the procedure adopted to derive the equilibrium path:

**Remark 1** The equilibrium path \(\{(b_1, \cdots, b_T), (x_1, \cdots, x_T)\}\) can be found by solving the recursive program

\[
V_t(x_{t-1}) = \max_{b_t, x_t \in [x_{t-1}, \overline{x}]} (v - b_t) P(x_{t-1}, x_t) + V_{t+1}(x_t) \quad (P3)
\text{s.t. } (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t),
\]
where \( x_0 = c \), \( V_{T+1}(x_T) = 0 \), and \( C_{T+1}(x_T) = 0 \). The value of the program with \( t = 1 \) is the buyer’s payoff in equilibrium.

Program (P3) shows that the equilibrium path \( \{(b_1, \ldots, b_T), (x_1, \ldots, x_T)\} \) maximizes the buyer’s payoff subject to two constraints. The first one is the sellers’ incentive-compatibility constraint, which ensures that sellers with different values will not deviate from their equilibrium choices. This constraint exists in every mechanism and is shown in the constraint part of the program. The second constraint is the buyer’s inability to commit to a bidding path over time, which amounts to the recursive form of the program where the buyer utilizes all of the available information to set a bid in each round. While this allows the buyer to adopt the \textit{ex post} optimal strategy, the lack of commitment on his part changes the prices that a seller, who expects the buyer’s strategy correctly, is willing to accept in early rounds, and hence prevents the buyer from achieving the \textit{ex ante} optimal outcome derived in Myerson (1981).

4 Equilibrium Bidding Behavior with No Lockout Period

In this section, we analyze the equilibrium bidding path when there is no lockout period so that the buyer can submit as many bids as desired, i.e., \( T \to \infty \). Two of our main results are established here. First, we characterize the equilibrium bidding path and show that any equilibrium can be classified into one of the following two classes: (i) a fully screening equilibrium and (ii) a price ceiling equilibrium. Second, we show that without a lockout period, the expected payoff of a buyer is weakly higher than that in a reverse auction without a reserve price, but lower than that in a reverse auction with the optimal reserve price.

4.1 Commitment and optimality

Before proceeding to the main result, we first consider the benchmark case where the buyer can commit to a bidding path in advance. We characterize the optimal path under commitment and show that, without commitment, Myerson’s optimal outcome for the buyer might not be attainable, even though the buyer can adjust the bid as frequently as he likes.
In our setting, a reverse Dutch auction with a reserve price $r$ such that $r + \frac{F_i(r)}{f_i(r)} = v$ implements Myerson’s optimal mechanism.\(^{14}\) In such a reverse Dutch auction, the price is continuously increased and stops at the reserve price. Therefore, if the buyer can commit and the number of rounds is large enough, the buyer can roughly duplicate the price path in a reverse Dutch auction and receive a payoff approximately the same as in an optimal mechanism.

Nonetheless, when commitment is not possible, the maximum payoff resulting from the optimal mechanism is not approximately achievable if the optimal auction design involves setting a reserve price. To see this, recall that on the equilibrium path, the last-round $b_T$ and $x_T$ can be found by solving

$$x_T = b_T = \arg \max_b (v - b) \left[ \tilde{F} (x_{T-1})^N - \tilde{F} (b)^N \right].$$

A necessary condition for $b_T$ is

$$\tilde{F} (x_{T-1})^N = \tilde{F} (b_T)^N + (v - b_T) N \tilde{F} (b_T)^{N-1} f (b_T). \tag{4}$$

Suppose that the optimal auction involves setting a reserve price $r < \bar{c}$. If the optimal auction can be approximately implemented when $T$ goes to infinity, then it must be that $\lim_{T \to \infty} b_T = \lim_{T \to \infty} x_T = r$ and $\lim_{T \to \infty} x_{T-1} = r$. Nevertheless, by equation (4), if $\lim_{T \to \infty} b_T = r$, $\lim_{T \to \infty} x_{T-1} < r$, so that the optimal auction cannot be approximately implemented.

### 4.2 Characterization of the equilibrium paths

In characterizing the equilibrium path, the key question is how the buyer designs the bidding path to screen out the sellers. When it is possible to commit to the entire bidding path, it is optimal for the buyer to differentiate the sellers by gradually raising the bid. However, this strategy of finely screening out the sellers may not be optimal when the buyer is unable to commit, as he would inevitably react to any new information that can be gained along the way. We show that, in general, all the possible equilibrium paths can be classified into two classes: either the sellers with different costs are almost fully separated so the sellers’ private information is revealed gradually over time, or they are pooled into several cost intervals and most information about the sellers’ costs is revealed just before the deadline.

\(^{14}\)If $v \geq \bar{c} + \frac{1}{f_i(\bar{c})}$, no reserve price is required to implement the optimal mechanism.
To obtain our characterization result, we restrict our attention to the environment in which there exists a continuous solution function $\pi^T_t(x_{t-1})$ for programs (P1) and (P2) (wherein the superscript $T$ represents the total number of rounds). We can show that such a continuous function exists if the distribution is of the form $\tilde{F}(x) = \left(\frac{v-x}{\bar{v}-\underline{v}}\right)^{a}$ where $x \in [\underline{c}, \bar{c}]$, $\bar{c} \leq v$, and $a \geq 1$. An equilibrium derived from continuous functions $\pi^T_t(x_{t-1})$ is a reasonable equilibrium to focus on, since it has the property that, in round $t$, the bid $b_t$ and the cutoff $x_t$ are continuous functions of the starting belief $x_{t-1}$. Moreover, we can prove that there exists, at most, one continuous solution function $\pi^T_t(x_{t-1})$ for each $t$, so that the equilibrium derived is unique. In the remaining analysis, we will focus on the unique equilibrium derived from continuous functions $\pi^T_t(x_{t-1})$.

**Condition 1** $\tilde{F}$ is such that (i) there exists a continuous solution function $\pi^T_t(x_{t-1})$ for programs (P1) and (P2) for all values of $T-t$, and (ii) the limit $\lim_{T-t \to \infty} \pi^T_t(x_{t-1})$ is also continuous.

**Remark 2** We focus on the unique equilibrium derived from continuous functions $\pi^T_t(x_{t-1})$.

To state the first main result, we introduce some notation. Let $x^T_t$ be the cutoff in round $t$ on the equilibrium path with total number of rounds $T$, let $X^T = \{x^T_t\}_{t=1}^T$, and denote by $\|A\|$ the number of elements in set $A$. The following defines a cluster point of the cutoff set $X^T$ when $T \to \infty$.

**Definition 2** $z \in [\underline{c}, \bar{c}]$ is a cluster point if for any $\epsilon > 0$ and $M > 0$, there exists $T'(\epsilon, M)$ such that for all $T > T'(\epsilon, M)$, $\|\{x \in X^T \mid |x - z| < \epsilon\}\| > M$.

Definition 2 implies that if $z$ is a cluster point, the number of $x_t$’s falling around $z$ increases with $T$. Let $B$ be the set of cluster points. Theorem 1 states how the cluster points are distributed in the cost interval $[\underline{c}, \bar{c}]$.

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15To derive equilibrium paths, we solve programs (P1) and (P2). The solution $\pi^T_t(x_{t-1})$ enters the constraint parts of all the round-$\tau$ programs, where $\tau < t$. Therefore, to apply the envelope theorem, we restrict our attention to the setting in which there exists a continuous solution function $\pi^T_t(x_{t-1})$. Since $\tilde{F}$ is smooth, the continuity of $\pi^T_t(x_{t-1})$ implies that $\pi^T_t(x_{t-1})$ is also smooth for all $t$ (as shown in Proposition 5 in the online appendix). Thus, the objective functions and the constraints of the programs in Section 3 are differentiable, and the envelope theorem can be applied.

16See Proposition 7 in the online appendix.

17See Proposition 6 in the online appendix.

18With uniqueness, given starting belief $x_{t-1}$, the same continuation equilibrium will be triggered in all continuation games with the same number of rounds left $T-t+1$, i.e., $\pi^T_t(x_{t-1}) = \pi^T_t(x_{t-1})$ if $T - t = T' - t'$. 

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Theorem 1 Assume Condition 1.

1. The cluster point set $B$ is either the whole interval $[c, \bar{c}]$ or a single point $\{c\}$, i.e., $B = [c, \bar{c}]$ or $\{c\}$.

2. The cluster point set $B$ is a single point $\{c\}$ if and only if the last period cutoff $x_T^T$ is bounded away from $\bar{c}$ when $T \to \infty$, i.e., $B = \{c\}$ if and only if $\lim_{T \to \infty} x_T^T < \bar{c}$.

3. If $B = [c, \bar{c}]$, the buyer’s payoff is approximately the same as that in a reverse auction without a reserve price.

Sketch of Proof. The details of the proof are in Appendix A. Here is the sketch.

We first show that if the number of rounds left in a continuation game starting with belief $x_{t-1}$ goes to infinity, then the difference between $x_t$ and $x_{t-1}$ goes to zero. So, if $a \in [c, \bar{c}]$ is a cluster point, any point $x < a$ must be a cluster point, too. However, we can also show that it cannot be the case wherein $a \in (c, \bar{c})$, $[c, a]$ belongs to the cluster point set $B$, and $(a, \bar{c})$ belongs to the complement of $B$ because such a path implies that the buyer does not choose the optimal strategy in a certain round, and thus cannot be an equilibrium path. Therefore, the cluster point set is either $[c, \bar{c}]$ or $\{c\}$. The third statement follows from the revenue equivalence theorem.

The theorem states that any equilibrium can be classified into either one of the two classes, each of which exhibits different properties and offers different implications. For expositional purposes, we refer to the equilibrium with the cluster point set $[c, \bar{c}]$ as the fully screening equilibrium and refer to the equilibrium with the cluster point set $\{c\}$ as the price ceiling equilibrium. To see why these two classes of paths arise in equilibrium, notice that in our setting, a reverse Dutch auction with the optimal reserve price implements Myerson’s optimal mechanism: if the buyer can commit, it is in his best interest to commit to a strategy in which he gradually raises the bid to fully price discriminate among the sellers until it reaches the optimal reserve price. This is not implementable, however, when the buyer is unable to commit to a bidding path because he cannot avoid responding to the information revealed by sellers’ rejections. To commit to a price ceiling, therefore, it is imperative to ensure that the buyer does not gain much information from rejections, which can be achieved by keeping the bids low enough because rejecting a low bid does not reveal much about the sellers’ costs. With the lack of commitment, the buyer
thus faces a cumbersome trade-off: on the one hand, he would like to raise the bid gradually to better screen out the sellers; on the other hand, he also needs to keep the bids low enough not to gain much information to respond to. If the benefit of having a price ceiling dominates the benefit of fine screening, the buyer will keep the bids sufficiently low in initial rounds, followed by large jumps at the end to the reserve price; if not, the buyer will increase the bid steadily over time until it reaches the upper bound $\overline{c}$.

The distribution of the cluster points also indicates several notable properties of each class of equilibrium. First, in the fully screening equilibrium where the cluster point set spans over the whole interval $[c, \overline{c}]$, the cutoffs are more evenly distributed over the interval. The sellers are more finely screened as the number of rounds $T$ increases, and fully separated in the limit. Information is revealed more gradually over time, and a transaction is more likely to occur earlier than in the price ceiling equilibrium.

By contrast, in the price ceiling equilibrium where the cluster point set is a single point $\{c\}$, while the number of cutoffs falling in interval $[c, c + \epsilon]$ for an arbitrarily small $\epsilon$ increases with $T$, the number falling in $[c + \epsilon, \overline{c}]$ does not. Only those sellers with costs arbitrarily close to $c$ are more finely separated, but all other types are roughly partitioned into a finite number of groups. Most information is revealed in the last few rounds, but in a rough manner because those remaining sellers are partially pooled, even in the limit. This also means that most types accept in the last few rounds, and a transaction is more likely to occur late. Moreover, since the bids are kept low in initial rounds followed by large discrete jumps at the end, the bidding path is convexly increasing. By doing so, the buyer can successfully commit to a price ceiling, i.e., $\lim_{T \to \infty} x_T^f < \overline{c}$.

These properties of the price ceiling equilibrium seem to capture important facets of real bargaining situations well. Spann and Tellis (2006) analyze bidding patterns in the data of a NYOP retailer in Germany that sells airline tickets for various airlines and allows multiple bidding. They argue that: (i) with a positive bidding cost, the pattern should be concavely increasing, because at the beginning consumers try to increase the probability of successful bidding by bidding higher, but when the bids are closer to their reservation value, the rate of increase slows down; (ii) with zero bidding cost, the pattern should be linearly increasing. They find, however, that only 36% of the data fit the first pattern (concavely increasing) and 5% fit the second pattern (linearly increasing). Moreover, they even find that 23% of the data fit the
pattern which is convexly increasing, prompting them to conclude that consumer behavior on the Internet is not so rational. Our analysis provides an explanation for this seemingly puzzling observation, i.e., a convexly increasing bidding path can actually occur in a fully rational environment. In addition to this, in the price ceiling equilibrium, a transaction is more likely to occur in the last few rounds. This is related to the deadline effect that has been observed in many negotiation processes such as bargaining during strikes and pretrial negotiations.

Based on the above analysis, we can also characterize the buyer’s payoff with different classes of equilibrium paths and obtain an upper bound and a lower bound for the buyer’s expected payoff. By Theorem 1, we know that in the fully screening equilibrium, the buyer’s payoff is approximately the same as that in a reverse auction without a reserve price. The following theorem further proves that if the equilibrium falls in the class of the price ceiling equilibrium, the buyer’s payoff is higher than that in a reverse auction without a reserve price. The intuition is as follows: in the price ceiling equilibrium, the buyer keeps the bid low in the initial rounds so that only sellers with costs around $c$ accept. The buyer could instead have a bidding path that increases more aggressively from the beginning while the sellers correctly anticipate the buyer’s behavior, so that in the end, $\lim_{T \to \infty} x_T = \bar{c}$. Since the buyer chooses not to do so, it must be that he can get a higher payoff by keeping the bid low in the initial rounds.

**Theorem 2** Assume Condition 1. When $T \to \infty$, if the equilibrium falls in the class of the price ceiling equilibrium, the buyer’s expected payoff is strictly greater than that in a reverse auction without a reserve price. Thus, when $T \to \infty$, the buyer’s expected payoff is between the payoff in a reverse auction without a reserve price and the payoff in a reverse auction with the optimal reserve price.

**Proof.** Note that when $T \to \infty$, a path that almost fully separates sellers and satisfies the sellers’ IC constraint is a feasible solution candidate to program (P3) (it is the stationary solution to program (P3) when $T = \infty$, see Appendix C for detail) and it brings the buyer almost the same expected payoff as in a reverse auction with no reserve price. Therefore, if the solution to program (P3) is the path with $\lim_{T \to \infty} x_T < \bar{c}$, it must yield a higher value to the program than in a reverse auction with no reserve price. This proves the first statement. The second statement follows from Theorem 1, the discussion in Section 4.1, and the first statement.

As we show in Section 4.1, when the optimal auction design involves setting a
reserve price, the inability to commit to a bidding path prevents the buyer from achieving the outcome derived from Myerson’s optimal mechanism, which can be implemented by a reverse auction with the optimal reserve price. However, the presence of a deadline endows the buyer with certain commitment power and thus enables the buyer to achieve a payoff higher than in a reverse auction without any reserve price.

4.3 Factors in determining the class of the equilibrium path

Which class of equilibrium would occur depends on the distribution of the sellers’ cost $F$, the buyer’s value $v$, and the number of sellers $N$, parameters that affect the buyer’s trade-off between finely screening the sellers’ costs and having a reserve price. With a higher $v$, the buyer values the good more highly and cannot stand the risk of not getting the good, so setting a reserve price is less profitable for the buyer. On the other hand, when $N$ is larger, the probability that there exists a seller with a low cost increases, and competition among the sellers forces them to accept lower prices, too, so that a buyer also benefits less from setting a reserve price. We know that if the benefit of screening finely dominates the benefit of having a reserve price, the fully screening equilibrium arises. Therefore, we expect that the fully screening equilibrium is more likely to occur when $v$ is large and when $N$ is large.\footnote{The intuition regarding how $N$ affects the equilibrium path is confirmed in Horner and Samuelson (2011). They prove that when $v = 1$ and a seller’s cost is uniformly distributed on $[0,1]$, an equilibrium path with $\lim_{T \to \infty} x_T^* < \overline{c}$ occurs if and only if $N < 6$.}

The relationship between $v$ and the class of the equilibrium path is illustrated in Figure 1. Figure 1 shows the path of $x_t$ for different values of $v$ when $T = 20$, $N = 2$, and a seller’s cost is uniformly distributed on $[0,1]$. With $v = 1$, $v = 1.2$, and $v = 1.4$, the optimal reserve prices are 0.5, 0.6, and 0.7, respectively. So when $v = 1$, the buyer is more inclined to have $x_T$ much lower than $\overline{c} = 1$, and the price ceiling equilibrium arises; in equilibrium, a seller with a cost higher than 0.1 will not sell the good until the last two rounds, which implies that transactions are much more likely to occur in the last two rounds. On the other hand, when $v = 1.4$, the loss from having $x_T$ much lower than $\overline{c} = 1$ dominates the benefit, so the fully screening equilibrium arises; in equilibrium, the buyer raises bids gradually to a price close to 1, and a transaction is almost equally likely to occur in every round. The result that the fully screening equilibrium is more likely to occur when $v$ is large is formally proved in Proposition 2.
Proposition 2 Assume Condition 1. There exists a threshold \( \bar{v} \) such that the price ceiling equilibrium occurs if and only if \( v < \bar{v} \).

Proof. In round \( T - 1 \), given \( v \) and belief \( x_{T-2} \), the buyer’s payoff is

\[
V_{T-1}(x_{T-2}, v) = (v - \bar{x}_{T-1}(x_{T-2}, v)) \left[ F(x_{T-2}) - \bar{F}(x_{T-1}(x_{T-2}, v)) \right] \\
- C_T(\bar{x}_{T-1}(x_{T-2}, v)) \left[ \bar{F}(x_{T-2}) - \bar{F}(x_{T-2}, v) \right] + V_T(\bar{x}_{T-1}(x_{T-2}, v), v),
\]

where \( V_{T-1}(x_{T-2}, v) \) and \( \bar{x}_{T-1}(x_{T-2}, v) \) are as defined in (P2) but with one more argument \( v \). By the envelope theorem, \( \frac{\partial V_{T-1}}{\partial v} = \left( F(x_{T-2}) - \bar{F}(x_{T-1}) \right) + \frac{\partial V_T}{\partial v} \),

where \( \frac{\partial V_T}{\partial v} = \bar{F}(x_{T-1}) - \bar{F}(x_T) \), so \( \frac{\partial V_{T-1}}{\partial v} = F(x_{T-2}) - \bar{F}(x_T) \). For clarification, given \( v \), let \( x_T(v) \) be the equilibrium cutoff in round \( T \), and \( \pi^T(v) \) and \( \pi(v) \) be the buyer’s payoffs in our model with \( T \) rounds and in a reverse auction without a reserve price, respectively. Applying similar procedures to derive \( \frac{\partial V_{T-2}}{\partial v}, \frac{\partial V_{T-3}}{\partial v}, \ldots \), we can conclude that \( \frac{\partial \pi^T(v)}{\partial v} = 1 - \bar{F}(x_T(v)) \leq 1 \). Note that \( \frac{\partial \pi}{\partial v} = 1 \).

We next show that it cannot be the case that for \( v' > v \), \( \lim_{T \to \infty} x_T(v') < \bar{c} \) and \( \lim_{T \to \infty} x_T(v) = \bar{c} \). If that is the case, by Theorem 2, \( \lim_{T \to \infty} \pi^T(v') > \pi(v') \) and
\lim_{T \to \infty} \pi^T(v) = \pi(v).

\begin{align*}
\lim_{T \to \infty} \pi^T(v) &= \lim_{T \to \infty} \pi^T(v) - \int_v^{v'} \frac{d\lim_{T \to \infty} \pi^T(x)}{dx} dx \\
\pi(v) &= \pi(v) - \int_v^{v'} \frac{d\pi(x)}{dx} dx.
\end{align*}

(5)

and

(6)

Since \(\lim_{T \to \infty} \pi^T(v') > \pi(v')\) and \(\frac{d\lim_{T \to \infty} \pi^T(x)}{dx} \leq \frac{d\pi(x)}{dx}\), by (5) and (6), \(\lim_{T \to \infty} \pi^T(v) > \pi(v)\), a contradiction. Therefore, there is a threshold \(\bar{v}\) such that \(\lim_{T \to \infty} x^T_T(v) < \bar{c}\), i.e., the price ceiling equilibrium occurs, if and only if \(v < \bar{v}\).

5 The Role of the Lockout Period

In this section, we extend the model to consider the situation in which the buyer discounts the future, i.e., the buyer wants to close the transaction as early as possible.\(^{20}\) We show that with discounting, setting an appropriate lockout period rule may benefit the buyer, and we use numerical examples to examine the conditions under which having a lockout period benefits the buyer.

5.1 Model with discounting

The model is modified as follows. The buyer realizes his demand for the good at time 0 and tries to fulfill the demand in time period \([0, M]\). After time \(M\), the buyer no longer needs the good. If the buyer gets the good at price \(B\) at time \(t\), his utility is \(\delta \pi (v - B)\), where \(\delta \in (0, 1)\) is the discount factor for the whole time period \([0, M]\). A lockout period rule is set to regulate how frequently the buyer can submit a bid. If the lockout period is \(s\), the buyer can submit bids for \(\left\lfloor \frac{M}{s} \right\rfloor\) times, that is, \(T = \left\lfloor \frac{M}{s} \right\rfloor\).

With discounting, the buyer is more anxious to close the transaction early, but the buyer’s trade-off between finely screening the sellers’ costs and successfully committing to a reserve price still exists. If the buyer tries to commit to a reserve price, he has to bear the cost of delaying the transaction as well as not being able to screen the sellers. On the contrary, if the buyer gives up having a reserve price, he can raise the bid aggressively to close the transaction; moreover, if the buyer can revise

\(^{20}\)We continue to assume that the sellers have no preference for early or late transactions.
his bids extremely frequently, the sellers’ costs can be finely screened in a minute.

**Example 1** To see how $\delta$ affects the equilibrium path, Figure 2 shows the paths of $x_t$ for different values of $\delta$ when $v = 1$, $T = 50$, $N = 2$, and a seller’s cost is uniformly distributed on $[0,1]$. We can see that with lower $\delta$, the path becomes more concave, and a trade is more likely to occur in early rounds. This is consistent with our intuition because a buyer with a lower $\delta$ (who discounts the future more heavily) will find delaying the transaction to commit to a reserve price more costly, and hence will choose to raise bids aggressively.

**Summary of Example 1** A buyer with a lower $\delta$ raises bids more aggressively, and the equilibrium path of $x_t$ becomes more concave.

### 5.2 Examples for the optimal lockout period

When there is no discounting, having more rounds does not hurt the buyer because the buyer can choose to waste the additional bidding chances in the beginning. However, this strategy postpones the trade and cannot be implemented without
cost when the buyer discounts the future. Therefore, with a discount factor $\delta < 1$, the buyer’s payoff might not monotonically increase with the number of rounds, and setting a lockout period might increase the buyer’s payoff.

We use numerical examples to illustrate the circumstances under which having a lockout period benefits the buyer. We consider scenarios with different values of $\delta$ and $N$, and summarize the results of the numerical examples in Summary of Example 2 and Summary of Example 3. Notice that we focus on the environment in which Myerson’s optimal mechanism involves setting a reserve price. If setting a reserve price is unnecessary, having more rounds always benefits the buyer because it helps the buyer to separate the sellers better and to close the transaction earlier.

**Example 2** Suppose that $N = 2$, $v = 1.2$, and a seller’s cost is uniformly distributed on $[0,1]$. We consider the cases with $\delta = 1$, 0.96, 0.9, and 0.85, and the results are shown in the following table. The second column of the table shows the cost cutoff in the last round when $T$ goes to infinity.$^{21}$ When $\delta = 0.9$ and 0.85, $\lim_{T \to \infty} x_T = 1$, so the buyer raises the bid aggressively to $\bar{c}$ from the beginning, and the equilibrium paths of $x_t$ are concave. The third column shows the number of rounds $T^*$ that maximizes the buyer’s payoff, and the corresponding buyer’s payoff given $T^*$ is shown in the last column.$^{22}$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\lim_{T \to \infty} x_T$</th>
<th>$T^*$</th>
<th>$\pi(T^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.680</td>
<td>$\infty$</td>
<td>0.5556</td>
</tr>
<tr>
<td>0.96</td>
<td>0.808</td>
<td>3</td>
<td>0.5473</td>
</tr>
<tr>
<td>0.90</td>
<td>1</td>
<td>2</td>
<td>0.5399</td>
</tr>
<tr>
<td>0.85</td>
<td>1</td>
<td>$\infty$</td>
<td>0.5333</td>
</tr>
</tbody>
</table>

The result shows that when $\delta = 0.96$ and 0.9, setting a lockout period to allow the buyer to bid three and two times respectively maximizes the buyer’s payoff. With $\delta = 0.96$, when there is no lockout period, $\lim_{T \to \infty} x_T < 1$, so the equilibrium path of $x_t$ is mostly convex, and the transaction is very likely to occur late. This is

$^{21}$We derive the results numerically, so $T$ cannot increase unboundedly. However, we check all the cases when $T \leq 1000$ and find that the last-round cutoffs $x_T$ are constant for $T \geq 25$.

$^{22}$We know that for $\delta = 1$, the buyer’s payoff increases with $T$. For $\delta = 0.96$, 0.9, and 0.85, although we do not derive the buyer’s payoffs when $T$ goes to infinity, we check how the payoffs change when $T$ increases from 1 to 1000. For $\delta = 0.96$, when $T \geq 10$, the payoff falls below 0.54 and shows a trend of decline in $T$. For $\delta = 0.9$ and 0.85, the payoffs show a trend of increase when $T$ is large enough, and since $\lim_{T \to \infty} x_T = 1$, the payoffs converge to the payoff in a reverse auction with no reserve price.
illustrated by the dotted line in Figure 3, which shows the cutoff path when there are 50 rounds. The three stars in Figure 3 represent the equilibrium path when there are only 3 rounds. By comparing the two paths, we can see that by setting a lockout period so that the buyer is allowed to bid three times, the buyer benefits from having early transactions but suffers from not being able to separate sellers with costs around $c$. When $\delta = 1$, $\lim_{T \to \infty} x_T < 1$ as well, so the pros and cons of a lockout period rule are similar. However, waiting is more costly in the case of $\delta = 0.96$ than in the case of $\delta = 1$. Therefore, when $\delta = 0.96$, the benefit of setting an appropriate lockout period dominates the loss; but when $\delta = 1$, the loss dominates the benefit.

With $\delta = 0.9$, when there is no lockout period, $\lim_{T \to \infty} x_T = 1$, so the equilibrium path of $x_t$ is concave, and transactions occur early. This is illustrated by the dotted line in Figure 4, which shows the cutoff path when there are 50 rounds. The two stars in Figure 4 represent the equilibrium path when there are only 2 rounds. We can observe that, by setting a lockout period, the buyer benefits from having a last-round bid lower than $\overline{c}$, which functions like a reserve price, but suffers from not being able to close the transaction early and separate sellers finely. When $\delta \leq 0.85$, $\lim_{T \to \infty} x_T = 1$ as well, so the pros and cons of the lockout period rule are similar. Since waiting is more costly when $\delta$ is smaller, the loss caused by not being able to close the transaction early dominates the benefit when $\delta \leq 0.85$, while the benefit dominates the loss when $\delta = 0.9$. The example thus indicates that the buyer benefits from setting an appropriate lockout period when $\delta$ is in the middle range.

**Summary of Example 2** Setting an appropriate lockout period increases the buyer’s payoff when $\delta$ is in the middle range.

In addition to the situation mentioned above, if having a reserve price is valuable for the buyer (so that even with only one bidding chance allowed and no other chance to screen the sellers, the buyer still gets a higher payoff than in a reverse auction with no reserve price), then setting an appropriate lockout period benefits the buyer when $\delta$ is not too large. For instance, consider the case where $v = 1$, $N = 2$, and a seller’s cost is uniformly distributed on $[0, 1]$. The optimal reserve price is 0.5. Having a reserve price can improve the buyer’s payoff greatly. When $\delta$ is lower than 0.62, $\lim_{T \to \infty} x_T = 1$, so that the buyer’s payoff when there is no lockout period is at most $\frac{1}{2}$ (the payoff in a reverse auction with no reserve price). On the other hand, the buyer’s payoff when only one bidding chance is allowed is 0.3849 for all
Figure 3: Path of $x_t$ when there are 50 rounds and when there are 3 rounds, $\delta = 0.96$.

Figure 4: Path of $x_t$ when there are 50 rounds and when there are 2 rounds, $\delta = 0.9$. 
\( \delta \). Therefore, when \( \delta < 0.62 \), setting an appropriate lockout period always benefits the buyer.

**Example 3** Suppose that \( v = 1, \delta = 0.95 \), and a seller’s cost is uniformly distributed on \([0, 1]\). We consider the cases with \( N = 2, 3, \) and \( 4 \), and as in the previous example, the following table shows the limit of the last round cutoff when \( T \) goes to infinity \( \lim_{T \to \infty} x_T \), the number of rounds that maximizes the buyer’s payoff \( T^* \), and the corresponding buyer’s payoff \( \pi(T^*) \).\(^{23}\)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \lim_{T \to \infty} x_T )</th>
<th>( T^* )</th>
<th>( \pi(T^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.630</td>
<td>2</td>
<td>0.3949</td>
</tr>
<tr>
<td>3</td>
<td>0.917</td>
<td>4</td>
<td>0.5065</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>( \infty )</td>
<td>0.6000</td>
</tr>
</tbody>
</table>

The result shows that when the number of sellers is 2 or 3, setting an appropriate lockout period so that the buyer has only a few bidding chances increases the buyer’s payoff; however, if the number of sellers is larger than 3, having more rounds is better for the buyer. The reasoning is as follows. As we discussed in Section 4.3, when \( N \) is large, having a reserve price does not increase the buyer’s payoff much, so without a lockout period, the buyer does not hesitate to raise the bid from the beginning. Therefore, \( \lim_{T \to \infty} x_T = 1 \), the equilibrium path of \( x_t \) is concave, and transactions occur early. Similar to what is shown in Figure 4, by setting a lockout period, the buyer benefits from having a last-round bid lower than \( \overline{c} \), which functions like a reserve price, but suffers from not being able to close the transaction early and separate sellers finely. However, when \( N \) is large, having a reserve price is not important to the buyer, so that the loss dominates the benefit, and the buyer is better off without a lockout period.

**Summary of Example 3** Setting a lockout period increases the buyer’s payoff when \( N \) is small.

The above discussion shows that the optimal lockout period varies with the environment. An appropriate lockout period increases the buyer’s payoff when the

\(^{23}\) We derive the last-round cutoffs and the payoffs numerically when \( T \) increases from 1 to 1200. For \( N = 2 \), the cutoff and the payoff converge to 0.63 and 0.3834, respectively; and for \( N = 3 \), the cutoff and the payoff converge to 0.917 and 0.5, respectively. For \( N \geq 4 \), the cutoff converges to 1, and the payoff converges to the payoff in a reverse auction with no reserve price.
number of sellers is small and when the buyer’s discount factor is in the middle range. Priceline’s lockout period rule seems to hurt customers by restricting their rebidding opportunities, but in fact, a customer with waiting costs might find it beneficial.

6 Conclusion and Discussion

This paper analyzes the Name Your Own Price (NYOP) mechanism adopted by Priceline. We characterize the buyer’s and the sellers’ equilibrium strategies and show that Priceline’s lockout period restriction, a design alleged to protect sellers, can actually benefit customers with moderate discount factors. The analysis also suggests why, in reality, bargaining parties often take measures to make their offers rigid and consequently force themselves to make fewer offers, and why last-minute deals have become more prevalent with the advent of the Internet.

We show that when there is no lockout period and no waiting cost, the equilibria can be classified into two classes: (i) a fully screening equilibrium and (ii) a price ceiling equilibrium. In the fully screening equilibrium, sellers with different costs are almost fully separated with information about the sellers’ cost revealed gradually over time. In this case, the buyer raises bids continuously, ending, if necessary, with a price equal to the highest possible cost $\bar{c}$. In this equilibrium, which is efficient, the buyer’s payoff is approximately the same as the payoff in a reverse auction without a reserve price. In the price ceiling equilibrium, while sellers with costs arbitrarily close to the minimum cost level are finely separated, all other types of sellers are roughly partitioned into a finite number of groups, and information about the sellers’ costs is barely revealed in the initial rounds. In this case, the buyer does not raise the bid much until the very end, the ending bid is lower than $\bar{c}$, and the buyer’s payoff is greater than the payoff in a reverse auction without a reserve price. In the price ceiling equilibrium, most transactions occur just before the deadline. The delay of transactions incurs waiting cost if the buyer discounts the future. Therefore, setting a lockout period might actually benefit a buyer by moving transactions forward.

Discussion on the modeling approach and Priceline’s mechanism As in many other applied papers, our model is an abstraction of the real world, so it misses many features of reality. First of all, the buyer’s value is probably not known by the sellers. In reality, however, the information seems less imperfect
than the information about the sellers’ costs (the lowest prices they are willing to accept), which depend on the amounts of the excess inventory left over from their traditional retail channels. Therefore, while assuming that the sellers’ costs are private information, we assume that the buyer’s valuation is publicly known. This assumption also allows us to get around the signaling issue and focus on how the bidding path is designed to elicit the sellers’ information.

Next, although Priceline classifies hotels into different star ratings to ease customers’ quality concerns when buying through NYOP, there might still be some diversity within the same rating class. Hence, a customer might worry about the adverse selection problem, that is, if he bids low, only those whose quality is at the low end will accept. The adverse selection problem might increase a customer’s starting bid as well as add twists to the subsequent bidding path. By accounting for the quality dimension, as suggested in the procurement literature (see, e.g., Manelli and Vincent (1995), Morand and Thomas (2002), and Asker and Cantillon (2010)), a more sophisticated trading procedure might be necessary to increase the total surplus and possibly achieve full efficiency. This can explain why we have recently seen offer-counter-offer negotiation mechanisms arising on eBay and iOffer.com.

Extensions This paper indicates several interesting directions for future research. In addition to the adverse selection problem indicated previously, based on our analysis, one might be curious about whether Priceline could do better by adopting other measures, such as restricting the number of bidding chances instead of the frequency of bidding. Moreover, one might extend the model to consider the cases where the buyer has private information and where there are multiple buyers bidding at the same time. The following are some of our conjectures about the equilibrium path in such extended circumstances.

When the buyer’s value is private information, signaling issues arise. Since types with lower value are more inclined to bid low and have $x_T$ lower than $\overline{c}$, if sellers know that the buyer is of a lower type, they will accept lower prices. Knowing this, high types might try to imitate low types, and low types might try to bid lower in order to be distinguished from high types. Therefore, a convexly increasing path is still likely to occur, and the lockout period rule might still benefit the buyer.

When there are multiple buyers bidding at the same time (but the number of sellers is still larger than the number of buyers), everyone will try to get a unit from the seller with the lowest cost first. In the beginning, competition among the buyers
raises the price higher than it would have been if there were only one buyer, and
the sellers also raise the price thresholds that they are willing to accept. Once a
buyer’s offer is accepted by a seller, both leave the market, and the price drops to
a new level. Note that the price drops suddenly and is lower than it would have
been if no buyer had obtained a unit, which also means that the remaining sellers
are willing to accept a lower price after a buyer and a seller leave the market. This
is because the remaining sellers have rejected all prices offered before, so all players
believe that the remaining units will be sold at higher costs; if at the same time,
the sellers asked for a higher price after a buyer had left, then the remaining buyers
would have to pay an even higher price. This would make the buyers try harder
to get the first unit, and the prices in the beginning would be driven up further.
Therefore, along the equilibrium path, when there are more buyers staying in the
market, the sellers ask for higher prices but the units sold are provided by sellers
with lower costs; and when fewer buyers are left in the market, the sellers ask for
lower prices and the units are sold by sellers with higher costs. This is an interesting
phenomenon waiting for future theoretical and empirical investigation.

Exploring these extensions will bring us one step closer to reality and a bet-
ter understanding of the NYOP mechanism as well as other bargaining and price
determination processes.

A Appendix: the proofs

In this Appendix, we provide the proofs of Proposition 1 and Theorem 1.

Proof of Proposition 1. In the last round, recall that

$$
V_T(x_{T-1}) = \max_{x_T \in [x_{T-1}, \bar{x}]} (v - x_T) P(x_{T-1}, x_T),
$$

$$
\bar{x}_T(x_{T-1}) \in X_T(x_{T-1}) = \arg \max_{x_T \in [x_{T-1}, \bar{x}]} (v - x_T) P(x_{T-1}, x_T),
$$

and

$$
C_T(x_{T-1}) = (\bar{x}_T(x_{T-1}) - x_{T-1})G(x_{T-1}, \bar{x}_T(x_{T-1))).
$$

By Berge’s maximum theorem, we know that $V_T(x_{T-1})$ is continuous and $X_T(x_{T-1})$
is upper hemi-continuous. In round \( t \), \( t < T \), let

\[
\phi_t (x_{t-1}, x_t) = (v - x_t) \left[ \bar{F} (x_{t-1})^N - \bar{F} (x_t)^N \right]
- C_{t+1} (x_t) \left[ \bar{F} (x_{t-1}) - \bar{F} (x_t) \right] + V_{t+1} (x_t),
\]

\[
\alpha (x_{t-1}) = [x_{t-1}, c].
\]

Then

\[
V_t (x_{t-1}) = \max_{x \in \alpha (x_{t-1})} \phi_t (x_{t-1}, x_t),
\]

\[
\bar{x}_t (x_{t-1}) \in X_t (x_{t-1}) = \arg \max_{x \in \alpha (x_{t-1})} \phi_t (x_{t-1}, x_t).
\]

We show that by picking a proper \( \bar{x}_t (x_{t-1}) \) from \( X_t (x_{t-1}), t \leq T \), each round-\( t \) program has a solution.

First, observe that for the upper hemi-continuous correspondence \( X_T \), we are able to find (i) \( n_T \) closed intervals \([a_k, a_{k+1}]\), \( k = 1, \ldots, n_T \), such that \( \cup_k [a_k, a_{k+1}] = [\underline{c}, \overline{c}] \), and (ii) \( n_T \) continuous functions \( \bar{x}_{T,k} : [a_k, a_{k+1}] \to [a_k, \overline{c}], k = 1, \ldots, n_T \), such that \( \bar{x}_{T,k} (x) \in X_T (x), \forall x \in [a_k, a_{k+1}] \). Let

\[
\bar{x}_T (x_{t-1}) = \begin{cases} 
\bar{x}_{T,k} (x_{t-1}), & \text{if } x_{t-1} \in (a_k, a_{k+1}) \\
\arg \min_{x \in \{ \bar{x}_{T,k} (x_{t-1}), \bar{x}_{T,k+1} (x_{t-1}) \}} (x - x_{t-1}) G (x_{t-1}, x), & \text{if } x_{t-1} = a_{k+1}, k < n_T
\end{cases},
\]

\[
C_T (x_{t-1}) = (\bar{x}_T (x_{t-1}) - x_{t-1}) G (x_{t-1}, \bar{x}_T (x_{t-1})),
\]

\[
\bar{b}_T (x_{t-1}) = \bar{x}_T (x_{t-1}).
\]

\( C_T \) is lower semi-continuous and \( V_T \) is continuous, so \( \phi_{T-1} \) is upper semi-continuous. Note that \( \phi_{T-1} \) is graph-continuous with respect to \( \alpha \), which is defined in Leininger (1984). So by Leininger’s generalized maximum theorem, \( V_{T-1} \) is upper semi-continuous, and \( X_{T-1} \) is upper hemi-continuous.

Similarly, since \( X_{T-1} \) is upper hemi-continuous, we are able to find (i) \( n_{T-1} \) closed intervals \([a'_k, a'_{k+1}]\), \( k = 1, \ldots, n_{T-1} \), such that \( \cup_k [a'_k, a'_{k+1}] = [\underline{c}, \overline{c}] \), and (ii) \( n_{T-1} \) continuous functions \( \bar{x}_{T-1,k} : [a'_k, a'_{k+1}] \to [a'_k, \overline{c}], k = 1, \ldots, n_{T-1} \), such that
In the following proof, we occasionally add superscript 

Step 2: characterize the distribution of the cutoffs under Condition 1. To show this, it is easy to check that 

Step 4: show that the cluster point set is either 

Step 1: establish the existence of 

we divide it into the following steps: 

we only provide the proofs of the other two statements in this appendix. 

Under Condition 1, the following result shows a convergence property of 

Lemma 1 Assume Condition 1. Then \( \lim_{T \to \infty} x_{T-\tau}^T \) exists for all \( \tau \in \{0, 1, \cdots \} \). 

Proof. We first show that \( x_{T-1}^T (x_{t-1}) \) (defined in program (P2)) increases in \( x_{t-1} \) under Condition 1. To show this, it is easy to check that \( x_{T-1}^T (x_{T-1}) \) defined in program (P1) increases in \( x_{T-1} \). For \( t < T \), \( x_{T-1}^T (x_{t-1}) \) is derived from program (P2).
Let
\[
\varphi(x_t, x_{t-1}) = (v - b_t(x_t; x_{t-1})) \left[ \bar{F}(x_{t-1})^N - \bar{F}(x_t)^N \right] + V_{t+1}^T(x_t),
\]
where
\[
b_t(x_t; x_{t-1}) = \frac{C_{t+1}^T(x_t)}{F(x_t)^{N-1} + F(x_t)^{N-2} F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1}} + x_t.
\]

Then
\[
\frac{\partial \varphi(x_t, x_{t-1})}{\partial x_t} = - \left[ \bar{F}(x_{t-1}) - \bar{F}(x_t) \right] \left[ \left( \bar{F}(x_{t-1})^{N-1} + \cdots + \bar{F}(x_t)^{N-1} \right) + \frac{dC_{t+1}^T(x_t)}{dx_t} \right]
\]
\[
+ f(x_t) (\pi_{t+1}(x_t) - x_t) \left[ N \bar{F}(x_t)^{N-1} - (\bar{F} (\pi_{t+1}(x_t)))^{N-1} + \cdots + \bar{F}(x_t)^{N-1} \right]
\]
and
\[
\frac{\partial^2 \varphi(x_t, x_{t-1})}{\partial x_t \partial x_{t-1}} = f(x_t) \left[ N \bar{F}(x_t-1)^{N-1} + \frac{dC_{t+1}^T(x_t)}{dx_t} \right].
\]

For any \( x_t, x_{t-1}, \) and \( x'_{t-1} \in (x_{t-1}, x_t) \), if \( \frac{\partial^2 \varphi}{\partial x_t \partial x_{t-1}} (x_t, x_{t-1}, x'_{t-1}) \leq 0 \), which implies that
\[
\left( \bar{F}(x_{t-1})^{N-1} + \cdots + \bar{F}(x_t)^{N-1} \right) + \frac{dC_{t+1}^T(x_t)}{dx_t} > 0,
\]
then \( \frac{\partial^2 \varphi}{\partial x_t \partial x_{t-1}} (x_t, x_{t-1}, x'_{t-1}) > 0 \) for \( x \in [x_{t-1}, x'_{t-1}] \), and hence, \( \frac{\partial \varphi}{\partial x_t} (x_t, x_{t-1}) < 0 \). Therefore, \( \frac{\partial \varphi}{\partial x_t} (x_t, x_{t-1}) > 0 \) implies \( \frac{\partial \varphi}{\partial x_t} (x_t, x_{t-1}) > 0 \).

Since, given \( x_{t-1} \) and \( x'_{t-1} \), \( \varphi(x_t, x_{t-1}) \) and \( \varphi(x_t, x'_{t-1}) \) are maximized at \( x = \pi_T^T(x_{t-1}) \) and \( x = \pi_T^T(x'_{t-1}) \), respectively, the facts that (i) \( \frac{\partial \varphi}{\partial x_t} (x_t, x_{t-1}) > 0 \) implies \( \frac{\partial \varphi}{\partial x_t} (x_t, x'_{t-1}) > 0 \) and that (ii) \( \pi_T^T(x_{t-1}) \) is continuous lead to the conclusion that \( \pi_T^T(x_{t-1}) \) increases in \( x_{t-1} \).

Given this, we can prove the lemma. Note that given any \( t \) and \( T \), \( \pi_T^T(\cdot) = \pi_{t+1}^T(\cdot) \) (defined in program (P2) on page 14). When we increase the number of rounds from \( T \) to \( T+1 \), \( x_{t+1}^T = x_0^T = g \) and \( x_{t+1}^T = x_0^T \), so \( x_{t+1}^T \geq x_0^T \). Since \( \pi_T^T(x_{t-1}) \) increases in \( x_{t-1} \), \( x_{t+1}^T \geq x_0^T \) implies that \( x_{t+1}^T \geq x_{t-1}^T \) for all \( T \leq T \).

Hence, \( x_{t+1}^T \) increases in \( T \). Furthermore, \( x_{t+1}^T \) has an upper bound \( \bar{c} \), and so we conclude that \( \lim_{T \to \infty} x_{t+1}^T \) exists.

Since the sequence \( \{ \lim_{T \to \infty} x_{t+1}^T \} \) is weakly decreasing and bounded, \( \lim_{T \to \infty} \lim_{T \to \infty} x_{t+1}^T \), which we denote by \( a \), exists. This concludes Step 1.

**Step 2:** Given the existence of \( a \), we now characterize how the cutoffs \( x_{t+1}^T \) are distributed on \( (a, \bar{c}] \) when \( T \to \infty \). The following results are useful to this end.

**Lemma 2** Assume Condition 1. Let \( \epsilon \equiv x_{t+1}^T - x_t^T \) and \( \Delta \equiv x_t^T - x_{t-1}^T \). In equilibrium, if \( \epsilon > 0 \), \( \Delta > 0 \). Given \( x_t^T \), \( \epsilon \), and \( T - t \), the value of \( \Delta \) is determined.
Proof. When deriving the equilibrium path, we solve the first-order condition of program (P2). From the first-order condition, we show that if the difference between the two cutoffs $x_{t+1}^T$ and $x_t^T$ is positive, then the difference between $x_t^T$ and $x_{t-1}^T$ is also positive.

Given any $t$, $T$ and given belief $x_{t-1}$, the continuation equilibrium $x_t^*$ and $b_t^*$ are derived from

$$V_t^T(x_{t-1}) = \max_{x_t} (v - b_t(x_t; x_{t-1})) [\bar{F}(x_{t-1})^N - \bar{F}(x_t)^N] + V_{t+1}^T(x_t),$$

(P4)

where $b_t(x_t; x_{t-1}) = \frac{C_{t+1}^T(x_t)}{F(x_t)^N} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1} + x_t$.

The solution $x_t^*$ must satisfy the first-order condition

$$0 = \left[ \bar{F}(x_{t-1}) - \bar{F}(x_t^*) \right] - \left[ \left( \bar{F}(x_{t-1})^N - \cdots - \bar{F}(x_t^*)^{N-1} \right) - C_{t+1}^T(x_t^*) \right]$$

$$- C_{t+1}^T(x_t^*) f(x_t^*) + N (v - x_t^*) \bar{F}(x_t^*)^{N-1} f(x_t^*) + V_{t+1}^T(x_t^*).$$

(7)

Note that

$$V_{t+1}^T(x_t) = \max_{\{b_{t+1}, x_{t+1}\}} (v - b_{t+1}(x_{t+1}; x_t)) [\bar{F}(x_t)^N - \bar{F}(x_{t+1})^N] + V_{t+2}^T(x_{t+1}),$$

(8)

where $b_{t+1}(x_{t+1}; x_t) = \frac{C_{t+2}^T(x_{t+1})}{F(x_{t+1})^{N-1} + F(x_{t+1})^{N-2}F(x_t) + \cdots + F(x_t)^{N-1} + x_{t+1}}$.

Let $x_{t+1}^*(x_t)$ be the solution to program (8). By the envelope theorem,

$$V_{t+1}^{T_t}(x_t) = -N \bar{F}(x_t)^{N-1} f(x_t) (v - x_{t+1}^*) + f(x_t) C_{t+2}^T(x_{t+1}^*).$$

(9)

Plugging into (7), we get

$$0 = \left[ \bar{F}(x_{t-1}) - \bar{F}(x_t^*) \right] - \left[ C_{t+1}^T(x_t^*) - C_{t+2}^T(x_{t+1}^*) \right] f(x_t^*) + N \bar{F}(x_t^*)^{N-1} f(x_t^*) \left[ x_t^* - x_{t+1}^* \right]$$

$$+ f(x_t^*) x_{t+1}^* - x_t^* + \frac{N \bar{F}(x_t^*)^{N-1} - \left( \bar{F}(x_{t+1}^*)^{N-1} \cdots + \bar{F}(x_t^*)^{N-1} \right)}{(x_{t+1}^* - x_t^*)}.$$

(10)

If $x_{t+1}^* - x_t^* = \epsilon > 0$, $(x_{t+1}^* - x_t^*) \left[ N \bar{F}(x_t^*)^{N-1} - \left( \bar{F}(x_{t+1}^*)^{N-1} + \cdots + \bar{F}(x_t^*)^{N-1} \right) \right]$ is strictly positive, and
so by (10),
\[
[F(x_{t-1}) - \bar{F}(x^*_i)] \left[ \left( \bar{F}(x_{t-1})^{N-1} + \cdots + \bar{F}(x^*_i)^{N-1} \right) + c'_{T-t}(x^*_i) \right]
\]
is strictly positive. Condition 1 and Proposition 5 in the online appendix imply that \(c'_{T-t}(x)\) exists and is bounded. Therefore, by (10), if \(x^*_{t+1} - x^*_t > 0\), \(x^*_t - x_{t-1} > 0\). Moreover, the difference between \(x_{t-1}\) and \(x^*_t\) only depends on \(x^*_t\), \(\epsilon = x^*_{t+1} - x^*_t\), and \(T-t\).

**Lemma 3** Given any \(T\) and \(\tau < \infty\), if \(x^*_{T-\tau} < \bar{c}\), then \(x^*_T < \bar{c}\) and \(x^*_T - x^*_{T-1} > 0\).

**Proof.** Suppose that \(x^*_{T-\tau} < \bar{c}\). When \(\tau = 1\), given that \(x^*_{T-1} < \bar{c}\), by (4), \(x^*_T < \bar{c}\) and \(x^*_T - x^*_{T-1} > 0\). When \(\tau = 2\), given that \(x^*_{T-2} < \bar{c}\), if \(x^*_{T-1} = \bar{c}\), then \(x^*_T = \bar{c}\). This implies that the buyer pays for the good at a price higher than or equal to \(\bar{c}\), which is not optimal for the buyer and cannot happen in equilibrium. Therefore, \(x^*_{T-1} < \bar{c}\), and then we can apply the previous result for the case where \(\tau = 1\).

Applying the same argument for \(\tau = 3, 4, \cdots\), we can conclude that, for any \(\tau\), if \(x^*_{T-\tau} < \bar{c}\), then \(x^*_T < \bar{c}\) and \(x^*_T - x^*_{T-1} > 0\).

If \(a < \bar{c}\), there exists \(s < \infty\) such that \(\bar{c} - \lim_{T \to \infty} x^*_{T-s} > 0\). By Lemma 3, \(\lim_{T \to \infty} x^*_T < \bar{c}\) and \(\lim_{T \to \infty} x^*_T - \lim_{T \to \infty} x^*_{T-1} > 0\). By Lemma 2, \(\lim_{T \to \infty} x^*_{T-\tau+1} - \lim_{T \to \infty} x^*_{T-\tau} > 0\), for all \(\tau < \infty\). The two lemmas thus imply that either \(\lim_{T \to \infty} x^*_{T-\tau} = \bar{c}\) for all \(\tau \in \{0, 1, \cdots\}\), or \(\lim_{T \to \infty} x^*_T < \bar{c}\) and \(\lim_{T \to \infty} x^*_{T-\tau} - \lim_{T \to \infty} x^*_{T-\tau-1} > 0\) for all \(\tau \in \{0, 1, \cdots\}\). Therefore, \((a, \bar{c}] \subset (\underline{c}; \bar{c}] \setminus B\), which concludes step 2.

**Step 3:** To characterize how the cutoffs are distributed on \([\underline{c}; a]\), we first show that in a sequence of continuation games with increasing numbers of rounds and with convergent starting beliefs, the difference between the first-round cutoff and the starting belief converges to zero.

**Lemma 4** Assume Condition 1. Consider a set of continuation games starting with belief \(x\) and indexed by the number of rounds left, \(T\). In the continuation equilibrium, the cutoff in the first round of the continuation game approaches \(x\) when \(T \to \infty\), i.e., \(\lim_{T \to \infty} x^*_{T} (x) - x = 0\) (\(x^*_{T} (x)\) is defined in (P2)).

**Proof.** As \(T \to \infty\), the buyer’s payoff converges, so the additional payoff a buyer can get by adding one more round goes to 0. In the following proof, we show that when one more round is added to the continuation game, the additional payoff the buyer can get does not go to 0 as \(T \to \infty\) if there exists \(\epsilon > 0\) such that \(x^*_{T} (x) - x \geq \epsilon\) for an unlimited number of \(T\)’s. Since the buyer’s payoff must converge, when \(T \to \infty\), \(x^*_{T} (x) \to x\).
Suppose that in the continuation game with \( T \) rounds left, \( \pi_1^T (x) - x = \epsilon \). Then when there are \( T + 1 \) rounds left, if we impose a constraint \( x_1 = x \), where \( x_1 \) is the cutoff in the first round of the continuation game, while deriving the remaining continuation path from program (P4), then the buyer’s payoff is the same as when there are \( T \) rounds left, and \( \pi_2^{T+1} (x_1) - x_1 = \epsilon \). In (10), if we let \( t = 1 \), and let \( x_0 = x_1 = x \) and \( x_2 = \pi_2^{T+1} (x_1) \), then

\[
- [\tilde{F} (x) - \tilde{F} (x_1)] \left[ (\tilde{F} (x)^{N-1} + \cdots + \tilde{F} (x_1)^{N-1}) + c_T^t (x_1) \right]
+ f (x_1) \left( \pi_2^{T+1} (x_1) - x_1 \right) \left[ N \tilde{F} (x_1)^{N-1} - (\tilde{F} (\pi_2^{T+1} (x_1)))^{N-1} + \cdots + \tilde{F} (x_1)^{N-1} \right] > 0.
\]

By Condition 1, \( c_T^t (\cdot) \) exists, and \( \{c_T^t\}_T \) is uniformly bounded. Therefore, there exists a \( \Delta \) such that for \( x_1 \in (x, x + \Delta) \), (11) still holds for all \( T \). This implies that choosing \( x_1 = x + \Delta \) instead of \( x_1 = x \) can increase the buyer’s payoff by a positive number, which does not vanish when \( T \to \infty \). Therefore, if \( \pi_1^T (x) - x \geq \epsilon \), increasing the number of rounds from \( T \) to \( T+1 \) strictly increases the buyer’s payoff, and the additional payoff does not go to 0 as \( T \to \infty \). □

**Corollary 1** Assume Condition 1. Consider a set of continuation games indexed by the number of rounds left, \( T \), and starting with belief \( x^T \) such that \( \lim_{T \to \infty} x^T = x \). In the continuation equilibrium, the cutoff in the first round of the continuation game approaches \( x \) when \( T \to \infty \), i.e., \( \lim_{T \to \infty} \pi_1^T (x^T) - x = 0 \).

**Proof.** By Condition 1, which implies the continuity of the value function, and the fact that given starting belief \( x \), the buyer’s payoff converges as the number of rounds left \( T \) goes to infinity, it is implied that in the set of continuation games indexed by the number of rounds left \( T \) and starting with belief \( x^T \) such that \( \lim_{T \to \infty} x^T = x \), the buyer’s payoff also converges as \( T \to \infty \). Then, by applying similar arguments to those in the proof of Lemma 4, we can conclude that \( \lim_{T \to \infty} \pi_1^T (x^T) - x = 0 \). □

With the result of Corollary 1, we now show that the cutoffs are distributed more and more densely on \([c, a)\) when \( T \to \infty \). The following two results can be used to prove this.

**Lemma 5** The set of cluster points \( B \) is closed.

**Proof.** If \( B \) is not closed, there exists \( y \in [c, a] \setminus B \) such that for any \( \epsilon > 0 \), there is \( z \in B \) such that \( |z - y| < \frac{\epsilon}{2} \). Since \( z \in B \), given \( \epsilon \), for any \( M \), there exists \( T' \) such
that for all \( T > T' \), \( \| \{ x \in X^T \mid |x - z| < \frac{\epsilon}{2} \} \| > M \). However, then for all \( T > T' \), \( \| \{ x \in X^T \mid |x - y| < \epsilon \} \| > M \), so \( y \in B \), a contradiction. Therefore, \( B \) is closed.

\[ \blacksquare \]

**Lemma 6**  Assume Condition 1. \([c, a] \subset B\).

**Proof.** We prove the lemma by contradiction. First, assume that not the whole interval \([c, a]\) is contained in the cluster point set. Then we find an interval \((z - \epsilon_1, z + \epsilon) \subset [c, a]\) such that for \( T = R_1, R_2, \cdots \) (an infinite sequence), the number of cutoffs falling in \((z - \epsilon_1, z + \epsilon)\) does not exceed a finite number \( M + 1 \), and moreover, there exists a sequence \( \{ y_i \mid y_i \in X^{R_i} \}_{i \in \mathbb{N}} \) such that \( \lim_{i \to \infty} y_i = z - \xi_1 \). Since \( a \) is a cluster point, an unbounded number of cutoffs will fall in \((z - \epsilon_1, \overline{z})\) when \( T \to \infty \). So by Corollary 1, unboundedly many cutoffs will approach \( z - \xi_1 \), and the number of cutoffs in \((z - \epsilon_1, z + \epsilon)\) will exceed \( M + 1 \) when \( T \) is large enough, a contradiction. The details of the proof are shown below.

By Lemma 4, \( c \in B \). If not the whole interval \([c, a]\) belongs to \( B \), by Lemma 5, there exist \( b, c \) such that \((b, c) \subset [c, \overline{c}] \setminus B \) and \( b, c \in B \). For any \( z \in (b, c) \), there exist \( \epsilon \) and \( M \) such that we cannot find such \( T' \) so that for all \( T > T' \), \( \| \{ x \in X^T \mid |x - z| < \epsilon \} \| > M \).

Given \( z, \epsilon, \) and \( M \), let \( E_1 \equiv \{ \epsilon_1 \mid \text{There does not exist } T' \text{ such that for all } T > T', \| \{ x \in X^T \mid x \in (z - \epsilon_1, z + \epsilon) \} \| > M \} \)

and let \( \xi_1 \equiv \sup (E_1) \).\(^{24}\) Note that there exists an infinite sequence \( \{ T_1, T_2, \cdots \} \) such that \( \| \{ x \in X^T \mid x \in (z - \xi_1, z + \epsilon) \} \| \leq M + 1 \) for \( T = T_1, T_2, \cdots \). Since \( a \in B \) (the number of \( x_i \)'s falling around \( a \) increases as \( T \to \infty \)), there exists an infinite subsequence \( \{ S_1, S_2, \cdots \} \subset \{ T_1, T_2, \cdots \} \) such that for those \( T = S_1, S_2, \cdots \), the number of \( x_i \)'s falling in \((z - \xi_1, \overline{z})\) increases with \( T \).\(^{25}\) Moreover, since \( \xi_1 \equiv \sup (E_1) \), there exists a sequence \( \{ y_i \mid y_i \in X^{R_i} \}_{i \in \mathbb{N}} \), where \( \{ R_1, R_2, \cdots \} \subset \{ S_1, S_2, \cdots \} \), such that \( \lim_{i \to \infty} y_i = z - \xi_1 \). So, we can consider continuation games starting with belief \( \{ y_i \}_{i \in \mathbb{N}} \) and apply Corollary 1. By Corollary 1, unboundedly many \( x_i \)'s in \((z - \xi_1, \overline{z})\) will approach \( z - \xi_1 \) when \( T = R_1, R_2, \cdots \) goes to infinity. Therefore, the number of \( x_i \)'s falling in \((z - \xi_1, z + \epsilon)\) cannot be bounded, a contradiction. \( \blacksquare \)

It follows from Lemma 6 and Step 2 that \([c, a]\) is the cluster point set, and \((a, \overline{a}]\)

\(^{24}\)\( E_1 \) is not empty because \( \epsilon \in E_1 \).

\(^{25}\)If there does not exist such a subsequence, this implies that the number of \( x_i \)'s in \([z - \xi_1, \overline{z}]\) is bounded as \( T \to \infty \). Therefore, the number of \( x_i \)'s in \([z - \xi_1, \overline{z}] \setminus (a - \epsilon, a + \epsilon) \), where \( \epsilon \) is a small positive number, decreases as \( T \to \infty \), but this cannot be true because \( T \) is unbounded, and the number of \( x_i \)'s in \([z - \xi_1, \overline{z}] \setminus (a - \epsilon, a + \epsilon) \) is non-negative.
Step 4: To prove the theorem, we establish Lemma 9 with the assistance of Lemma 7 and Lemma 8. First let \( C' (x) = \lim_{\tau \to -\infty} c'_\tau (x) \) (where \( c_\tau (x) \) is defined on page 33).

Lemma 7 Assume Condition 1. When \( \tau \to \infty \) and \( \lim_{\tau \to -\infty} x_\tau = a \), if \( c'_{\tau -1} (x'_1 (x_\tau)) \) does not converge to \(-N F (a)^{N-1} \), then \( x'_1 (x_\tau) \) converges to 1.

Proof. We apply the Implicit Function Theorem to derive \( \lim_{\tau \to -\infty} x'_1 (x_\tau) \). By the first-order condition (10)

\[
0 = \phi_{\tau +1}(x'_1 (x_\tau), x_\tau)
= -F(x_\tau) - F(x'_1 (x_\tau)) \left[ (F(x_\tau)^{N-1} + \cdots + F(x'_1 (x_\tau)^{N-1}) + c'_{\tau -1} (x'_1 (x_\tau)))
+ f(x'_1 (x_\tau)) [(x'_2 (x_\tau) - x'_1 (x_\tau))]
\times \left[ N F(x'_1 (x_\tau))^{N-1} - (F(x'_2 (x'_1 (x_\tau)))^{N-1} + \cdots + F(x'_1 (x_\tau))^{N-1}) \right]
\]

and the Implicit Function Theorem, \( \frac{d \phi_{\tau +1}(x'_1 (x_\tau), x_\tau)}{d x_\tau} = -\frac{\partial \phi_{\tau +1}(x'_1 (x_\tau), x_\tau)}{\partial x_\tau} \). When \( \tau \to \infty \) and \( \lim_{\tau \to -\infty} x_\tau = a \), by Corollary 1, \( x'_1 (x_\tau) \) and \( x'_2 (x'_1 (x_\tau)) \) both converge to a. Therefore, \( \frac{\partial \phi_{\tau +1}(x'_1 (x_\tau), x_\tau)}{\partial x_\tau} \) and \( -\frac{\partial \phi_{\tau +1}(x'_1 (x_\tau), x_\tau)}{\partial x_\tau} \) both converge to \( f(a) \left[ N F (a)^{N-1} + \lim_{\tau \to -\infty} c'_{\tau -1} (x'_1 (x_\tau)) \right] \), and \( \frac{d \phi_{\tau +1}(x'_1 (x_\tau), x_\tau)}{d x_\tau} = -\frac{\partial \phi_{\tau +1}(x'_1 (x_\tau), x_\tau)}{\partial x_\tau} \) converges to 1.

Lemma 8 Assume Condition 1. Given \( a = \lim_{\tau \to -\infty} \lim_{T \to -\infty} x_{T - \tau} \), if \( \lim_{x \to a} C' (x) \neq -N F (a)^{N-1} \), then \( C' (x) \) is continuous at \( a \).

Proof. Note that \( c_\tau (x) = C'_{T - \tau} (x) = C'_{\tau} (x) \), so

\[
c_1 (x) = C'_{\tau} (x) = (x'_1 (x_\tau) - x) G(x, x'_1 (x_\tau)),
\]

and for \( \tau \geq 2 \)

\[
c_\tau (x_\tau) = C'_{\tau} (x_\tau) = (x'_1 (x_\tau) - x) G(x_\tau, x'_1 (x_\tau)) + C'_{\tau+1} (x'_1 (x_\tau))
= (x'_1 (x_\tau) - x) G(x_\tau, x'_1 (x_\tau)) + c_{\tau -1} (x'_1 (x_\tau)).
\]
Then
\[
c'_\tau (x_\tau) = \left[ (\overline{\pi}'_1(x_\tau) - 1) G(x_\tau, \overline{\pi}'_1(x_\tau)) \\
+ (\overline{\pi}'_1(x_\tau) - x_\tau) \left( \frac{\partial G}{\partial x_\tau} + \frac{\partial G}{\partial \overline{\pi}'_1(x_\tau)} \right) \right] + c'_{\tau - 1} (\overline{\pi}'_1(x_\tau)) \overline{\pi}''_1(x_\tau).
\]

(12)

First, let \( x_\tau = a \) for all \( \tau \geq 2 \). Because \( \overline{\pi}'_1(x_\tau) > a \), given the assumption that \( \lim_{x_1 a} C'(x) = \lim_{x \to \infty} c'_\tau(x) \neq -N \overline{F}(a)^{N-1} \), \( c'_{\tau - 1} (\overline{\pi}'_1(x_\tau)) \) does not converge to \(-N \overline{F}(a)^{N-1}\) when \( \tau \to \infty \). Thus by Lemma 7, \( \overline{\pi}'_1(x_\tau) \) converges to 1. Moreover, \( \lim_{\tau \to \infty} \overline{\pi}'_1(x_\tau) = x_\tau = a \), and \( \{G(x_\tau, \overline{\pi}'_1(x_\tau))\}_\tau \) and \( \left\{ \frac{\partial G}{\partial x_\tau} + \frac{\partial G}{\partial \overline{\pi}'_1(x_\tau)} \right\}_\tau \) are uniformly bounded, so (12) implies that \( \lim_{\tau \to \infty} c'_\tau(a) = \lim_{x_1 a} \lim_{\tau \to \infty} c'_\tau(x) \).

Next, because \([c, a] \subset B\) and \( \lim_{\tau \to \infty} \lim_{\tau \to \infty} x_{T - \tau}^T = a \), we can find a sequence \( \{x_\tau \in X^T\}_\tau \) such that \( x_\tau < a \), \( \lim_{\tau \to \infty} x_\tau = a \), \( \overline{\pi}'_1(x_\tau) \geq a \), and \( \lim_{\tau \to \infty} \overline{\pi}'_1(x_\tau) = a \). Similarly, \( \lim_{\tau \to \infty} \overline{\pi}'_1(x_\tau) = 1 \), \( \lim_{\tau \to \infty} \overline{\pi}'_1(x_\tau) = a \), and \( \{G(x_\tau, \overline{\pi}'_1(x_\tau))\}_\tau \) and \( \left\{ \frac{\partial G}{\partial x_\tau} + \frac{\partial G}{\partial \overline{\pi}'_1(x_\tau)} \right\}_\tau \) are uniformly bounded, so (12) implies that \( \lim_{x_1 a} \lim_{\tau \to \infty} c'_\tau(x) = \lim_{x_1 a} \lim_{\tau \to \infty} c'_\tau(x) \). Therefore, \( C'(x) = \lim_{\tau \to \infty} c'_\tau(x) \) is continuous at \( a \).

These two results lead to the following lemma which can be used to prove the theorem.

**Lemma 9** Assume Condition 1. Either \( a = \underline{c} \) or \( a = \bar{c} \).

**Proof.** The proof is by contradiction. Suppose that \( a \in (\underline{c}, \bar{c}) \). By the facts that \([\underline{c}, a] = B\), \( \lim_{\tau \to \infty} \lim_{\tau \to \infty} x_{T - \tau}^T = a \), and the first-order condition (10), we first conclude that \( \lim_{x_1 a} \lim_{\tau \to \infty} c'_\tau(x) \neq \lim_{x_1 a} \lim_{\tau \to \infty} c'_\tau(x) = -N \overline{F}(a)^{N-1} \). However, Lemmas 7 and 8 show that \( \lim_{\tau \to \infty} c'_\tau(x) \) is continuous at \( a \), a contradiction.

The details of the proof are shown below.

Since \([\underline{c}, a] = B\) and \( \lim_{\tau \to \infty} \lim_{\tau \to \infty} x_{T - \tau}^T = a \), we can rewrite the necessary condition (10) for the optimality problem as

\[
0 = \left[ F(x - dx^-) - F(x) \right] \left[ -F(x - dx^-)^{N-1} - \cdots - F(x)^{N-1} - C'(x) \right] \\
- f(x) dx^+ \left[ F(x + dx^+)^{N-1} + \cdots + F(x)^{N-1} - N \overline{F}(x)^{N-1} \right]
\]

(13)

where \( x \in [\underline{c}, a] \), \( C'(x) = \lim_{\tau \to \infty} c'_\tau(x) \), and \( dx^- \) and \( dx^+ \) are two positive numbers which can be arbitrarily small. For \( x \in (\underline{c}, a) \), since \([\underline{c}, a] = B\), \( dx^- \in O(dx^+) \) and \( dx^+ \in O(dx^-) \), i.e., \( dx^+ \) and \( dx^- \) converge to 0 at the same rate. Therefore, an
approximation of equation (13) is
\[
0 = f(x) dx^+ \left[ -N \bar{F}(x)^{N-1} - \frac{(N-1)}{2} \bar{F}(x)^{N-2} f(x) dx^+ - C'(x) \right]
\]
\[
-f(x) dx^- \left[ N \bar{F}(x)^{N-1} - \frac{(N-1)}{2} \bar{F}(x)^{N-2} f(x) dx^- - N \bar{F}(x)^{N-1} \right],
\]
and it implies \( C'(x) = -N \bar{F}(x)^{N-1} \) for \( x \in (\underline{c}, a) \).

Since \( (a, \bar{c}) = [\underline{c}, \bar{c}] \setminus B \) and \( [\underline{c}, a] = B \), if the starting belief of a continuation game decreases from \( a \) to \( a - \epsilon \), where \( \epsilon \) is a small positive number, \( \lim_{T \to \infty} x_T^{T-\tau} \) does not change for all \( \tau \); but if the starting belief increases from \( a \) to \( a + \epsilon \), \( \lim_{T \to \infty} x_T^{T-\tau} \) increases as well for all \( \tau \). Given starting belief \( x \), \( C'(x) \) is composed of \( \lim_{T \to \infty} \frac{dx_T^{T-\tau}}{dx} \) and \( \lim_{T \to \infty} x_T^{T-\tau} \), \( \tau = 1, 2, \ldots \). Therefore, the asymmetric behavior of \( x_T^{T-\tau} \) when the starting belief changes from \( a \) to \( a - \epsilon \) and from \( a \) to \( a + \epsilon \) leads to the result that \( \lim_{x \downarrow a} C'(x) \neq \lim_{x \uparrow a} C'(x) = -N \bar{F}(a)^{N-1} \), and \( C'(x) \) is not continuous at \( a \). However, by Lemma 8, \( C'(x) \) should be continuous at \( a \), a contradiction. Therefore, \( a \in (\underline{c}, \bar{c}) \) cannot occur in equilibrium.  

As we have seen in step 3, \( B = [\underline{c}, a] \) for some \( a \in [\underline{c}, \bar{c}] \). Lemma 9 shows that \( a \) can only be \( \underline{c} \) or \( \bar{c} \) which proves the first statement.

**Proof of the second statement.** For the second statement, note that if \( \lim_{T \to \infty} x_T^T < \bar{c} \), \( B \) cannot be \([\underline{c}, \bar{c}]\), so \( B = \{\underline{c}\} \). On the other hand, if \( B = \{\underline{c}\} \), there exists \( \tau < \infty \) such that \( \bar{c} - \lim_{T \to \infty} x_T^{T-\tau} > 0 \). By Lemma 3, \( \lim_{T \to \infty} x_T^T < \bar{c} \).

**Appendix: the cutoff property of a seller’s strategy**

**Proposition 3** In a symmetric pure strategy equilibrium, if a seller with cost \( x \) accepts in round \( t \), then a seller with cost \( x' < x \) also accepts in round \( t \).

**Proof.** Suppose that at the beginning of round \( t \), a seller believes that the other sellers’ costs are in set \( S_t \subset [\underline{c}, \bar{c}] \). After seeing the current bid \( b_t \), it is expected that a seller with cost in set \( A_t \subset S_t \) accepts in round \( t \); if no seller accepts, the buyer

\[
c'_t(x) = \sum_{t=1}^{T} \left( \frac{dx_T^{T-\tau+1}}{dx} - \frac{dx_T^{T-\tau}}{dx} \right) G \left( x_T^{T-\tau}, x_T^{T-\tau+1} \right) + (x_T^{T-\tau+1} - x_T^{T-\tau}) \left( \frac{\partial G}{\partial x_T^{T-\tau}} \frac{dx_T^{T-\tau}}{dx} \right) + \frac{\partial G}{\partial x_T^{T-\tau+1}} \frac{dx_T^{T-\tau+1}}{dx}.
\]
will offer \( b_{t+1} \) in round \( t + 1 \), and a seller with cost in set \( A_{t+1} \subset S_{t+1} \equiv S_t \setminus A_t \) will accept. If a seller with cost \( x \) accepts in round \( t \) instead of round \( t + 1 \), it implies that the seller gets a higher expected payoff by accepting in round \( t \) than by accepting in round \( t + 1 \), i.e.,

\[
(b_t - x) \sum_{n=0}^{N-1} \frac{1}{n+1} \frac{(N-1)!}{n!(N-n-1)!} \Pr(x \in A_t \mid x \in S_t)^n (1 - \Pr(x \in A_t \mid x \in S_t))^{N-n-1} > \Pr(x \in S_{t+1} \mid x \in S_t)^{N-1} (b_{t+1} - x) \times \sum_{n=0}^{N-1} \frac{1}{n+1} \frac{(N-1)!}{n!(N-n-1)!} \Pr(x \in A_{t+1} \mid x \in S_{t+1})^n (1 - \Pr(x \in A_{t+1} \mid x \in S_{t+1}))^{N-n-1}
\]

\[
\Rightarrow (b_t - x) \frac{\sum_{n=0}^{N-1} (\Pr(x \in S_t) - \Pr(x \in A_t))^{N-n-1} \Pr(x \in S_t)^n}{N \Pr(x \in S_t)^{N-1}} > (b_{t+1} - x) \frac{\sum_{n=0}^{N-1} (\Pr(x \in S_{t+1}) - \Pr(x \in A_{t+1}))^{N-n-1} \Pr(x \in S_{t+1})^n}{N \Pr(x \in S_t)^{N-1}}.
\]

Since \( \Pr(x \in S_t) - \Pr(x \in A_t) > \Pr(x \in S_{t+1}) - \Pr(x \in A_{t+1}) \) and \( \Pr(x \in S_t) > \Pr(x \in S_{t+1}) \), if \( x' < x \),

\[
(x - x') \frac{\sum_{n=0}^{N-1} (\Pr(x \in S_t) - \Pr(x \in A_t))^{N-n-1} \Pr(x \in S_t)^n}{N \Pr(x \in S_t)^{N-1}} > (x - x') \frac{\sum_{n=0}^{N-1} (\Pr(x \in S_{t+1}) - \Pr(x \in A_{t+1}))^{N-n-1} \Pr(x \in S_{t+1})^n}{N \Pr(x \in S_t)^{N-1}}.
\]

By (14) and (15),

\[
(b_t - x') \frac{\sum_{n=0}^{N-1} (\Pr(x \in S_t) - \Pr(x \in A_t))^{N-n-1} \Pr(x \in S_t)^n}{N \Pr(x \in S_t)^{N-1}} > (b_{t+1} - x') \frac{\sum_{n=0}^{N-1} (\Pr(x \in S_{t+1}) - \Pr(x \in A_{t+1}))^{N-n-1} \Pr(x \in S_{t+1})^n}{N \Pr(x \in S_t)^{N-1}},
\]

so a seller with cost \( x' \) also gets a higher expected payoff by accepting in round \( t \) than by accepting in round \( t + 1 \). The same arguments apply when we compare a seller’s payoffs if he accepts in round \( t \) and round \( \tau \geq t + 1 \). So, if a seller with cost \( x \) accepts in round \( t \), a seller with cost \( x' < x \) also accepts in round \( t \). ■
C Appendix: the stationary solution when $T = \infty$

**Proposition 4** A path that fully separates sellers is a stationary solution to program (P3) when $T = \infty$.

**Proof.** If the buyer fully separates sellers, we can rewrite the necessary condition (10) for a stationary solution as

$$0 = \left[ \bar{F}(x - dx) - \bar{F}(x) \right] \left[ -\bar{F}(x - dx)^{N-1} - \cdots - \bar{F}(x)^{N-1} - C'(x) \right]$$

$$- f(x) dx \left[ \bar{F}(x + dx)^{N-1} + \cdots + \bar{F}(x)^{N-1} - N \bar{F}(x)^{N-1} \right]$$

$$\Rightarrow 0 = f(x) dx \left[ -N \bar{F}(x)^{N-1} - \frac{(N - 1)N}{2} \bar{F}(x)^{N-2} f(x) dx - C'(x) \right]$$

$$- f(x) dx \left[ N \bar{F}(x)^{N-1} - \frac{(N - 1)N}{2} \bar{F}(x)^{N-2} f(x) dx - N \bar{F}(x)^{N-1} \right],$$

where $dx$ is a positive number that can be arbitrarily small. Note that $\frac{C(x)}{N}$ can be regarded as the information rent given to a seller with cost $x$. In our setting, in an incentive compatible mechanism that fully separates sellers with different costs, the information rent $R(x)$ has the property that $R'(x) = -\bar{F}(x)^{N-1}$, so that $\frac{C'(x)}{N} = -\bar{F}(x)^{N-1}$. Therefore, the necessary condition holds. Given $x_{t-1}$ and that $dx = x_{t+1} - x_t$ is arbitrarily small, if $x_t - x_{t-1}$ does not go to 0 at the same rate as $dx$, then (10) is negative. So the necessary condition is also sufficient, and a path that fully separates sellers is a stationary solution. $\blacksquare$

**References**


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